13/10/20

UNIT-2

RANDOM VARIABLES AND DISTRIBUTION FUNCTIONS

Random Variable Definition:

*. Let S be a sample space associated with a given random experiment. A real valued function defined on X, and taking values in $R(-\infty, \infty)$ is called one-dimensional random variable.

Another definition:

*. A random variable is a function $X(\omega)$ with domain S and range $(-\infty, \infty)$ such that for every real number "a" the event $[\omega : X(\omega) \leq a] \in B$.

Simply:

*. Associating the outcome of the experiment to the real numbers is called Random Variable.

Distribution Function:

D.f: Let X be a random variable the function F defined for all real x by

*. $F(x) = P(X \leq x) = P\{\omega : X(\omega) \leq x\}, -\infty < x < \infty$

is called the distribution function of the random variable (X). It is denoted by $F(x)$.

Simply:

*. Showing the random variable in function form is called distribution function.

Properties of Distribution Function:

*. The properties of the distribution function $F(x)$ is given below.

1.) If F is the distribution function of random variable X and if $a < b$, then

$$P(a < X \leq b) = F(b) - F(a).$$

2.) If F is distribution function of one-dimensional random variable X, then

i) $0 \leq F(x) \leq 1$

ii) $F(x) \leq F(y)$; if $x < y$.

(x) 2m (3.) If F is d.f of the one-dimensional random variable X, then

$$F(-\infty) = \lim_{x \to -\infty} F(x) = 0$$

$$F(\infty) = \lim_{x \to \infty} F(x) = 1.)$$

Discrete random variable :

D.f: A variable which can assume only a countable or single value number of real values and for which the variable takes depend on chance, is called a discrete random variable.

Another definition :

*. A real valued function defined on a discrete sample space is called a discrete random variable.

Examples for discrete random variable :

*. Accidents per month, no. of telephone calls per unit time, no. of success in young trial and so on.

*. Consider, tossing a coin twice, the sample space is

$$S = \{HH, HT, TH, TT\}$$

*. Let X be the random variable, That is

$X(HH) = 2$

$X(HT) = 1$

$X(TH) = 1$

$X(TT) = 0.$

*. Here X is a Discrete random variable Assuming the values 0, 1 and 2.

Probability Mass function:

*. If X is a discrete random variable with distinct values $x_1, x_2, \ldots, x_n, \ldots$ Then the function $p(x)$ defined as:

$$p(x) = \begin{cases} P(X = x_i) = p_i & ; \text{ if } x = x_i \\ 0 & ; \text{ if } x \neq x_i ; i = 1, 2, \ldots \end{cases}$$

is called the probability mass function and it satisfies following 2 conditions

i) $p(x_i) \geq 0 \; \forall i$

ii) $\sum_{i=1}^{\infty} p(x_i) = 1.$

Example;

In the above coin tossing example to $\boxed{S=4}$

$X = 2$ if $P = 0.25$

$X = 1$ if $P = 0.5$

$X = 0$ if $P = 0.25.$

$\Rightarrow X(HH) = 2 \Rightarrow 1/4 = 0.25$

$\left.\begin{array}{l} X(HT) \\ X(TH) \end{array}\right\} = 1 \Rightarrow 2/4 = 0.5$

$X(TT) = 0 \Rightarrow 1/4 = 0.25.$

$P[X=2] = 0.25$

$P[X=1] = 0.5$

$P[X=0] = 0.25$

$\overline{\quad 1 \quad}$

Hence X : 0 1 2

$P(x)$: 0.25 0.5 0.25 is probability Mass Function.

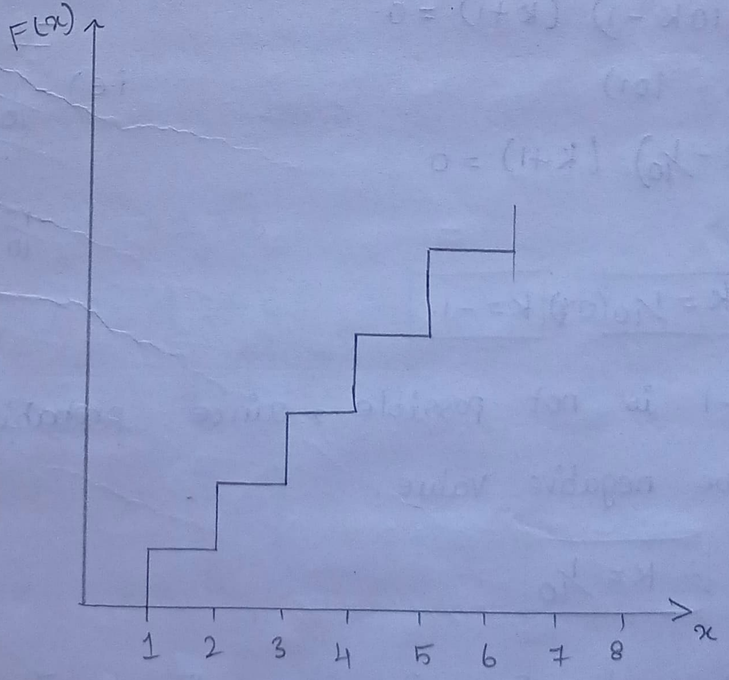
Distribution Function:

*. Let X be a Random Variable is D.F is defined by

$$F_X(x) = p[X \leq x]$$

$$= \sum_{i : x_i \leq x} p_i.$$

*. This Distribution Function is a step function as given below:



Example:

X : 0 1 2

$P(x)$: 0.25 0.5 0.25.

$F(X) = 0.25$ if $X \leq 0$

$F(X) = 0.75$ if $X \leq 1$.

$F(x) = 1$ if $x \leq 2$.

ii)

Example:

1.) A random variable X has the following probability functions:

X :	0	1	2	3	4	5	6	7
$P(x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Soln: i) Find the value of k

ii) Evaluate prob. of i) $P(X<6)$ ii) $P(X \geq 6)$

iii) $P(0<X<5)$.

Soln:

1.) Since given function is a part,

$$\Sigma p_i = 1.$$

$$k+2k+2k+3k+k^2+2k^2+7k^2+k=1$$

$$10k^2+9k-1=0.$$

-10

9

-1 10

$$\Rightarrow (10k-1)(k+1)=0.$$

(or)

i.e.) $\frac{-1}{10k}$; $\frac{\cancel{10}}{\cancel{10}k} = \frac{1}{k}$.

(or)

$$(k-1/10)(k+1)=0$$

$\frac{-1}{10}$; $\frac{\cancel{10}}{\cancel{10}} = 1$

where ;

$$\boxed{k=1/10} \text{ (or) } \boxed{k=-1.}$$

$\therefore k=-1$ is not possible, since probability can never be negative value.

$$\boxed{\therefore k = 1/10}$$

i)

X :	0	1	2	3	4	5	6	7
$P(x)$:	0	$1/10$	$2/10$	$2/10$	$\frac{3}{10}$	$1/100$	$2/100$	$17/100$

ii) 1.) $P(X<6) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$= 0 + 1/10 + 2/10 + 2/10 + 3/10 + 1/100$

$= \frac{10+20+20+30+1}{100}$

$P(X<6) = \frac{81}{100}$.

2.) $P(X \geq 6) = P(X=6) + P(X=7)$

$= 2/100 + 17/100$

$P(X \geq 6) = \frac{19}{100}$.

3.) $P(0 < x < 5) = P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= 1/10 + 2/10 + 2/10 + 3/10$.

$= \frac{1+2+2+3}{10}$

$= \cancel{8}^{4}/\cancel{10}_{5}$.

$P(0<x<5) = 4/5$

∴ Probability distribution:

$F_X(x) = P[X \leq x]$

$= 0$ if $x \leq 0$

$= 1/10$ if $x \leq 1$

$= 3/10$ if $x \leq 3$

$= 5/10$ if $x \leq 4$

$= 8/10$ if $x \leq 5$

$= \frac{81}{100}$ if $x \leq 5$

$= \frac{83}{100}$ if $x \leq 6$

$= \cancel{\frac{100}{100}} = 1$ if $x \leq 7$.

2) 2 dice are thrown. Let the random variable X denote sum of the two numbers on the dice.

i) Find the probability mass function

ii) Find the following probabilities.

1.) $\Rightarrow P(x \leq 3)$

2.) $\Rightarrow P(2 < x \leq 7)$

3.) $\Rightarrow P(x \geq 6)$

4.) $\Rightarrow P(10 < x < 12)$.

Soln:

When throwing 2 dice ;

$$S = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6)$$
$$(2,1)\ (2,2)\ (2,3)\ (2,4)\ (2,5)\ (2,6)$$
$$(3,1)\ (3,2)\ (3,3)\ (3,4)\ (3,5)\ (3,6)$$
$$(4,1)\ (4,2)\ (4,3)\ (4,5)\ (4,6)\ (4,7)$$
$$(5,1)\ (5,2)\ (5,3)\ (5,4)\ (5,5)\ (5,6)$$
$$(6,1)\ (6,2)\ (6,3)\ (6,4)\ (6,5)\ (6,6)\}$$

i) Let X denote the sum of the numbers on the 2 dice.

X :	2	3	4	5	6	7	8	9	10	11	12
P(x) :	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

$\therefore$ This is the probability mass function.

ii) 1.) $P(x \leq 3) = P(x=2) + P(x=3)$

$= \frac{1}{36} + \frac{2}{36} = \frac{\cancel{3}^{1}}{\cancel{36}_{12}}$

$= \frac{1}{12}$

2.) $P(2 < x \leq 7) = P(x=3) + P(x=4) + P(x=5) + P(x=6) + P(x=7)$

$= \frac{2}{36} + \frac{3}{36} + \frac{4}{36} + \frac{5}{36} + \frac{6}{36}$

$= \frac{\cancel{20}^{5}}{\cancel{36}_{9}} = \frac{5}{9}.$

3.) $P(x \geq 6) = P(x=6) + P(x=7) + P(x=8) + P(x=9) + P(x=10)$

$+ P(x=11) + P(x=12)$

$= 5/36 + 6/36 + 5/36 + 4/36 + 3/36 + 2/36 + 1/36$

$= \cancel{26}^{13}/\cancel{36}_{18} = \frac{13}{18}$

4.) $P(10 < x < 12) = P(x=11)$

$= 2/36.$

Probability distributions:

$F_{x}(x) = 1/36$ if $x \leq 2$

$= 3/36$ if $x \leq 3$

$= 6/36$ if $x \leq 4$

$= 10/36$ if $x \leq 5$

$= 15/36$ if $x \leq 6$

$= 21/36$ if $x \leq 7$

$= 26/36$ if $x \leq 8$

$= 30/36$ if $x \leq 9$

$= 33/36$ if $x \leq 10$

$= 35/36$ if $x \leq 11$

$= \cancel{36}^{1}/\cancel{36}_{1} = 1$ if $x \leq 12.$

3.) A random variable x has the following probability distribution.

x :	0	1	2	3	4	5	6	7	8
p(x):	a	3a	5a	7a	9a	11a	13a	15a	17a.

i) Find the value a.

ii.) Find the following probability:

1.) $p(x>3)$ 2.) $p(x\leq 3)$ 3.) $p(0<x\leq 5)$

iii) Distribution $f(x)$.

Soln:

i) Since given in a part;

$$\Sigma p_i = 1$$

$$a+3a+5a+7a+9a+11a+13a+15a+17a=1.$$

$$81a=1$$

$$\boxed{a=1/81}$$

$\therefore$

X :	0	1	2	3	4	5	6	7	8
p(x) :	1/81	3/81	5/81	7/81	9/81	11/81	13/81	15/81	17/81

ii) 1.) $p(x>3) = p(x=4)+p(x=5)+p(x=6)+p(x=7)+p(x=8)$

$$= 9/81 + 11/81 + 13/81 + 15/81 + 17/81.$$

$$= 65/81.$$

2.) $p(x\leq 3) = p(x=0)+p(x=1)+p(x=2)+p(x=3)$

$$= 1/81 + 3/81 + 5/81 + 7/81.$$

$$= 16/81.$$

3.) $p(0<x\leq 5) = p(x=1)+p(x=2)+p(x=3)+p(x=4)+p(x=5)$

$$= 3/81 + 5/81 + 7/81 + 9/81 + 11/81$$

$$= 35/81$$

iii) probability distribution:

$$F_X(x) = P[X\leq x]$$

$$= 1/81 \text{ if } x\leq 0$$

$$= 4/81 \text{ if } x\leq 1$$

$$= 9/81 \text{ if } x\leq 2$$

$= 16/81$ if $x \le 3$

$= 25/81$ if $x \le 4$

$= 36/81$ if $x \le 5$

$= 49/81$ if $x \le 6$

$= 64/81$ if $x \le 7$.

$= \cancel{81/81} = 1$ if $x \le 8$

4) A random variable X has the following probability distribution:

X :	-2	-1	0	1	2	3
p(x) :	0.1	K	0.2	2K	0.3	K

i) Find the value of K

ii) Find : 1.) $P(x \le 0)$ 2.) $P(-1 < x \le 3)$ 3.) $P(x > 1)$

iii) Also find distribution function.

Soln:

i) Since given is a part;

$\Sigma P_i = 1$.

$0.1 + K + 0.2 + 2K + 0.3 + K = 1$.

$4K + 0.6 = 1$

$4k = 1 - 0.6 = 0.4$

$k = 0.4/4$.

$\boxed{K = 0.1}$

$\therefore$

X :	-2	-1	0	1	2	3
p(x) :	0.1	0.1	0.2	0.2	0.3	0.1

ii) 1.) $p(x \le 0) = p(x=-2) + p(x=-1) + p(x=0)$

$= 0.1 + 0.1 + 0.2 = 0.2 + 0.2 = 0.4$.

2.) $p(-1 \le x \le 3) = p(x=0) + p(x=1) + p(x=2) + p(x=3)$

$= 0.2 + 0.2 + 0.3 + 0.1$

$= 0.8$

3.) $P(x > 1) = P(x = 2) + P(x = 3)$

$= 0.3 + 0.1$

$= 0.4.$

iii) Probability distribution function:

$F_X(x) = (X \leq x)$

$= 0.1$ if $x \leq -2$

$= 0.2$ if $x \leq -1$

$= 0.4$ if $x \leq 0$

$= 0.6$ if $x \leq 1$

$= 0.9$ if $x \leq 2$

$= 1.0$ if $x \leq 3.$

Continuous Random Variable:

*. The Random variable X is said to be continuous, if it can take all possible value between certain limits.

*. In other words, a Random variable is said to be continuous when its different values cannot be put in 1-1 correspondence with a set of positive integers.

Probability Density function:

*. Associated with each continuous Random variable X is a function called Probability density function namely $f(x)$ and it satisfies following two conditions:

1.) $f(x) \geq 0$

2.) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Various Measures of Central tendency, Dispersion etc.

1.) Arithmetic mean $= \int_a^b x f(x)\, dx$

2.) Geometric mean $\Rightarrow \log G = \int_a^b \log x . f(x) dx$

3.) Harmonic mean $\Rightarrow 1/H = \int_a^b 1/x . f(x) dx.$

4.) $\mu_r' = \int_a^b x^r . f(x)\, dx$

$\mu_r = \int_a^b (x - mean)^r . f(x)\, dx.$

5.) In particular;

$\mu_1' = \int_a^b x . f(x)\, dx = Mean$

$\mu_2' = \int_a^b x^2 . f(x) dx.$

and $\mu_1 = 0$

$\mu_2 = variance = \mu_2' - \mu_1'^2$

$= \int_a^b x^2 f(x) dx - \left[\int_a^b x . f(x)\, dx\right]^2$

6.) Medians:

*. Medians is the point which it divides the entire distribution in two equal parts. In case of continuous distribution, median is the point which divides the total area into two equal parts.

$\therefore$ M is the median;

$\int_0^M f(x) . dx = \int_M^b f(x) . dx) = 1/2.$

7.) Mean deviation:

$$M.D = \int_a^b |x - \text{mean}| f(x)\, dx$$

8.) Quartile deviation:

$$\int_a^{Q_1} f(x)\,dx = 1/4 \quad ; \quad \int_a^{Q_3} f(x)\,dx = 3/4.$$

5.) Example:

X be a continuous variable have the probability distribution.

$$f(x) = 6x(1-x), \quad 0 \le x \le 1$$
$$= 0 \text{ otherwise.}$$

i) verify whether $f(x)$ is the probability density function.

ii) Determine a number b such that prob. of $P(x<b) = P(x>b)$

Soln:

i) To verify whether $f(x)$ is the pdf:

$$\int_0^1 f(x)\,dx = \int_0^1 6x(1-x)\,dx$$

$$= 6\int_0^1 (x - x^2)\,dx. = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= 6\left[\frac{1}{2} - \frac{1}{3}\right] = \cancel{6}\left[\frac{3-2}{\cancel{6}}\right]$$

$$\int_0^1 f(x)\,dx = 1.$$

$\therefore \int_0^1 f(x)\,dx = 1$ therefore $f(x)$ is the probability density function.

ii) $P(x<b) = \int_0^b f(x)\,dx$

L.H.S

$$= \int_0^b 6x(1-x)\,dx$$

The limit is 0 to 1 and b is the intermittent between 0 and 1.

$$= 6\int_0^b (x - x^2)\,dx \qquad = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^b$$

$$= 6\left[\frac{b^2}{2} - \frac{b^3}{3}\right] = \cancel{6}\left[\frac{3b^2 - 2b^3}{\cancel{6}}\right]$$

$$p(x<b) = 3b^2 - 2b^3.$$

R.H.S $\quad p(x>b) = \int_b^1 6x(1-x^2)\cdot dx$

$$= 6\int_b^1 (x - x^2)\,dx = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_b^1$$

$$= 6\left[\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{b^2}{2} - \frac{b^3}{3}\right)\right]$$

$$= 6\left[\left(\frac{3-2}{6}\right) - \left(\frac{3b^2 - 2b^3}{6}\right)\right]$$

$$= \cancel{6}\left[\frac{1 - 3b^2 + 2b^3}{\cancel{6}}\right]$$

$$p(x>b) = 1 - 3b^2 + 2b^3.$$

Given $\quad p(x<b) = p(x>b).$

$\Rightarrow 3b^2 - 2b^3 = 1 - 3b^2 + 2b^3.$

$4b^3 - 6b^2 + 1 = 0.$

It is an cubic form.

$4b^3 - 6b^2 + 1 = 0$

For 4 the possible roots are $(2,2)\ (1,4)\ (4,1)\ (-2,-2)$

Try with 2 so $\cancel{2}/\cancel{4}_2 = 1/2.$

$b = 1/2 \Rightarrow 4(1/2)^3 - 6(1/2)^2 + 1 = 0.$

$$= \frac{\cancel{4}^1}{\cancel{8}_2} - \frac{\cancel{6}^3}{\cancel{4}_2} + 1. = \frac{1-3+2}{2}$$

$= 0$

$\therefore$ The value $b = 1/2$ can be possible root.

6.) A continuous random variable x has the following variable $f(x) = 3x^2, 0 \le x \le 1$.

i) verify $f(x)$ is a pdf.

ii) find a & b such that $p(x \le a) = p(x > a) = 0.05$.

iii) $p(x > b) = 0.05$.

Soln:

i) To verify whether $f(x)$ is pdf.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 3x^2 . dx.$$

$$= 3\left[\frac{x^3}{3}\right]_0^1 = \cancel{3}\left[\frac{1}{\cancel{3}}\right]$$

$$\int f(x) dx = 1$$

$\therefore f(x)$ is pdf.

ii) $p(x \le a) = p(x > a) = 0.05$.

$$p(x \le a) = a^3$$

$$a^3 = 0.05$$

$$a = (0.05)^{1/3}.$$

$$p(x > a) = \int_a^1 3x^2 . dx$$

$$= 1 - b^3.$$

iii) $p(x > b) = 0.05$.

$$1 - b^3 = 0.05.$$

$$b^3 = 1 - 0.05 = 0.95$$

$$b = (0.95)^{1/3}.$$

$p(x \le a) = p(x > a)$. since each must be equal to its $1/2$ as the total probability should be always 1

ii) $P(x \le a) = 1/2$; $P(x > a) = 1/2$.

$$\int_0^a f(x) . dx = 1/2.$$

$$\cancel{3}\left[\frac{x^3}{\cancel{3}}\right]_0^a = 1/2.$$

$$a^3 = 1/2 \Rightarrow a = (1/2)^{1/3}$$

$$a = (0.5)^{1/3}.$$

iii) $P(x > b) = 0.05$.

$$\int_b^1 f(x) . dx = 0.05.$$

$$\cancel{3}\left[\frac{x^3}{\cancel{3}}\right]_b^1 = 0.05$$

$$1 - b^3 = 0.05.$$

$$1 - 0.05 = b^3.$$

$$0.95 = b3 \Rightarrow b = (0.95)^{1/3}$$

$$b = (19/20)^{1/3}$$

7) Let x be a continuous variable with Pdf.

$$f(x) = \begin{cases} ax & , 0 \leq x \leq 1 \\ a & , 1 \leq x \leq 2 \\ -ax + 3a & , 2 \leq x \leq 3 \\ 0 & , \text{else.} \end{cases}$$

i) determine a   ii) Find $p(x \leq 1.5)$.

i) Soln: The total probability is unity.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

$$\Rightarrow \int_0^1 ax \cdot dx + \int_1^2 a\,dx + \int_2^3 (-ax + 3a)\,dx = 1.$$

$$a\int_0^1 x \cdot dx + a\int_1^2 dx + a\int_2^3 (-x + 3) \,dx = 1.$$

$$a\left[\frac{x^2}{2}\right]_0^1 + a\left[x\right]_1^2 + a\left[-\frac{x^2}{2} + 3x\right]_2^3 = 1.$$

$$a\left[\frac{1}{2}\right] + a\left[2-1\right] + a\left[\left(-\frac{9}{2} + 9\right) - \left(-\cancel{\frac{4}{2}}^{2} + 6\right)\right] = 1$$

$$a/2 + a + a\left[\left(\frac{-9+18}{2}\right) - (-2+6)\right] = 1.$$

$$a/2 + a + a\left[\frac{9}{2} - 4\right] = 1$$

$$a/2 + a + a\left[\frac{9-8}{2}\right] = 1.$$

$$a/2 + a + a/2 = 1.$$

$$\frac{a + 2a + a}{2} = 1.$$

$$4a/2 = 1.$$

$$a = \cancel{2}^{1}/\cancel{4}_{2}.$$

$$\boxed{\therefore a = 1/2}$$

ii) $p(x \leq 1.5)$

$$= \int_{-\infty}^{1.5} f(x)\,dx.$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{1.5} f(x)\,dx\}$$

$$= 0 + \int_{0}^{1} ax.dx + \int_{1}^{1.5} a\,dx = a\left[\frac{x^2}{2}\right]_0^1 + a[x]_1^{1.5}$$

$$= a\left[\tfrac{1}{2}\right] + a[1.5-1].$$

$$= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)(0.5).$$

$$= \frac{1}{4} + \frac{0.5}{2} = \frac{1+0.5\times 2}{4} = \frac{1+1}{4}.$$

$$= \cancel{2}/\cancel{4}_{2} = \frac{1}{2}.$$

$\therefore p(x \le 1.5) = \frac{1}{2}.$

5th sum continuation:

Hence $b = \frac{1}{2} \Rightarrow (2b-1)$ will the term.

$$
\begin{array}{r|l}
 & 2b^2 - 2b - 1 \\
\hline
2b-1 & 4b^3 - 6b^2 + 0b + 1 \\
 & \cancel{4b^3} - 2b^2 \\
 & (-) \quad (+) \\
\hline
 & -4b^2 + 0b \\
 & \cancel{-4b^2} + 2b \\
 & (+) \quad (-) \\
\hline
 & -2b + 1 \\
 & \cancel{-2b} + \cancel{1} \\
 & (+) \quad (-) \\
\hline
 & 0
\end{array}
$$

$\therefore$ The factorisation for $4b^3 - 6b^2 + 1$ is $(2b-1)(2b^2-2b-1)$

Now $2b-1=0$ (or) $2b^2-2b-1=0$.

$b = \frac{1}{2}$ (or) $\cancel{x} = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}.$

$$= \frac{+2 \pm \sqrt{4-4(2)(-1)}}{2(2)}$$

$$= \frac{2 \pm \sqrt{4+8}}{4} = \frac{2 \pm 2\sqrt{3}}{4}.$$

$$= \cancel{2}\left(\frac{1 \pm \sqrt{3}}{\cancel{4}\,2}\right) = \frac{1 \pm \sqrt{3}}{2}. = \frac{1 \pm \sqrt{3}}{2}.$$

because a real no lying between 0 and 1 hence b cannot be $\frac{1 \pm \sqrt{3}}{2}$.

$\frac{1+\sqrt{3}}{2} = 1.366.$

$\frac{1-\sqrt{3}}{2} = -0.366.$

$$\boxed{\therefore b = \tfrac{1}{2}.}$$

8) Prove that the geometric mean G of the distribution d.F (or) $f(x) = 6(2-x)(x-1)dx$; $1 \le x \le 2$ is given by $6 \log 16G = 19$.

Soln:

By def of G.M $\Rightarrow \log G = \int_a^b \log x . f(x) dx.$

$$= \int_1^2 \log x . 6(2-x)(x-1) dx.$$

$$= 6 \int_1^2 (2x - x^2 - 2 + x) \log x \, dx.$$

$$= 6 \int_1^2 (3x - x^2 - 2) \log x \, dx.$$

$$\log G = -6 \int_1^2 (x^2 - 3x + 2) \log x \, dx. \text{ --- (1)}$$

using the above integration by parts method.

WKT, $\int u dv = uv - \int v du. = \int_1^2 (x^2 - 3x + 2) \log x \, dx.$

$u = \log x$; $dv = \cancel{\log x \, dx}\,(x^2 - 3x + 2)$

$du = \frac{1}{x} dx$; $v = \frac{x^3}{3} - \frac{3x^2}{2} + 2x.$

$$\int u dv = \left[(\log x)\left(\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right)\right]_1^2 - \int_1^2 \left(\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right) \frac{1}{x} dx$$

First we take the uv values to simplify;

$$= \left[(\log x)\left(\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right)\right]_1^2$$

$a[x]_1^{1.5}$

$(2b^2 - 2b - 1)$

$= \log 2 \left[8/3 - 12/2 + 4 \right] - \log 1.$ $\quad [\because \log 1 = 0]$

$\cancel{= \log 2 \left[\frac{8 - 18 + 12}{3} \right]} \cancel{= \log 2 \left[2/3 \right]}$

$= \log 2 \left[\frac{16 - 36 + 24}{6} \right] = \log 2 \left[4/6 \right].$

Then we take the $\int v du$ values:

$$-\int v du = -\int_1^2 \left(x^3/3 - 3x^2/2 + 2x \right) 1/x \, dx.$$

$$= -\int_1^2 \left(\frac{x^{\cancel{3}2}}{3\cancel{x}} - \frac{3x^{\cancel{2}}}{2\cancel{x}} + \frac{2\cancel{x}}{\cancel{x}} \right) dx. = -\int_1^2 \left(x^2/3 - \frac{3x}{2} + 2 \right) dx$$

$$= -\left[\left(\frac{x^3}{3 \times 3} - \frac{3x^2}{2 \times 2} + 2\cancel{x} \right) \right]_1^2 = -\left[\frac{x^3}{9} - \frac{3x^2}{4} + 4 \right]_1^2$$

$$= -\left[\left(8/9 - \frac{12}{4} + 4 \right) - \left(1/9 - 3/4 + 2 \right) \right].$$

$$= -\left[8/9 - \frac{12}{4} + 4 - 1/9 + 3/4 - 2 \right]$$

$$= -\left[7/9 - 9/4 \right] + 2 = \left[9/4 - 7/9 \right] - 2$$

$$= \left[\frac{81 - 28 - 72}{36} \right] = \left[\frac{81 - 100}{36} \right]$$

$$= -19/36. \Rightarrow \int u dv = \log 2 \left[4/6 \right] - (19/36).$$

$\therefore$ Subs in ①;

$$\log G = -6 \left[(4/6) \log 2 - 19/36 \right].$$

$$= \left[-\cancel{6} (4/\cancel{6}) \log 2 \right] - \left[-\cancel{6} \left(\frac{19}{\cancel{36}_6} \right) \right].$$

$$= -4 \log 2 + 19/6.$$

$\log G + 4 \log 2 = 19/6.$

$(\because \log a + b = \log ab)$

$\log G + \log 2^4 = 19/6. \Rightarrow \log (G * 2^4) = 19/6$

$\log(16G) = 19/6$

$\boxed{6 \log 16 G = 19}$

Hence proved.

9.) A random variable x is distributed at random between the value 0 and 1 so that its probability density function is $f(x) = kx^2(1-x^3)$ where k is constant. Find the value of k, mean and variance.

Soln:

i) wkt,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1. \Rightarrow \int_0^1 kx^2(1-x^3)\,dx = 1.$$

$$k\int_0^1 (x^2 - x^5)\,dx = 1$$

$$k\left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^1 dx = 1.$$

$$k\left[\frac{1}{3} - \frac{1}{6}\right] - 0 = 1. \Rightarrow k\left[\frac{2-1}{6}\right] = 1.$$

$$k\left(\frac{1}{6}\right) = 1.$$

$$\boxed{k=6}$$

ii) Mean $= \int_a^b x f(x)\,dx = \mu_1'$

$$= \int_0^1 x\,(k\,x^2(1-x^3))\,dx. = 6\int_0^1 (x^3 - x^6)\,dx$$

$$= 6\left[\frac{x^4}{4} - \frac{x^7}{7}\right]_0^1 = 6\left[\frac{1}{4} - \frac{1}{7}\right]$$

$$= 6\left[\frac{7-4}{28}\right] = {}^{3}\cancel{6}\left[\frac{3}{\cancel{28}_{14}}\right] = \frac{9}{14}.$$

$$\boxed{\therefore \text{Mean} = 9/14}$$

iii) Variance $= \mu_2' - \mu_1'^2$

$$\mu_2' = \int_{-\infty}^{\infty} x^2 . f(x)\,dx \quad \text{(By formula)}$$

$$= \int_0^1 x^2 \left(Kx^2(1-x^3)\right)dx = 6\int_0^1 (x^4 - x^7)\,dx.$$

$$= 6\left[\frac{x^5}{5} - \frac{x^8}{8}\right]_0^1 = 6\left[\frac{1}{5} - \frac{1}{8}\right]$$

$$= 6\left[\frac{8-5}{40}\right] = \cancel{6}^{3}\left[\frac{3}{\cancel{40}_{20}}\right].$$

~~mean~~ $\mu_2' = 9/20$

$$\therefore \text{Variance} = 9/20 - \left(9/14\right)^2.$$

$$= 9/20 - \frac{81}{196} = \frac{1764-1620}{3920} = \frac{144}{3920}.$$

$$\therefore \text{Variance} = \frac{9}{245}$$

10.) In a continuous distribution whose relative frequency density is given by $f(x) = y_0\, x(2-x)$, $0 \le x \le 2$.

i) Find mean, variance, β_1, β_2 also show the distribution is symmetrical.

ii) Find mean deviation about mean.

iii) Show that for this distribution $\mu_{2n+1} = 0$.

iv) Find mode, harmonic mean and median.

Soln:

$\therefore$ The total probability is unity $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$y_0 \int_0^2 x(2-x)\,dx = y_0 \int_0^2 (2x - x^2)\,dx = 1$$

$$= y_0\left[\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3}\right]_0^2 = y_0\left[\cancel{x^2}\, 4 - \frac{8}{3}\right]$$

$$y_0\left[\frac{12-8}{3}\right] = 1$$

$y_0(4/3) = 1$

$\boxed{y_0 = 3/4}$

Then; we take a raw mean formula;

$$\mu_r' = \int_0^2 x^r f(x)\,dx$$

[when sub $r = 1, 2, 3, 4 \ldots$
$r = 1$ is mean
$r = 2$ is variance
for finding β_1 and β_2 we put some value on r]

$$= 3/4 \int_0^2 x^r \cdot x(2-x)\,dx$$

$$= 3/4 \int_0^2 x^{r+1}(2-x)\,dx$$

$$= 3/4 \int_0^2 (2 \cdot x^{r+1} - x^{r+2})\,dx = 3/4\left[2\frac{x^{r+2}}{r+2} - \frac{x^{r+3}}{r+3}\right]_0^2$$

$$= \left[\frac{3}{\cancel{4}_2} \times \cancel{2}^1 \frac{x^{r+2}}{r+2} - 3/4\,\frac{x^{r+3}}{r+3}\right]_0^2$$

$$= \frac{3}{2} \times \frac{2^{r+2}}{r+2} - 3/4\,\frac{2^{r+3}}{r+3}$$

$$= \left(\overbrace{3 \times 2^{-1} \times 2^{r+2}}^{3\times 2^{r+1}}\Big/ (r+2)\right) - \overbrace{3 \times 2^{-2} \times 2^{r+3}}^{3\times 2^{r+1}}\Big/(r+3)$$

$$= 3 \times 2^{r+1}\left[\frac{1}{r+2} - \frac{1}{r+3}\right]$$

$$= 3 \cdot 2^{r+1}\left[\frac{\cancel{r}+3-\cancel{r}-2}{(r+2)(r+3)}\right]$$

$$\mu_r' = \frac{3 \cdot 2^{r+1}}{(r+2)(r+3)}$$

$\therefore$ Mean: i) $\mu_1' = \dfrac{3 \cdot 2^{1+1}}{(1+2)(1+3)} = \dfrac{3 \cdot 2^2}{(3)(4)} = \dfrac{\cancel{12}}{\cancel{12}} = 1.$

ii) $\mu_2' = \dfrac{3 \cdot 2^{3}}{(4)(5)} = 6/5.$

iii) $\mu_3' = \dfrac{\cancel{3} \cdot 2^{4}}{(5)(\cancel{6})_2} = 8/5.$

iv) $\mu_4' = \dfrac{\cancel{3} \cdot 2^5}{(\cancel{6})(7)} = 16/7.$

Variance $= \mu_2' - (\mu_1')^2 = 6/5 - (1)^2$
(μ_2)

$= \frac{6-5}{5} = 1/5$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$= (8/5) - 3(6/5)(1) + 2(1)^3 = 8/5 - 18/5 + 2$

$= \frac{8-18+10}{5} = 0$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$= (16/7) - 4(8/5)(1) + 6(6/5)(1)^2 - 3(1)^4$

$= 16/7 - 32/5 + 36/5 - 3 = 16/7 + 4/5 - 3$

$= \frac{80+28-105}{35} = 3/35$

$\therefore \beta_1 = \frac{\mu_3^2}{\mu_2^2} = 0 \quad ; \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{(3/35)}{(1/5)^2} = \frac{3}{\cancel{35}_7} \times \cancel{25}^5$

$= 15/7$

$\therefore \beta_1 = 0$ the distribution is symmetrical.

Symmetrical Definition:

*. When the mean value (M), median value ($\bar{x}$) and mode value (Z) are equal is called symmetrical.

Mean deviation about mean:

$M.D = \int_a^b |x - \text{mean}| f(x) dx$

$= \int_0^1 |x-1| f(x) dx + \int_1^2 |x-1| f(x) dx$

$= 3/4 \left[\int_0^1 (1-x) x (2-x) dx + \int_1^2 (x-1) x (2-x) dx \right]$

$$= \frac{3}{4}\left[\int_0^1 (x-x^2)(2-x)\,dx + \int_1^2 (x^2-x)(2-x)\,dx\right]$$

$$= \frac{3}{4}\left[\int_0^1 (2x-2x^2-x^2+x^3)\,dx + \int_1^2 (2x^2-x^3-2x+x^2)\,dx\right]$$

$$= \frac{3}{4}\left[\int_0^1 (2x-3x^2+x^3)\,dx + \int_1^2 (3x^2-x^3-2x)\,dx\right]$$

$$= \frac{3}{4}\left[\left(\int_0^1 \frac{\cancel{2}x^2}{\cancel{2}} - \frac{\cancel{3}x^3}{\cancel{3}} + \frac{x^4}{4}\right) + \left(\int_1^2 \frac{\cancel{3}x^3}{\cancel{3}} - \frac{x^4}{4} - \frac{\cancel{2}x^2}{\cancel{2}}\right)\right]$$

$$= \frac{3}{4}\left(\left[x^2 - x^3 + x^4/4\right]_0^1 + \left[x^3 - x^4/4 - x^2\right]_1^2\right)$$

$$= \frac{3}{4}\left(\left[\cancel{1}-\cancel{1}+1/4\right] + \left[(\cancel{8}-\cancel{16}/4-\cancel{4}) - (\cancel{1}-1/4-\cancel{1})\right]\right)$$

$$= \frac{3}{4}\left(\frac{1}{4}+\frac{1}{4}\right) = \frac{3}{4}\left(\frac{2}{4}\right) = \frac{3}{8}.$$

Then $\mu_{2n+1} = 0$.

$$\mu_{2n+1} = \int_0^2 (x-\text{mean})^{2n+1} f(x)\,dx.$$

$$= \frac{3}{4}\int_0^2 (x-1)^{2n+1}\, x(2-x)\,dx$$

By substitution method;

Let $t = x-1 \Rightarrow x = t+1$

$2-x = 2-t-1$. ($\because$ from x)

$= 1-t$.

$\therefore$ The limits of x are 0 and 2.

if $x=0$; $t=0-1 \Rightarrow t=-1$.

if $x=2$; $t=2-1 \Rightarrow t=1$.

$\therefore$ The limits of t are -1 and 1

Hence;

$$\mu_{2+1} = \left[\int_{-1}^{1} t^{2n+1}(t+1)(1-t)\,dt\right]\frac{3}{4}.$$

$$= \left[\int_{-1}^{1} t^{2n+1}(1-t^2)\,dt\right].\frac{3}{4}. \quad (\because (a+b)(a-b) = (a^2-b^2))$$

$\therefore$ Here (t^{2n+1}) is and odd function of t and $(1-t^2)$ is an even function of t.

Integration of odd and even function is an odd function of t and is equal to 0.

$$\boxed{\therefore \mu_{2+1} = 0}$$

Then;

$\Rightarrow$ Mode = most frequently occuring value.

$$f(x) = \frac{3}{4}\,x(2-x)$$

$$= \frac{3}{4}(2x-x^2).$$

$$f'(x) = \frac{3}{4}(2-2x) = \frac{3}{4}\times 2 - \frac{3}{4}\times 2x. \quad (\because \text{Diff } f(x))$$

$$= \frac{3}{2} - \frac{3}{2}x.$$

$$f'(x) = 0 \Rightarrow \frac{3}{2} - \frac{3}{2}x = 0.$$

$$\frac{3}{2} = \frac{3}{2}x.$$

$$\boxed{\therefore x = 1}.$$

$\therefore f''(x) = -\frac{3}{2} < 0$ hence $x=1$ is the maximum value

$$\boxed{\therefore \text{Mode} = 1}$$

Then;

Harmonic mean : $\frac{1}{H} = \int_a^b \frac{1}{x} f(x)\,dx.$

$$\frac{1}{H} = \frac{3}{4}\int_0^2 \frac{1}{x}\cdot x(2-x)\,dx.$$