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II BSc. - O.R - V. Queueing Theory

Introduction

Ordinarily the line that forms in front of service facilities is called a Queue or a waiting line. A queue, thus involves arriving Customers (or items) who wait to be serviced at the facility which provide the service they want to have. In brief, the word queue refers to waiting in line.

The idea about a queue may be expressed as under:

Customers arrival → Possible waiting line or queue → Getting Serviced → Departure of Customers

Object of the Queuing Theory

If there are queues then Customers have to wait for some time before service. The time lost in waiting is often expensive in terms of money, equipments etc., As such there are costs associated with waiting in line, commonly known as waiting-time cost. On the otherhand if there are no queues, members of the servicing stations might stay idle. costs associated with service on the facility are known as service cost.

The object of the queing theory is to acheive a good economic balance between these two types of costs and the optimum soln.. is arrived at a point where the sum of the waiting time and service costs are minimum. In brief, the object of any queing problem is to minimize the total waiting and service costs.

It is important to note that queing theory does not directly solve the problem of minimizing the total waiting and service costs but the theory provides the management with information necessary to take relevant decisions for the purpose. It does the job by estimating different characteristics of the waiting line such as

1. The average arrival rate
2. the average service rate
3. the average length of the queue
4. the average waiting time in the queue and
5. the average time spent in the system.

Characteristics of Queuing Models

A queing model is specified completely by six main characteristics:

1. Arrival distribution
2. ~~Depar~~ Service distn..
3. Service Channels
4. ~~Arrival~~ Queue discipline
5. Queue size
6. Calling source or population

1. Arrival distribution

The job or a person entering the system is known as customer to the system. The customer entry to the system is given by the arrival distn.. The arrival of customer is defined by the arrival rate ie.. the no. of customers entering per unit time. In almost all the queueing problems customer entry is random. In the random arrivals, the poisson distn. is used to describe the arrival distn.. Based on the Poissons distn.., the prob.. of 'n' arrivals in time 't' is given by

$$P(n \text{ arrivals in time } t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

where λ = mean arrival rate.

It is also necessary to know the reaction of a customer upon entering the system. There are three types.

a) Balking

If a customer decides not to enter the queue because of its huge length, he is said to have balked

b) Reneging

A customer may enter the queue, but after sometime loses patience and decides to leave. In this case, the customer is said to have reneged.

c. Jockeying

In the Cases when there are two or more parallel queues, the Customer may move from one queue to another for his personal economic gains, that is jockey for position.

2. Service distribution

It describes the pattern by which the service is provided to the Customers for the pattern by which the Customers leave the system. The departure of a Customer depends on the service rate. Service rate is defined on the no. of Customers leaving the system per unit time. In the random service time, the exponential prob. distn. of 'n' complete services in time 't' is ~~given by~~ written, without proof.

$$P(n \text{ complete services in time } t) = \frac{e^{-\mu t}(\mu t)^n}{n!}$$

where μ = mean service rate

3. Service channels

The queing system may be single channel system in which there is one server or multi Channel system in which there are more than one server

4. Service (or) Queue discipline

The service discipline describes the way in which the Customer is selected from the queue for service. The different service disciplines are

a) First Come First Served - FCFS

b) Last Come First Served - LCFS

c) Random selection Service In Random order - SIRO

d) Selection based on priority

The most Common service discipline is FCFS.

5. Queue size

The queue size gives the maximum nos of Customers in the system. This may be finite or infinite.

6. Calling Source or population

The population or calling source of the queueing system may be finite or infinite.

Classification of Queues

Generally Queueing model may be completely specified in the following symbol form : $(a/b/c) : (d/e/f)$

where a = ~~Probability law of arrival~~ arrival distribution

b = service distribution

c = no. of service channels in the system

d = service discipline

e = max. no. of Customers allowed in the system

f = Calling ~~source~~ or population.

Transient State

A system is said to be in Transient State when its operating characteristics (like input, output, mean queue length etc.) are dependent on time

Steady State

A system is said to be in Steady state when the behaviour of the system is independent of time.

Traffic intensity (or) Utilisation factor

This is the ratio between arrival and service rate and is given by

$$\rho = \frac{\lambda}{\mu}.$$

If $\rho > 1$, then the queue will grow without end

$\rho = 1$, then no change in queue length, will be noticed.

$\rho < 1$, then the length of the queue ~~length~~ will go on diminishing gradually.

Terms Commonly used in Queueing Theory

1. Customer : Persons or Units arriving at a station for sevile; Customeers may be either persons or machines or other items.

2. Service Station : Point where service is provided

3. Waiting time : Time a Customer spends in the queue before being serviced

4. Time spent by a Customer in the system :

Waiting time + Service time

5. Number of Customers in the system : No.. of Customers in the queue plus the no.. of Customers being serviced

6. Queue length : No. of Customers waiting in the queue

7. Queueing System : System consisting arrival of Customers, Waiting in queue, Picked up for service according to a certain discipline, being serviced and the departure of Customers.

Classification of Queues

D.G. Kendall and A. Lee introduced useful notation for queuing models. The Complete notation can be expressed as

$$(a/b/c) : (d/e/f)$$

Where a = arrival distribution

b = Service distribution

c = no.. of service channels in the system

d = Service discipline

e = max. no.. of Customers allowed in the system

f = Calling source or population.

~~For example, (M/M/1) : (FCFS/N/∞)~~
~~represents~~

The following conventional codes are generally used to replace the symbols a, b, c, d, e & f.

For a & b

M = Markovian (Poisson) arrival (or) departure distr.. (or exponential service time distr.)

For c

1 = single service channel

c = multi service channel

For d

FCFS = first come, first served

LCFS = Last come, first served

For e & f

N = finite no.. of customers in the system

∞ = infinite "

For example, (M/M/1) : (FCFS/N/∞)

represents Poisson arrival, Exponential service, single server, 'first come, first served' discipline, max. allowable customers 'N' in the system and infinite population model.

—x

→ ⊗ Let us consider a single channel system with Poisson arrivals and exponential service time distr.. Both the arrivals and service rates are indept of the no.. of customers in the waiting line. Arrivals are handled in FCFS basis. Also the arrival rate λ is lesser than the service rate μ.

Model I : Single channel Poisson Arrivals with Exponential service Infinite Popln. model ~~[(M/M/1) : (FCFS/∞/∞)]~~

(M/M/1 : FCFS/∞/∞)

Assumptions

1. One queue and it is sufficiently long
2. One service station
3. Queue discipline: 'First Come first served'.
4. The popln. from which the queue does arise is sufficiently large
5. Arrivals and services are coming individually.
6. Arrivals and services occur in accordance with a Poisson process. The mean arrival rate is often represented by λ and the mean service rate by μ.
7. As a result of above, the time intervals between successive arrivals (i.e., the waiting time between events or what is called 'inter-arrival time') follow exponential distn.. and as such the mean inter-arrival time is represented as

$$\text{Mean inter-arrival time} = \frac{1}{\text{mean arrival time}} = \frac{1}{\lambda}$$

Similarly the time-intervals between services follow exponential distn.. and as such the mean time taken to service a unit is represented as

$$\text{Mean Inter-service time} = \frac{1}{\text{mean service rate}} = \frac{1}{\mu}.$$

The following formulae are ~~seen~~ often used when the ratio λ/μ is less than one.

$\boxed{\mu > \lambda}$

1. Expected no.. of Customers in the queue $\Big\} = E(n_q) = \dfrac{\lambda^2}{\mu(\mu-\lambda)}$

2. Expected no.. of Customers in the system (waiting line + Service) $\Big\} = E(n_s) = \dfrac{\lambda}{\mu-\lambda}$

3. Expected waiting time per Customer in the queue $\Big\} = E(W_q) = \dfrac{\lambda}{\mu(\mu-\lambda)}$

4. Expected waiting time per Customer in the system $\Big\} = E(W_s) = \dfrac{1}{\mu-\lambda}$

5. The prob.. that the system is busy (or) Utilisation factor. $\Big\} = \rho = \dfrac{\lambda}{\mu}$

This ratio is also known as traffic intensity.

6. The prob.. that the system is empty $\Big\} = P_0 = 1-\rho = 1-\dfrac{\lambda}{\mu}$

7. The steady state probability of 'n' Customers in the system $\Big\} = P_n = \left(\dfrac{\lambda}{\mu}\right)^n \left(1-\dfrac{\lambda}{\mu}\right)$

8. Average length of the non-empty queue (length of the queue that is formed from time to time) (Expected no.. of Customers waiting in line excluding those times when the line is empty) $\Big\} = L_n = \dfrac{\mu}{\mu-\lambda}$

9. Average waiting time in non-empty queue $\Big\} = \cancel{W_q = \dfrac{1}{\mu-\lambda}}$

9. Probability of Queue size $\geq N = \rho^N$.

10. Probability [Waiting time in the Queue $\geq t$] $= \int_t^{\infty} \left(\frac{\lambda}{\mu}\right)(\mu-\lambda)e^{-(\mu-\lambda)t}\,dt$

11. Prob [w.time in the system $\geq t$] $= \int_t^{\infty} (\mu-\lambda)e^{-(\mu-\lambda)t}\,dt$

① A firm has a single machinist in a repair shop. He works eight hours and an average four machines break each day. It takes on the average one hour to repair a machine. Using Poisson-exponential model, determine

i) the expected no. of machines in a repair shop

ii) the expected no. of machines in the shop on which the machinist has not started to work

iii) the average waiting time in the system

iv) the average time a machine waits for service

v) the expected proportion of time a facility will be idle.

Soln:

We are given λ = Average no. of arrivals per unit time

$= \frac{4}{8} = 0.5$ machines / hour.

μ = Average no. of Customers that can be serviced per unit time

= 1 machine / hour

i) Expected no. of machines in the repair shop $\}$ = Expected no. of Customers in the system

$$E(n_s) = \frac{\lambda}{\mu-\lambda} = \frac{0.5}{1-0.5} = \frac{0.5}{0.5} = 1 \text{ machine}$$

ii) Expected no. of machines in the shop on which the machinist has not started to work = Expected no. of Customer in the queue

$$E(n_q) = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{(0.5)^2}{1\times(1-0.5)} = 0.5 \text{ machine}$$

iii) Average waiting time in the system

$$E(W_s) = \frac{1}{\mu-\lambda} = \frac{1}{1-0.5} = 2 \text{ hrs}$$

iv) Average time a machine waits for service = Average waiting time in the queue

$$E(W_q) = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{0.5}{1(1-0.5)} = 1 \text{ hr.}$$

v) The expected proportion of time a facility will be idle

$$P_0 = 1-\rho = 1-\frac{\lambda}{\mu} = 1-\frac{0.5}{1} = 0.5 \text{ hr.}$$

Q) Customers arrive at a box office window, being manned by single individual according to poisson's input process with a mean rate of 30 per hour. Time required to serve a Customer has an exponential distn. with a mean of 90 seconds. Find the average waiting time of a Customer. Also determine the average no. of Customer in the system, queue and average queue length of non empty queue.

Soln:

Mean arrival rate $= \lambda = 30/\text{hour}$

Mean Service rate $= \mu = \frac{3600}{90} = 40/\text{hour}$.

i) Average waiting time of a customer in the queue

$$E(W_q) = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{30}{40(40-30)} = 0.074 \text{ hrs.}$$

ii) Average no. of customers in the system

$$E(n_s) = \frac{\lambda}{\mu-\lambda} = \frac{30}{40-30} = 3 \text{ customers}$$

iii) Average no. of customers in the queue

$$E(n_q) = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{30^2}{40(40-30)} = 2.25 \text{ customers}$$

iv) Average length of non-empty queue

$$L_n = \frac{\mu}{\mu-\lambda} = \frac{40}{40-30} = 4$$

Q) Consider a self service store with one cashier. Assume Poisson arrivals and exponential service times. Suppose 9 customers arrive on the average every 5 minutes and the cashier can serve 10 in 5 minutes. Find

i) the avg. no. of customers queueing for service

ii) the prob. of having more than 10 customers in the queue?

iii) the prob.. that a Customer has to queue for more than 2 minutes

iv) if the service can be speeded upto 12 in 5 minutes by using different Cashier what will be effect on the quantities (i), (ii) & (iii).

Soln :

We are given Arrival rate $\lambda = \frac{9}{5}$ per minute.

Service rate $\mu = \frac{10}{5}$. per minute.

i). Avg. no.. of Customers in the queue,

$$E(n_q) = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{(9/5)^2}{\frac{10}{5}\left(\frac{10}{5}-\frac{9}{5}\right)} = 8.1$$

ii) The prob.. of having more than 10 Customers in the queue

$$\rho^n = \rho^{10} = \left(\frac{\lambda}{\mu}\right)^{10} = \left(\frac{9/\cancel{5}}{10/\cancel{5}}\right)^{10} = (0.9)^{10} = 0.35.$$

iii) The prob.. that a Customer has to queue for more than 2 minutes is

$$P = \int_t^\infty \left(\frac{\lambda}{\mu}\right)(\mu-\lambda)\, e^{-(\mu-\lambda)t}\, dt$$

$$= \int_2^\infty (0.9)\left(\frac{10}{5}-\frac{9}{5}\right) e^{-\left(\frac{10}{5}-\frac{9}{5}\right)t} dt$$

$$= \int_2^\infty (0.9)(0.2)\, e^{-0.2t}\, dt$$

$$= 0.18 \int_2^\infty e^{-0.2t}\, dt$$

$$= 0.18 \left(\frac{e^{-0.2t}}{-0.2} \right)_2^{\infty} \qquad \int e^{-x} dx = \frac{e^{-x}}{-1}$$

$$= \frac{0.18}{-0.2} \left[e^{-0.2t} \right]_2^{\infty}$$

$$= -0.9 \left[e^{-\infty} - e^{-0.4} \right]$$

$$= -0.9 \left[0 - 0.67 \right]$$

$$= \underline{0.6033}$$

If $\mu = \frac{12}{5}$ per minute $E(n_q) = 2.25$

$$\rho^n = 0.056$$

$$p = 0.226.$$

① At a public telephone booth in a post office arrivals are considered to be Poisson with an average inter arrival time of 12 minutes. The length of the telephone call may be assumed to be distributed exponentially with an average of 4 minutes. Calculate the following :

1) what is the prob.. that a fresh arrival will not have to wait for the telephone

ii) what is the prob.. that an arrival will have to wait more than 10 minutes before the telephone is free

iii) what is the average length of queue that form from time to time.

Soln :

We are given Arrival rate $\lambda = \frac{1}{12}$ per min.

and

Service rate $\mu = \frac{1}{4}$ per min.

i) The prob. that a fresh arrival will not have to wait is

$$P_0 = 1 - \rho = 1 - \frac{\lambda}{\mu} = 1 - \frac{1/12}{1/4} = 0.667$$

ii) The prob. that an arrival will have to wait more than 10 minutes before the phone is free

$$\rho = \frac{\lambda}{\mu} = 0.33$$

$$P = \int_{10}^{\infty} \left(\frac{\lambda}{\mu}\right)(\mu - \lambda)\, e^{-(\mu-\lambda)t}\, dt$$

$$= \int_{10}^{\infty} (0.333)\left(\frac{1}{4} - \frac{1}{12}\right) e^{-\left(\frac{1}{4}-\frac{1}{12}\right)t}\, dt$$

$$= 0.0555 \int_{10}^{\infty} e^{-0.1666t}\, dt$$

$$= 0.0555 \left(\frac{e^{-0.1666t}}{-0.1666}\right)_{10}^{\infty}$$

$$= \frac{0.0555}{-0.1666}\left(e^{-\infty} - e^{-1.666}\right)$$

$$= \frac{0.0555}{-0.1666}\left(-0.1890\right)$$

$$= 0.06296$$

iii) Avg. length of the queue that form from time to time

$$L_n = \frac{\mu}{\mu - \lambda} = \frac{1/4}{\frac{1}{4} - \frac{1}{12}} = 1.5$$

APR 2000 – Sec – C – ...

(1) On an average 96 patients per 24 hour day require the service of an emergency clinic. Also on average a patient requires 10 minutes of active attention. Assume that the facility can handle only one emergency at a time. Suppose that it cost the clinic Rs. 100 per patient treated to obtain an average servicing time of 10 minutes and thus each minute of decrease in this average time would cost Rs. 10 per patient treated. How much would have to be budgeted by the clinic to decrease the average size of the queue from ~~the average size of the queue~~ from $1\frac{1}{3}$ patients to $\frac{1}{2}$ patient?

Soln:

Here mean arrival rate $= \lambda = \frac{96}{24} = 4$ patients/hr.

mean service rate $= \mu = \frac{1}{10} \times 60 = 6$ patients/hr

Average no. of Patients in the queue $\Big\} = E(n_q) = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{4^2}{6(6-4)} = \frac{16}{12} = \frac{4}{3}$ (improper fraction) $= 1\frac{1}{3}$ (Mixed fraction)

This no. is to be reduced from $1\frac{1}{3}$ to $\frac{1}{2}$.

$\therefore$ Now $E(n_q') = \frac{\lambda^2}{\mu'(\mu'-4)} = \frac{4^2}{\mu'(\mu'-4)} = \frac{1}{2}$

$4^2 = \frac{1}{2}\left[\mu'^2 - 4\mu'\right]$

$32 = \mu'^2 - 4\mu'$ (Factorization)

$\Rightarrow \mu'^2 - 4\mu' - 32 = 0$

(or) $(\mu'-8)(\mu'+4) = 0$

(or) $\mu' = 8$ patients/hr as $\mu' = -4$ is illogical

(*) Factorization

$x^2 - 4x - 32$

(multipliers: 2, 4, 8, 16, 32)

Select $8 \times 4 = 32$

difference $8 - 4 = 4$

Big no. ie., 8 – minus sign

$x^2 - 8x + 4x - 32$

$x(x-8) + 4(x-8)$

$(x-8)(x+4)$

c.n.

$\therefore$ Average time required by each patient $\Big\} = \frac{1}{8}$ hrs $= \frac{60}{8}$ minutes $= \frac{15}{2} = 7.5$ minutes

$\Rightarrow$ Decreasein the time required by each patient

$= 10 - 7.5 = 2.5$ minutes

$\therefore$ The budget required for each patient $\Big\} = 100 + (2.5 \times 10)$
$= $ Rs. 125

$\Rightarrow$ To decrease the size of the Queue from $1\frac{1}{3}$ Patients to $\frac{1}{2}$ Patient, the budget per patient should be increased from Rs. 100 to Rs. 125.

(or)

APR-2010. Sec-C - 8marks

① People arrive at a theatre ticket booth in Poisson distributed arrival rate of 25 per hour. Service time is constant at 2 minutes. Calculate

i) Mean no. in the waiting line
ii) Mean waiting time
iii) The utilisation factor.

Soln: We are given Mean arrival rate $= \lambda = 25/\text{hr}$

Mean service rate $= \frac{1}{2} \times 60 = 30/\text{hr}.$

i) Mean no. in the waiting line $=$ ~~Avg length in the queue~~

$$\cancel{\text{Avg} = \frac{\mu}{\mu-\lambda} = \frac{30}{30-25} = \frac{30}{5}}$$

$=$ Expected no. of Customers in the queue

$$= E(n_q) = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{25^2}{30(30-25)} = \frac{625}{150} = 4.16$$

$\simeq 4$ people

ii) Mean waiting time (in the queue) $= E(W_q) = \dfrac{\lambda}{\mu(\mu-\lambda)}$

$$= \frac{25}{30(30-25)} = \frac{25}{150}$$

$$= 0.167 \text{ hr} \simeq 10 \text{ minutes}$$

(iii) Utilisation factor $= \rho = \dfrac{\lambda}{\mu} = \dfrac{25}{30} = 0.833$

① A repair shop is attended by a single mechanic has an average of 4 customers an hour who bring small appliances for repair. He inspects them for defects and quite often can fix them right way or otherwise render (diagnosis) a diagnosis. This takes him 6 minutes on the average. Arrivals are Poisson and service time has the exponential distribution.

a) Find the proportion of time during which the shop is empty.

b) Find the prob. of finding at least one customer in the shop.

c) What is the avg. no. of customers in the system.

d) Find the avg time including services

e) If it was decided that a customer would not tolerate a wait of more than 10 minutes. What is the prob. that a customer would have to wait at least that length of time.

Sol: $\lambda = 4/\text{hr}$ $\quad \mu = \frac{1}{6} \times 60 = 10/\text{hr}.$

a) idle $= 1 - \dfrac{\lambda}{\mu} = 0.6$

b) Prob. of one customers in the system $= P(\text{system is busy}) = 1$

c) $E(n_s) = \dfrac{\lambda}{\mu-\lambda} = 0.667$

d) $E(W_s) = \dfrac{1}{\mu-\lambda} = 0.1667$

e) $\int_{t}^{10} \left(\frac{\lambda}{\mu}\right)(\mu-\lambda) e^{-(\mu-\lambda)t}\, dt = 0.4$

$e^{0} = 1$

ii) Mean waiting time (in the queue) $= E(W_q) = \dfrac{\lambda}{\mu(\mu-\lambda)}$

$$= \frac{25}{30(30-25)} = \frac{25}{150}$$

$$= 0.167 \text{ hrs} \simeq 10 \text{ minutes}$$

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○ A repair shop is attended by a single mechanic has an average of 4 customers an hour who bring small appliances for repair. He inspects them for defects and quite often can fix them right way or otherwise render (coniyyqi) a diagnosis. This takes him 6 minutes on the average. Arrivals are Poisson and service time has the exponential distribution.

a) Find the proportion of time during which the shop is empty.

b) Find the prob. of ~~[illegible]~~ at least one customer in the shop.

c) What is the avg. no. of customers in the system.

d) Find the avg time including service

e) If it was decided that a customer would not tolerate a wait of more than 10 minutes. What is the prob. that a customer would have to wait atleast that length of time.

Soln: $\lambda = 4/\text{hr}$. $\mu = \frac{1}{6} \times 60 = 10/\text{hr}$.

a) idle $= 1 - \dfrac{\lambda}{\mu} = 0.6$

b) Prob. of one customer in the system $=$ system is busy $= \rho$ ~~[illegible]~~

c) $E(n_s) = \dfrac{\lambda}{\mu-\lambda} = 0.667$

d) $E(W_s) = \dfrac{1}{\mu-\lambda} = 0.1667$

e) $\int_0^{10} \left(\frac{\lambda}{\mu}\right)(\mu-\lambda)e^{-(\mu-\lambda)t}\,dt = 0.4$

$e^0 = 1$

Model II : $(M/M/1 : FCFS/N/\infty)$ Finite Queue length model

Here the capacity of the system is limited, say N. Infact arrivals will not exceed N in any case. The Various measures of this model are

1. The Prob. that the system is empty $\Big\} = P_0 = \dfrac{1-\rho}{1-\rho^{N+1}}$ where $\rho = \dfrac{\lambda}{\mu}$, $\lambda > \mu$ is allowed

2. The Prob. of 'n' Customers in the system $\Big\} = P_N = \dfrac{1-\rho}{1-\rho^{N+1}} \times \rho^N$ (P_0)

3. Expected no. of Customers in the system $\Big\} = E(n_s) = P_0 \sum_{n=0}^{N} n \cdot \rho^n$

4. Expected no. of Customers in the queue $\Big\} = E(n_q) = E(n_s) - \dfrac{\lambda}{\mu}$

5. Expected waiting time per Customer in the system $\Big\} = E(W_s) = \dfrac{E(n_s)}{\lambda(1-P_N)}$

6. Expected waiting time per Customer in the queue $\Big\} = E(W_q) = \dfrac{E(n_q)}{\lambda(1-P_N)}$

7. Average queue length $= L_n = E(n_s) - 1 + P_0$

A barbershop has space to accommodate only 10 customers. He can service only one person at a time. If a customer comes to his shop and finds it full, he goes to the next shop. Customers randomly arrive at an average rate 10 per hour and the barbers service time is exponential with an average of 5 minutes per customer. Find P_0, P_N.

Soln :

$\lambda = 10$ cust/hr

Here $N = 10$, $\cancel{\lambda = \frac{10}{60} = \frac{1}{6}, \mu = \frac{1}{5}}$ $\mu = 12$ cust/hr

$$\therefore \rho = \frac{\lambda}{\mu} = \frac{10}{12} = \frac{5}{6}$$

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-5/6}{1-(5/6)^{10+1}} = \frac{0.1667}{0.8655} = 0.1926$$

$$P_N = \left(\frac{1-\rho}{1-\rho^{N+1}}\right)\rho^N$$

$$= \frac{1-5/6}{1-(5/6)^{10+1}}\left(5/6\right)^{10}$$

$$= (0.1926)\left(5/6\right)^{10},$$

$$= 0.031$$

① Patients arrive at a clinic according to Poissons distr. at the rate of 30 per hour. The waiting room doesnot accomodate more than 14 patients. Examination time per patient is exponential with mean rate 20/hr. Find, Avg no. of customers in the system, waiting time in the system, queue.

Soln:

We are given $N = 14$, $\lambda = 30/hr$ $\mu = 20/hr$.

$$\therefore \rho = \frac{\lambda}{\mu} = \frac{30}{20} = 1.5.$$

Prob. of 14 Customers in the system $\Big\} = P_N = \dfrac{1-\rho}{1-\rho^{N+1}} \cdot \rho^{n}$

$$P_{14} = \frac{1-1.5}{1-(1.5)^{15}} \times (1.5)^{14}$$

$$= 0.3328$$

Avg. no. of Customers in the system $\Big\} = E(n_s) = P_0 \sum_{n=0}^{N} n.\rho^n$

$$= \frac{1-\rho}{1-\rho^{N+1}} \sum_{n=0}^{N} n.\rho^n.$$

$$= \frac{1-1.5}{1-(1.5)^{15}} \sum_{n=0}^{14} n.\rho^n$$

$$= \frac{-0.5}{-436.89} \left[0+\rho+2\rho^2+3\rho^3+\cdots+14\rho^{14}\right]$$

$$= 0.001144 \times 10515.45 = 11.98$$

Expected waiting time of a Patient in the system $\Bigg\} = \dfrac{E(n_s)}{\lambda(1-P_N)} = \dfrac{10.519}{30(1-0.3328)} = 0.59$

Avg. No. of Customers in the queue $\Bigg\} = E(n_s) - \dfrac{\lambda}{\mu}$

$$E(n_q) = 11.98 - 1.5 = 10.48$$

$$E(W_s) = 0.597$$

Expected waiting time of a Patient in the queue $\Bigg\} = E(W_q) = \dfrac{E(n_q)}{\lambda(1-P_N)}$

$$= \frac{9.015}{30(1-0.3328)} = 0.[illegible]$$

⊙ If, for a period of 24 hours in a day trains arrive at the yard every 20 minutes but the service time continues to remain 36 minutes, If the capacity of the yard is limited to 4 trains calculate

i) the prob. that the yard is empty

ii) average queue length

Soln: We are given, mean arrival rate $\lambda = \frac{1}{20}$ per minute

service rate $\mu = 1/36$ per minute

Capacity of the yard $N = 4$

$$\therefore \rho = \frac{\lambda}{\mu} = \frac{1/20}{1/36} = 1.8$$

i) The prob. that the yard is empty $= P_0 = \dfrac{1-\rho}{1-\rho^{N+1}}$

$$= \frac{1-1.8}{1-(1.8)^5}$$

$$= 0.0447$$

ii) Average queue length $= L_n = E(n_s) - 1 + P_0$

$$E(n_s) = P_0 \sum_{n=0}^{N} n\rho^n$$

$$= \frac{1-\rho}{1-\rho^{N+1}} \sum_{n=0}^{N} n \cdot \rho^n$$

$$= \frac{1-1.8}{1-(1.8)^5} \sum_{n=0}^{4} n\rho^n$$

$$= \frac{-0.8}{-17.89} \left[\rho + 2\rho^2 + 3\rho^3 + 4\rho^4\right]$$

$$= 0.0447 \left[1.8 + 2(1.8)^2 + 3(1.8)^3 + 4(1.8)^4\right]$$

$$= 0.0447 \times 67.76$$

$$= 3.029$$

$$\therefore L_n = E(n_s) - 1 + P_0$$

$$= 3.029 - 1 + 0.0447$$

$$= \underline{2.0737}$$

Nov' 2000 ; Sec D

Customers arrive at a one window drive-in bank according to Poisson distn.. with mean 10 per hour. The space in front of the window including that for the serviced car can accomodate a max of 3 cars. Other can wait outside this space. Service time per customer is exponential with mean 5 minutes. Then

i) What is the prob.. that an arriving customer can drive directly to the space in front of the window. $(P_0) = 0.3219$

ii) What is the prob.. that an arriving customer will have to wait outside the indicated space $(P_N) = 0.1863$

iii) How long is an arriving customer is expected to wait before starting service? $[E(W_q)]$

Model. III : Multi-Service model – $(M/M/C):(FCFS/\infty/\infty)$

Here $C =$ No. of Service Channels & $\lambda > \mu$ allowed

1. The Prob. that the system is empty $\Big\} = P_0 = \dfrac{1}{\displaystyle\sum_{n=0}^{c-1} \frac{(\lambda/\mu)^n}{n!} + \left[\frac{(\lambda/\mu)^c}{c!} \times \frac{c\mu}{c\mu-\lambda}\right]}$

2. Expected no. of Customers in the system $\Big\} = E(n_s) = \dfrac{\lambda\mu(\lambda/\mu)^c}{(c-1)!\,(c\mu-\lambda)^2} P_0 + \dfrac{\lambda}{\mu}$

3. Expected no. of Customers in the queue $\Big\} = E(n_q) = E(n_s) - \dfrac{\lambda}{\mu}$

4. Expected waiting time per Customer in the system $\Big\} = E(W_s) = \dfrac{E(n_s)}{\lambda}$

5. Expected waiting time per Customer in the queue $\Big\} = E(W_q) = \dfrac{E(n_q)}{\lambda}$

6. The Prob. that a Customer has to wait $P(n \geq c) = \dfrac{\mu(\lambda/\mu)^c}{(c-1)!\,(c\mu-\lambda)} P_0$

7. The Prob. that a Customer enters the service without waiting $\Big\} = 1 - P(n \geq c)$

8. Utilisation factor $\rho = \dfrac{\lambda}{c.\mu}$

Scanned with CamScanner

① A telephone exchange has two long distance operators. The telephone Company finds that during peak load, long distance calls arrive in a Poissons fashion at an average rate of 15 per hour. The length of service on these calls are approximately exponentially distributed with mean length of 5 minutes.

a) What is the prob. that a subscriber will have to wait for his long distance call during peak hour of the day?

b) If the subscriber will wait and are serviced in turn, what is the expected waiting time?

Soln :

mean arrival rate $= \lambda = 15$ / hour

mean service rate $= \mu = \frac{60}{5} = 12$ / hour

No. of Channels $= c = 2$

a) The Prob. that a Customer has to wait

$$P(n \geq c) = \frac{\mu(\lambda/\mu)^c}{(c-1)!\,(c\mu-\lambda)} P_0$$

$$\therefore P_0 = \frac{1}{\sum_{n=0}^{c-1} \frac{(\lambda/\mu)^n}{n!} + \left[\frac{(\lambda/\mu)^c}{c!} \times \frac{c\mu}{c\mu-\lambda}\right]}$$

$$= \frac{1}{\sum_{n=0}^{1} \frac{(\lambda/\mu)^n}{n!} + \left[\frac{(\lambda/\mu)^2}{2!} \times \frac{2\mu}{2\mu-\lambda}\right]}$$

$$= \cfrac{1}{\cfrac{(1.25)^0}{0!} + \cfrac{(1.25)}{1!} + \left[\cfrac{(1.25)^2}{2!} \times \cfrac{2\times 12}{(2\times 12)-15}\right]}$$

$$= \frac{1}{1 + 1.25 + [0.78125 \times 2.667]}$$

$$= \frac{1}{4.33}$$

$$= 0.231$$

$$\therefore P(n \geq 2) = \frac{\mu (\lambda/\mu)^2}{(2-1)!\,(2\mu-\lambda)} P_0$$

$$= \frac{12 (15/12)^2}{1!\,(2\times 12 - 15)} (0.231)$$

$$= \frac{4.33125}{9}$$

$$= 0.48125$$

b) The expected waiting time

$$E(W_q) = \frac{E(n_q)}{\lambda} = \frac{\cancel{\lambda}\,\mu (\lambda/\mu)^c}{\cancel{\lambda}(c-1)!\,(c\mu-\lambda)^2} P_0$$

$$= \frac{12 (1.25)^2}{(2-1)!\,(2\times 12 - 15)^2} (0.231)$$

$$= \frac{4.33125}{81}$$

$$= 0.0535 \text{ hrs} = 3.20 \text{ min.}$$

⊙ A Car garage has 3 identical stalls each of whic Can service an average of 6 Cars per hour. An average of 12 Cars arrive each hour at the garage. Assuming that the arrivals following Poisson process, the service times are exponentially distributed and the parking facilities are Virtually (real-2ommowuron) Unlimited

Determine

i) the expected no. of Cars in the system

ii) the expected time spent by a Car waiting for service

and iii) the expected time of a Car spent in the system.

Soln:

mean arrival rate $\lambda = 12$ per hour.

mean service rate $\mu = 6$ per hour.

No. of Service channels $= C = 3$

$$P_0 = \frac{1}{\sum_{n=0}^{C-1} \frac{(\lambda/\mu)^n}{n!} + \left[\frac{(\lambda/\mu)^c}{C!} \times \frac{C\mu}{C\mu - \lambda}\right]}$$

$$= \frac{1}{\sum_{n=0}^{2} \frac{2^n}{n!} + \left[\frac{2^3}{3!} * \frac{3\times 6}{3\times 6 - 12}\right]}$$

$$= \frac{1}{\frac{2^0}{0!} + \frac{2}{1!} + \frac{4}{2!} + \left[\frac{8}{6} * \frac{18}{6}\right]}$$

$$= \frac{1}{1+2+2+4}$$

$$= \frac{1}{9} = 0.1111$$

i) Expected no. of cars in the system

$$E(n_s) = \frac{\lambda\mu(\lambda/\mu)^c}{(c-1)!\,(c\mu-\lambda)^2} \times P_0 + \frac{\lambda}{\mu}$$

$$= \frac{12\times6\times(2)^3}{2!\,(18-12)^2} \times (0.1111) + \frac{12}{6}$$

$$= \frac{63.9936}{72} + 2$$

$$= 2.8888 \text{ Cars}$$

(ii) The expected time spent by a car waiting for service $E(W_q) = \dfrac{\mu(\lambda/\mu)^c}{(c-1)!\,(c\mu-\lambda)^2} P_0$

$$= \frac{6(2)^3}{2!\,(3\times6-12)^2} \times 0.1111$$

$$= \frac{5.3328}{72}$$

$$= 0.074 \text{ hour}$$

iii) Expected time of a car spent in the system

$$E(W_s) = \frac{E(n_s)}{\lambda} = \frac{2.8888}{12} = 0.241 \text{ hours.}$$

Model IV : Multi Service Finite Queue length model

$(M/M/C) : (FCFS/N/\infty)$

This model is essentially the same as model III except that the maximum no. of customers in the system is limited to N, where $N>C$ (C - no. of Channels).

1. The Prob. that the system is empty } = $P_0 = 1 \Big/ \left[\sum_{n=0}^{c-1} \frac{1}{n!} (\lambda/\mu)^n + \sum_{n=c}^{N} \frac{1}{c^{n-c} c!} (\lambda/\mu)^n \right]$

2. The Prob. that 'n' customers in the system } $= P_n = \begin{cases} \frac{1}{n!} (\lambda/\mu)^n P_0 & : 0 \le n \le c-1 \\ \frac{1}{c^{n-c} c!} (\lambda/\mu)^n P_0 & : c \le n \le N \end{cases}$

3. Expected no. of customers in the queue } $= E(n_q) =$ $(c\rho)^c \rho$

$$= \frac{(c\rho)^c \rho}{c!(1-\rho)^2} \left[1 - \rho^{N-c+1} - (1-\rho)(N-c+1)\rho^{N-c} \right] \cdot P_0$$

4. Expected no. of customers in the system } $= E(n_s) = E(n_q) + c - P_0 \sum_{n=0}^{c-1} \frac{(c-n)(c\rho)^n}{n!}$

5. Expected waiting time per customer in the system } $= E(W_s) = \frac{E(n_s)}{\lambda(1-P_N)}$

7) $\rho = \frac{\lambda}{\mu}$

6. Expected waiting time per customer in the queue } $= E(W_q) = E(W_s) - \frac{1}{\mu}$

(1) A barber shop has two barbers and three chairs for customers. Assume that the customers arrives in Poisson fashion at a rate of 5 per hour and that each barber services customers according to an exponential distn with mean of 15 minutes. Further if a customer arrives and there are no empty chairs in the shop, he will leave (i) What is the prob. that the shop is empty? (ii) What is the expected no. of customers in the shop?

Soln: Here $c = 2, N = 3, \lambda = 5/\text{hour}, \mu = \frac{60}{15} = 4/\text{hr}$

(i) $$P_0 = \left[\sum_{n=0}^{c-1} \frac{1}{n!}(\lambda/\mu)^n + \sum_{n=c}^{N} \frac{1}{c^{n-c}\,c!}(\lambda/\mu)^n\right]^{-1}$$

$$= \left[\sum_{n=0}^{1} \frac{1}{n!}(5/4)^n + \sum_{n=2}^{3} \frac{1}{2^{n-2}\,2!}(5/4)^n\right]^{-1}$$

$$= \left[\frac{1}{0!}(5/4)^0 + \frac{1}{1!}(5/4)^1 + \frac{1}{2^0\,2!}(5/4)^2 + \frac{1}{2^1\,2!}(5/4)^3\right]^{-1}$$

$$= \left[1 + \frac{5}{4} + 0.78125 + 0.4882\right]^{-1}$$

$$= (3.51945)^{-1}$$

$$= \frac{1}{3.51945}$$

$$= 0.2841$$

(ii) $E(n_q) = \dfrac{(c\rho)^c \rho}{c!(1-\rho)^2}\left[1-\rho^{N-c+1}-(1-\rho)(N-c+1)\rho^{N-c}\right]\times P_0$

$$= \frac{\left(2\times\frac{5}{4}\right)^2\left(\frac{5}{4}\right)}{2!\left(1-\frac{5}{4}\right)^2}\left[1-\left(\frac{5}{4}\right)^{3-2+1}-\left(1-\frac{5}{4}\right)(3-2+1)\left(\frac{5}{4}\right)^{3-2}\right]\times 0.2841$$

$$= \frac{7.8125}{0.125}\left[1-1.5625-\{(-0.25)(2)(1.25)^1\}\right]\times 0.2841$$

$$= \frac{2.2195}{0.125}\left[0.0625\right]$$

$$= 1.1098$$

$$\therefore E(n_s) = E(n_q) + c - P_0\sum_{n=0}^{c-1}\frac{(c-n)(c\rho)^n}{n!}$$

$$= 1.1098 + 2 - 0.2841\sum_{n=0}^{1}\frac{(2-n)\left(\frac{5}{4}\right)^n}{n!}$$

$$= 3.1098 - 0.2841\left[\frac{2}{0!}+\frac{1\times 1.25}{1!}\right]$$

$$= 3.1098 - 0.9233$$

$$= 2.1865$$

Q) A car servicing station has 3 stalls where service can be offered simultaneously. The cars wait in such a way that when a stall becomes vacant, the car at the head of the line pulls up to it. The station can accomodate at most 4 cars at one time. The arrival pattern is Poisson with a mean of one car per 5 minutes during the peak hours. The service time is exponential with mean 6 minutes. Find i) avge no. of cars in the queue, system

ii) avge waiting time in the system

$E(n_q) = 2.80$ cars

$E(n_s) = 3.95$ "

$E(W_s) = 20.45$ minutes (or)

$P_0 = 0.00392$

$E(W_q) = 0.242$ hrs 0.34076 hrs