UNIT-V

Boundary value problems:

A problem in which we have to solve a differential equation along with conditions specified at the end points of an interval (i.e, at the boundary), is called boundary value problem. There are boundary value problems for ordinary differential equations (eg. bending of beams) and partial differential equations (eg. one dimensional wave equation, heat equation etc). Often, analytical solutions (closed form solutions) are not available for these problems and so we use numerical techniques.

Classification of second order linear partial differential equation:

The general form of second order linear PDE is,

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

where $A, B, \ldots\ldots G$ are functions of x and y.

The general form of second order quasi-linear PDE is,

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0.$$

Here terms involving second order partial derivatives alone are required to be linear.

The above equations are classified as

1. elliptic, if $B^2 - 4AC < 0$
2. parabolic, if $B^2 - 4AC = 0$
3. hyperbolic, if $B^2 - 4AC > 0$

We discuss here, the finite difference method for solving one-dimensional wave equation and one and two dimensional heat equations.

The classification of some important PDE's:

1. Two-dimensional heat equation in steady state (Laplace equation):

$$u_{xx} + u_{yy} = 0.$$

Here, $A = 1$, $B = 0$, $C = 1$

$\therefore B^2 - 4AC = -4 < 0$

$\therefore$ The equation is of elliptic type.

2. Poisson equation:

$$u_{xx} + u_{yy} = f(x, y)$$

$A = 1$, $B = 0$, $C = 1$

$\therefore$ The equation is of elliptic type.

3. One-dimensional heat equation:

$$\frac{\partial^2 u}{\partial x^2} - a\frac{\partial u}{\partial t} = 0.$$

Here $A = 1$, $B = 0$, $C = 0$.

$\therefore B^2 - 4AC = 0.$

$\therefore$ The equation is of parabolic type.

4. One-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$$

$$\therefore a^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0.$$

$A = a^2$, $B = 0$, $C = -1$. $\therefore B^2 - 4AC = 4a^2 > 0.$

$\therefore$ The equation is of hyperbolic type.

Problems:

Classify the following PDE'S:

1. $f_{xx} + 2f_{xy} + f_{yy} = 0.$

$A = 1,\ B = 2,\ C = 1$

$\therefore B^2 - 4AC = 4 - 4 = 0.$

$\therefore$ The equation is parabolic.

$f_{xx} - x f_{yy} = 0$

$A = 1,\ B = 0,\ C = -x$

$B^2 - 4AC = 4x.$

$\therefore$ The equation is

elliptic if $x < 0$

parabolic if $x = 0$

hyperbolic if $x > 0$

Elliptic Equations (Laplace equation and Poisson equation).

Laplace Equation:

Laplace equation, in two dimensions is given by,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ or } u_{xx} + u_{yy} = 0 \text{ (or) } \nabla^2 u = 0$$

Standard five-point formula:

Consider the equation, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

We replace the partial derivatives by the corresponding difference quotients.

(i.e.)
$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = 0$$

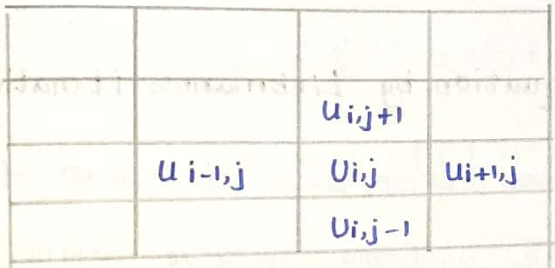
If we take $h = k$ (i.e.) Square mesh) we get,

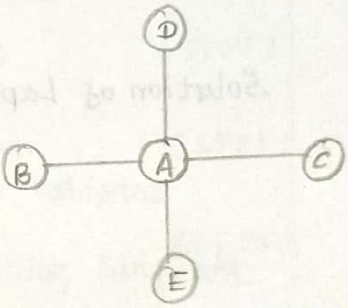
$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0.$$

$$u_{i,j} = \frac{1}{4}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right] \longrightarrow (1).$$

This is called Standard five point formula.

According to this formula. the value of u at any interior grid point is the average of the values of u at the four grid points near it, namely the two points just above and below the given point and the two point to the immediate left and right of the given point.





The value of u at A = average of the values of u at B, C, D and E.

Diagonal five point formula:

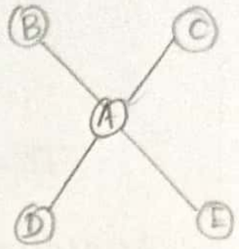
Laplace equation remains unchanged when the co-ordinate axes are rotated through 45°. Now, the axes co-inside with the diagonals through (i,j). So we may also take the value of $u_{i,j}$ as the average of the values of u at the nearest diagonal points.

(i,e) $$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j+1} + u_{i+1,j-1}\right]$$

This is called diagonal five point formula.

Scanned with CamScanner

$u_{i-1,j+1}$		$u_{i+1,j+1}$
	$u_{i,j}$	
$u_{i-1,j-1}$		$u_{i+1,j-1}$



The value of u at A = average of the values of u at B, C, D, E.

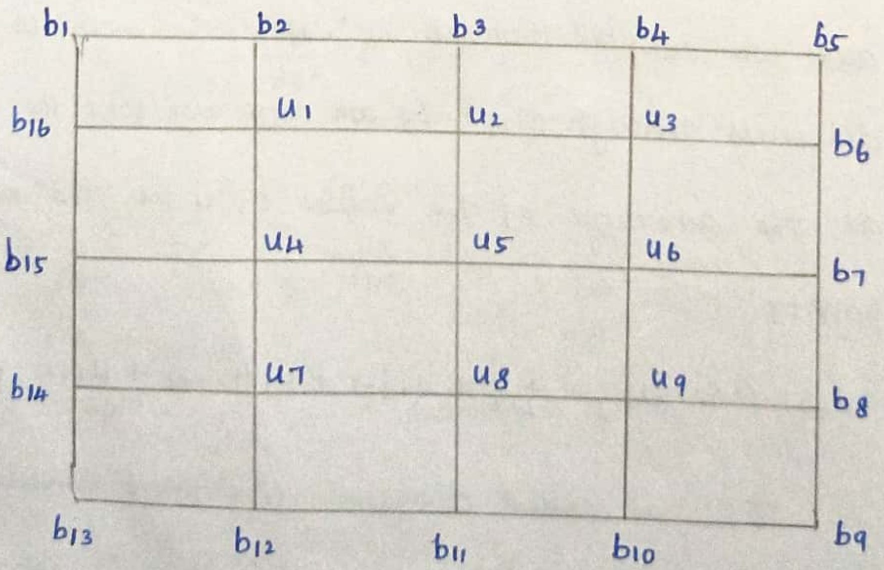
Note:

Standard five-point formula (SFPF) is preferred to diagonal five point formula (DFPF) since the error in DFPF is four times that in SFPF.

Solution of Laplace equation by Liebmann's iteration process:

Consider a square region, with the values of u given at the grid points on the boundary. To solve Laplace equation, given these boundary conditions, we divide the given square region into a network of 16 sub-squares.

Let the values of u on the boundary be $b_1, b_2, b_3, \ldots b_{16}$ and points be $u_1, u_2, \ldots u_9$ as in the figure.



We first find rough (initial) values of u using SFPF and DFPF and then improve the values by iteration using SFPF.

To find initial values of u:

We first find u_5 using SFPF, then we find u_1, u_3, u_7, u_9 using DFPF and then find u_2, u_4, u_6, u_8 using SFPF.

(i,e) $u_5 = \frac{1}{4}(b_3+b_7+b_{11}+b_{15})$ (SFPF)

$u_1 = \frac{1}{4}(b_1+b_3+u_5+b_{15})$ (DFPF)

$u_3 = \frac{1}{4}(b_3+b_5+b_7+u_5)$ (DFPF)

$u_7 = \frac{1}{4}(b_{15}+u_5+b_{11}+b_{13})$ (DFPF)

$u_9 = \frac{1}{4}(u_5+b_7+b_9+b_{11})$ (DFPF)

$u_2 = \frac{1}{4}(u_1+b_3+u_3+u_5)$ (SFPF)

$u_4 = \frac{1}{4}(b_{15}+u_1+u_5+u_7)$ (SFPF)

$u_6 = \frac{1}{4}(u_5+u_3+b_7+u_9)$ (SFPF)

$u_8 = \frac{1}{4}(u_7+u_5+u_9+b_{11})$ (SFPF)

Now that we have got the initial value of u at all the internal grid points, we iterate in the order $u_1, u_2, \cdots u_9$, using SFPF, always using the latest available values. (i.e.

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left(u_{i-1,j}^{(n+1)} + u_{i,j+1}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)}\right)$$

where the superscripts denote the iteration number. This is called Liebmann's iteration process. We continue the iteration till the values of u at each point are equal upto desired accuracy, in two successive iterations.

Note:

Using SFPF at the nine interior grid points, we can form 9 equations 9 unknowns $u_1, u_2, \cdots\cdots u_9$ and solve them using Gauss-Seidel method.

Poisson Equation:

The Second order partial differential equation.

$$\nabla^2 u = f(x,y) \text{ (or)}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

is called poisson equation.

Numerical solution of poisson Equation:

Consider a square region, with the values of u given at the grid points on the boundary. We wish to solve poisson equation with the given boundary conditions and obtain the values of u at the interior grid points.

Substitute the partial derivatives by the corresponding difference quotients and put $x = ih$, $y = jh$ (i.e take $x_0 = 0$, $y = 0$ and $h = k$) as in previous discussions).

We get,

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2} = f(ih, jh)$$

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh)$$

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh) \quad \rightarrow (1)$$

Apply equation (1) at each interior mesh point, and get a System of linear equations in $u_1, u_2, \ldots\ldots$ Solve these equations and get the required values of u.

Parabolic Equation:

One-dimensional heat equation:

Heat equation is a second order linear partial differential equation which describes the distribution of heat in a given region over a period of time. The one-dimensional heat equation models the flow of heat in a rod that is insulated along the sides so that there is no loss of heat. The temperature u in the rod depends on two-variables - x (position on the rod) and t (time). We consider only one dimension of the rod, the length. Hence the name one-dimensional since the thickness is assumed to be negligible compared to the length. Hence the name one-dimensional heat equation. The general one-dimensional heat equation is,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$(i.e)\ u_t = c^2 u_{xx}.$$

where $c^2 = \frac{k}{\rho\sigma}$ is the thermal diffusivity, a constant which depends on the material of the rod. K is the thermal conductivity, ρ is the density and σ is the specific heat capacity of the material of the rod.

To solve this equation, we require 3 conditions - two boundary condition and one initial condition namely.

$$u(0,t) = f_1(t),\ t \geq 0$$

$$u(l,t) = f_2(t),\ t \geq 0$$

$$u(x,0) = f_3(x),\ 0 \leq x < l.$$

where l is the length of the rod. $f_1(t)$ and $f_2(t)$ are functions of t (usually constants) and $f_3(x)$ is a function of x.

Numerical Solution of one-dimensional heat equation:

We discuss two methods to solve the one-dimensional heat-equation numerically.

1. Explicit method - Bender-Schmidt method.
2. Implicit method - Crank-Nicholson method.

Explicit method - Bendor Schmidt method:

To solve the one-dimensional heat equation, $u_t = c^2 u_{xx}$, with initial and boundary conditions as above, consider the step-sizes in x and t to be h and k respectively. Substituting $c^2 = \frac{1}{a}$ in the equation, we have.

$$u_{xx} = a u_t \longrightarrow ①$$

Now replace the partial derivatives by the corresponding difference quotients to get,

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = a\left(\frac{u_{i,j+1} - u_{i,j}}{k}\right)$$

$$\therefore u_{i,j+1} - u_{i,j} = \frac{k}{ah^2}(u_{i+1,j} - 2u_{i,j} + u_{i-i,j})$$

Now substitute, $\lambda = \frac{k}{ah^2}$

$$u_{i,j+1} - u_{i,j} = \lambda(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

$$\therefore u_{i,j+1} - u_{i,j} = \lambda(u_{i+1,j} - 2_{i,j} + u_{i-1,j} + u_{i,j})$$

The explicit formula for solving one-dimensional heat equation is given by,

$$u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j} \longrightarrow ②$$

If $0 < \lambda \le \frac{1}{2}$, the explicit formula is valid and the solution is stable. If $\lambda > \frac{1}{2}$, the solution is unstable. When $\lambda = \frac{1}{2}$, the above formule is simplified and reduces to,

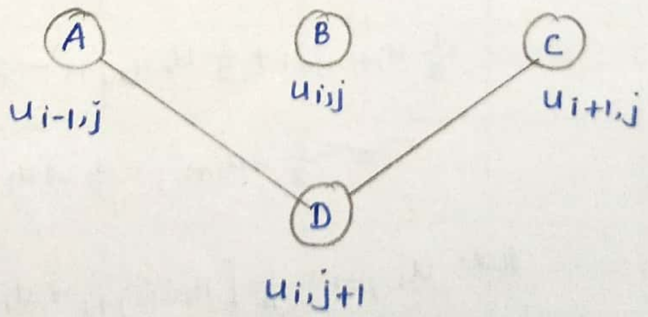
$$u_{i,j+1} = \frac{u_{i+1,j} + u_{i-1,j}}{2} \longrightarrow ③$$

Which is called Bender-schmidt recurrence equation. It is valid only when.

$$\lambda = \frac{k}{ah^2} \; : \; \lambda = \frac{1}{2} \text{ (i,e) } k = \frac{ah^2}{2}.$$

The various terms occurring in the formula can be represented using nodes. Since the problems involve time, we usually form the table starting with $t=0$ in the first row.

(i,e) The $(j+1)$th row comes below the jth row. So we represent the points as,



(i,e) Value of u at D = average of the values at A and C.

Scanned with CamScanner

Implicit method - Crank - Nicholson's methods

Consider the one-dimensional heat equation

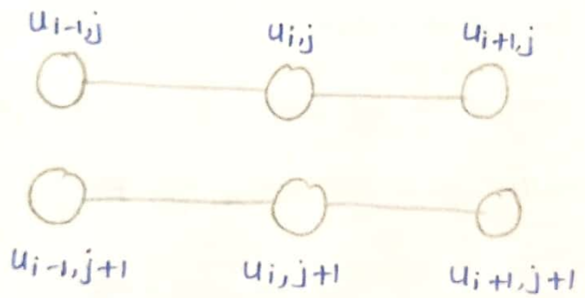
$$u_{xx} = a u_t \rightarrow (1)$$

We replace u_{xx} by the average of the values of the difference quotient at (i,j) and $(i, j+1)$ we get,

$$\lambda u_{i+1,j+1} + \lambda u_{i-1,j+1} - 2(\lambda+1) u_{i,j+1}$$

$$= -\lambda u_{i+1,j} - \lambda u_{i-1,j} + 2(\lambda-1) u_{i,j} \rightarrow (2)$$

Equation (2) gives (general) Crank-Nicholson difference Scheme. The terms occuring in the formula may be represented as nodes as shown below.



λ can take any value.

If we put $\lambda = 1$ in equation (2), the equation reduces to,

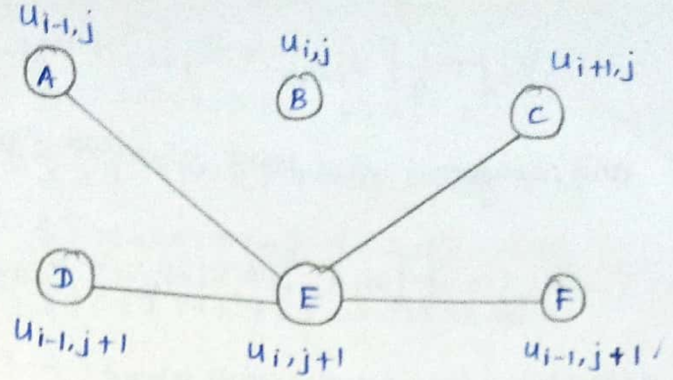
$$\frac{1}{2} u_{i+1,j+1} + \frac{1}{2} u_{i-1,j+1} - 2u_{i,j+1}$$

$$= -\frac{1}{2} \lambda u_{i+1,j} - \frac{1}{2} \lambda u_{i-1,j}$$

(i.e) $$u_{i,j+1} = \frac{1}{4} [u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}] \rightarrow (3)$$

The diagrammatic representation of the terms in equation (4) is,

Scanned with CamScanner



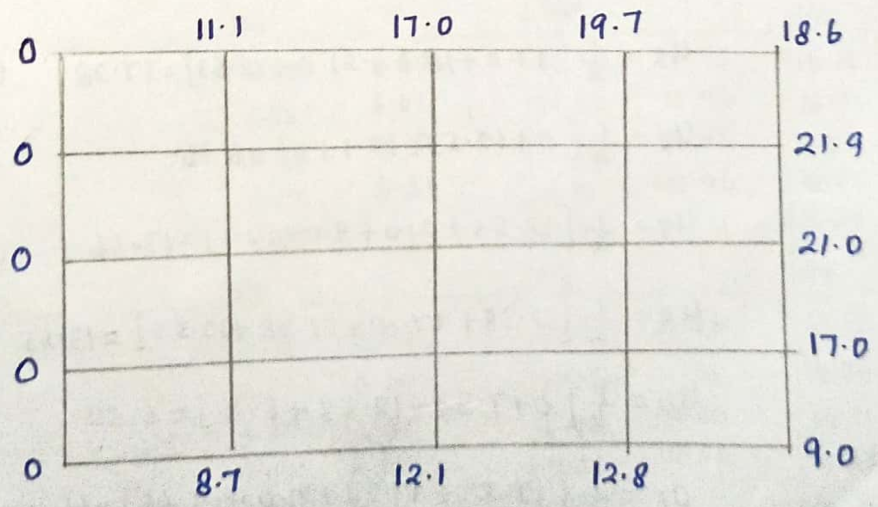
The value of u at E = average of the values of u at A, C, D and F.

This method is called implicit method since by applying the formula at the required points, we get equations, which have to solved to get the required values of u. The formula does not give the solution directly.

Problem.

1. Solve the eliptic equation $u_{xx} + u_{yy} = 0$ for the following square mess with boundary values as shown, using Liebman's iteration procedure:

Solution :



Let $u_1, u_2, \ldots\ldots u_9$ be the required values of u at the interior grid points. To obtain the initial values, we use Standard five point formula (SFPF),

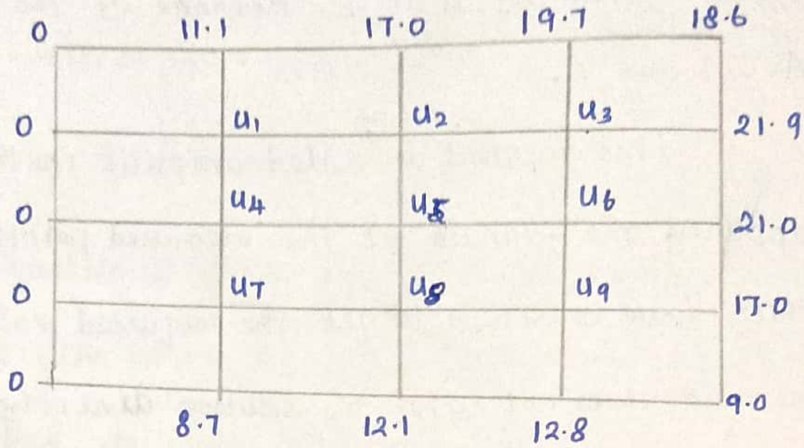
$$u_{i,j} = \frac{1}{4}[u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j+1}]$$

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]$$

and diagonal five point formula (DFPF)

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j+1}\right]$$

After this, we iterate using SFPF.



To find initial value:

The initial values are found in the order $u_5, u_1, u_3, u_7, u_9, u_2, u_4, u_6, u_8$.

$$u_5 = \frac{1}{4}[0 + 17.0 + 21.0 + 12.1] = 12.53 \qquad \text{(SFPF)}$$

$$u_1 = \frac{1}{4}[0 + 17.0 + 12.53 + 0] = 7.38 \qquad \text{(DFPF)}$$

$$u_3 = \frac{1}{4}[17.0 + 18.6 + 21.0 + 12.53] = 17.28 \qquad \text{(DFPF)}$$

$$u_7 = \frac{1}{4}[0 + 12.53 + 12.1 + 0] = 6.16 \qquad \text{(DFPF)}$$

$$u_9 = \frac{1}{4}[12.53 + 21.0 + 9.0 + 12.1] = 13.66 \qquad \text{(DFPF)}$$

$$u_2 = \frac{1}{4}[7.38 + 17.0 + 17.28 + 12.53] = 13.55 \qquad \text{(SFPF)}$$

$$u_4 = \frac{1}{4}[0 + 7.38 + 12.53 + 6.16] = 6.52 \qquad \text{(SFPF)}$$

$$u_6 = \frac{1}{4}[12.53 + 17.28 + 21.0 + 13.66] = 16.12 \qquad \text{(SFPF)}$$

$$u_8 = \frac{1}{4}[6.16 + 12.53 + 13.66 + 12.1] = 11.11 \qquad \text{(SFPF)}$$

We now iterate to find improved values in the order $u_1, u_2, \ldots u_9$ using SFPF and always using the latest available values.

First iteration :

$$u_1^{(1)} = \frac{1}{4}[0 + 11.1 + 13.55 + 6.52] = 7.79$$

$$u_2^{(1)} = \frac{1}{4}[7.79 + 17.0 + 17.28 + 12.53] = 13.65$$

$$u_3^{(1)} = \frac{1}{4}[13.65 + 19.7 + 21.9 + 16.12] = 17.84$$

$$u_4^{(1)} = \frac{1}{4}[0 + 7.79 + 21.53 + 6.16] = 6.62$$

$$u_5^{(1)} = \frac{1}{4}[6.62 + 13.65 + 16.12 + 11.11] = 11.88$$

$$u_6^{(1)} = \frac{1}{4}[11.88 + 17.84 + 21 + 13.66] = 16.10$$

$$u_7^{(1)} = \frac{1}{4}[0 + 6.62 + 11.11 + 8.7] = 6.61$$

$$u_8^{(1)} = \frac{1}{4}[6.61 + 11.88 + 13.66 + 12.1] = 11.06$$

$$u_9^{(1)} = \frac{1}{4}[11.06 + 16.10 + 17.0 + 12.8] = 14.24 .$$

We enter these values in the grid and continuous.

	0	11.1	17.0	19.7	18.6
0		u_1 7.38 7.84 7.79 7.84 7.84 7.85 7.83 7.85 7.83	u_2 13.55 13.67 13.65 13.67 13.64 13.68 13.63 13.68 13.65	u_3 17.28 17.89 17.84 17.89 17.84 17.90 17.87 17.90 17.88	21.9
0		u_4 6.52 6.60 6.52 6.61 6.58 6.61 6.57 6.61 6.59	u_5 12.53 11.95 11.88 11.96 11.85 11.96 11.91 11.96 11.94	u_6 16.12 16.30 16.11 16.30 16.23 16.30 16.27 16.30 16.29	21.0
0		u_7 6.16 6.64 6.61 6.64 6.59 6.64 6.62 6.64 6.63	u_8 11.11 11.26 11.06 11.26 11.20 11.26 11.24 11.26 11.25	u_9 13.66 14.34 14.24 14.34 14.31 14.34 14.33 14.34 14.34	17.0
0		8.7	12.1	12.8	9.0

All the values of u in the 8th iteration coincide with the corresponding value in the previous iteration. Hence we conclude that

$u_1 = 7.85$ $\quad u_3 = 17.90$ $\quad u_5 = 11.96$ $\quad u_7 = 6.64$ $\quad u_9 = 14.34.$

$u_2 = 13.68$ $\quad u_4 = 6.61$ $\quad u_6 = 16.30$ $\quad u_8 = 11.26$

2 Solve $u_{xx} + u_{yy} = 0$ over the square mesh of side 4 units satisfying the following boundary conditions.

1. $u(0,y) = 0$ for, $0 \le y \le 4$.

2. $u(4,y) = 12+y$, for $0 \le y \le 4$.

3. $u(x,0) = 3x$, for $0 \le x \le 4$

4. $u(x,4) = x^2$, for $0 \le x \le 4$

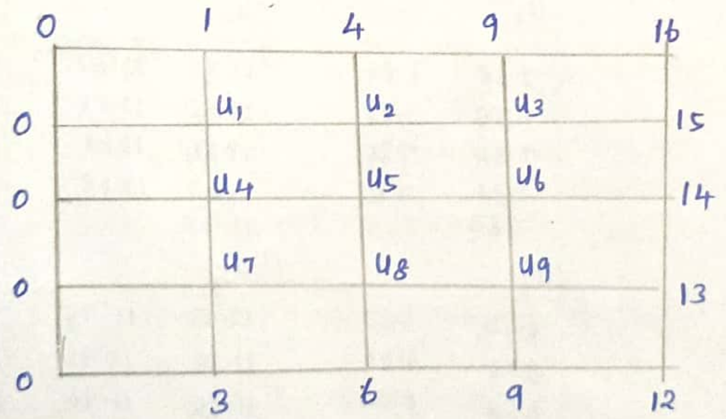
Solution:

We deive the mesh unto 16 smaller squares of the side one unit. The bottom left corner is taken as (0,0). The values of u on the boundary are got using the given functions. For Example,

$$u(4,1) = 12+1 = 13$$

$$u(3,1) - 3 \times 3 = 9$$

$$u(2,4) = 2^2 = 4.$$



Let $u_1, u_2, \cdots\cdots u_9$ be the required values of u at the interior grid points. To obtain the intial values, we use SFPF.

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i+1,j} + u_{i+1,j+1} + u_{i,j+1}\right]$$

and DFPF

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i,j+1}\right]$$

The we itenate using SFPF.

Initial values:

$$u_5 = \frac{1}{4}[0+4+14+6] = 6 \qquad (SFPF)$$

$$u_1 = \frac{1}{4}[0+4+6+0] = 2.5 \qquad (DFPF)$$

$$u_3 = \frac{1}{4}(4+16+14+6) = 10 \qquad (DFPF)$$

$$u_7 = \frac{1}{4}(0+6+6+0) = 3 \qquad (DFPF)$$

$$u_9 = \frac{1}{4}(6+14+12+6) = 9.5 \qquad (DFPF)$$

$$u_2 = \frac{1}{4}(2.5+4+10+6) = 5.625 \qquad (SFPF)$$

$$u_4 = \frac{1}{4}(0+2.5+6+3) = 2.875 \qquad (SFPF)$$

$$u_6 = \frac{1}{4}(6+10+14+9.5) = 9.875 \qquad (SFPF)$$

$$u_8 = \frac{1}{4}(3+6+9.5+6) = 6.125 \qquad (SFPE)$$

We now enter these values in the grid and iterate using SFPT-

$y=4$.

0	1	4	9	16
0	u_1 2.500 2.364 2.375 2.365 2.360 2.366 2.363 2.366	u_2 5.625 5.588 5.594 5.589 5.584 5.589 5.586 5.589	u_3 10.000 9.866 9.867 9.866 9.863 9.866 9.865 9.8666	15
0 $x=0$	u_4 2.875 2.873 2.844 2.874 2.866 2.875 2.871 2.875	u_5 6.000 6.124 6.110 6.124 6.118 6.125 6.122 6.125	u_6 9.875 9.875 9.869 9.875 9.872 9.875 9.874 9.875	14 $x=4$
0	u_7 3.000 3.008 2.992 3.008 2.004 3.009 3.007 3.009	u_8 6.125 6.160 6.151 6.160 6.157 6.161 6.159 6.161	u_9 9.500 9.509 9.505 14.34 9.507 9.509 9.508 9.509 9.509	13
0	3	6	9	12

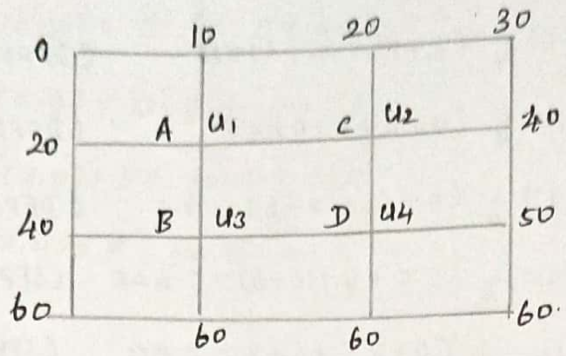
$y=0$

The values in the sixth and seventh iterations are the same. Hence we conclude that,

$u_1 = 2.366$, $u_2 = 5.589$, $u_3 = 9.866$, $u_4 = 2.875$, $u_5 = 6.125$

$u_6 = 9.875$, $u_7 = 3.009$, $u_8 = 6.161$, $u_9 = 9.509$.

2. Solve $u_{xx} + u_{yy} = 0$ at the nodal points of the following square grid using the boundary values indicated.



Solution:

Here, the initial values cannot be found using diagonal five-point formula or standard five point formula. So we have to proceed in one of the following methods:

1. Use standard five point formula at the four interior grid points, and get four equations in u_1, u_2, u_3, u_4 or

2. Assume one of the four grid values based on the boundary values and then proceed with iteration using standard five point formula.

Method (a):

Standard five-point formula is,

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]$$

using this at A, we get,

$$u_1 = \frac{1}{4}(20 + 10 + u_2 + u_3)$$

$$\text{(ie)}\ 4u_1 - u_2 - u_3 = 30 \longrightarrow (1)$$

At B, we have.

$$u_2 = \frac{1}{4}(u_1 + 20 + 40 + u_4)$$

$$\text{(i,e)}\ -u_1 + 4u_2 - u_4 = 60 \longrightarrow (2)$$

At C, we have,

$$u_3 = \frac{1}{4}(40 + u_1 + u_4 + 60)$$

(i.e) $-u_1 + 4u_3 - u_4 = 100 \longrightarrow (3)$

At D, we have,

$$u_4 = \frac{1}{4}(u_3 + u_2 + 50 + 60)$$

(i.e) $-u_2 - u_3 + 4u_4 = 110 \longrightarrow (4)$

$(1) - (4) \Rightarrow \quad u_4 = u_1 + 20 \longrightarrow (5)$

using (5) in (2), we get,

$$-u_1 + 2u_2 = 40 \longrightarrow (6)$$

using (5) in (3), we get.

$$-u_1 + 2u_2 = 60 \longrightarrow (7).$$

Solving (1), (6), (7), we get,

$$u_1 = 26.67,\ u_2 = 33.33,\ u_3 = 43.33,\ u_4 = u_1 + 20 = 46.67$$

Method (B)

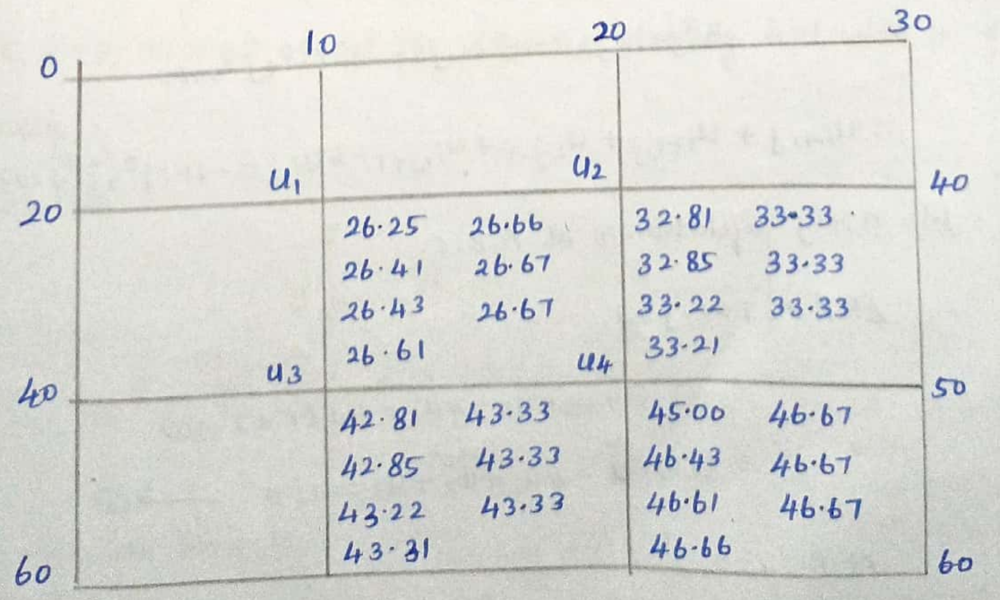
From the boundary values. Let us assume that $u_4 = 45$.

The initial values are,

$u_1 = \frac{1}{4}(0 + 20 + 45 + 40) = 26.25$ (DFPF)

$u_2 = \frac{1}{4}(26.25 + 20 + 40 + 45) = 32.81$ (SFPF)

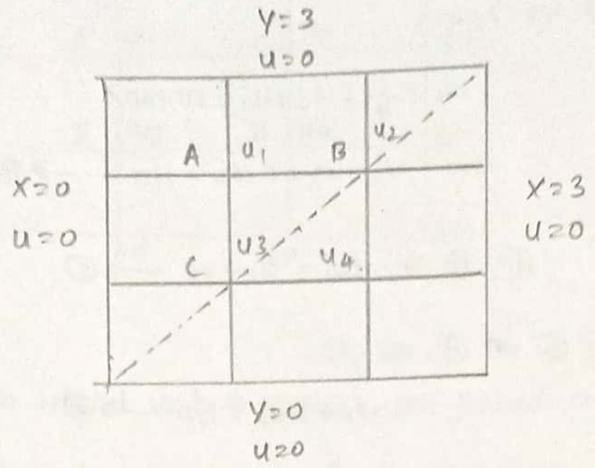
$u_3 = \frac{1}{4}(40 + 26.25 + 45 + 60) = 42.81$ (DFPF)



$u_1 = 26.67,\ u_2 = 33.33,\ u_3 = 43.33,\ u_4 = 46.67$

1. Solve the poisson equation $\nabla^2 u = -10(x^2+y^2+10)$ over the square mesh with Sides $x=y=0$, $x=y=3$ with $u=0$ on the boundary and mesh length 1 unit.

Solution:



The given PDE and boundary conditions are unaltered when x and y are interchanged. (i.e) The PDE and boundary values are symmetrical about the line $x=y$, Hence the interior values of u are also symmetrical about the line $x=y$.

$$\therefore u_1 = u_4.$$

We use the formula,

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih,jh)$$

at each grid point, where,

$$f(x,y) = -10(x^2+y^2+10), h=1$$

$$-h^2 f(ih,jh) = f(i,j) = -10(i^2+j^2+10)$$

$$\therefore u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -10(i^2+j^2+10) \rightarrow ①$$

We apply equation ① at A, B, C.

At A, $i=1, j=2$.

$$\therefore 0 + u_2 + u_3 - 4u_1 = -10(1^2+2^2+10)$$

(i.e) $-4u_1 + u_2 + u_3 = -150 \longrightarrow ②$

At B, $i=2, j=2$

$$u_1 + 0 + 0 + u_4 - 4u_2 = -10(2^2+2^2+10)$$

Scanned with CamScanner

$$u_1 - 4u_2 + u_4 = -180$$

$$2u_1 - 4u_3 = -180$$

$$u_1 - 2u_2 = -90 \longrightarrow (3)$$

At C, $i=1, j=1$.

$$0 + u_1 + u_4 + 0 - 4u_3 = -10(1^2 + 1^2 + 10)$$

$$u_1 - 4u_3 + u_4 = -120$$

$$2u_1 - 4u_3 = -120$$

$$u_1 - 2u_3 = -60 \longrightarrow (4)$$

We solve equations (2), (3), (4) to find u_1, u_2 and u_3.

$$(3) \Rightarrow u_2 = \frac{u_1 + 90}{2}$$

$$(4) \Rightarrow u_3 = \frac{u_1 + 60}{2}$$

Substituting in (2), we get,

$$-4u_1 + \frac{u_1 + 90}{2} + \frac{u_1 + 60}{2} = -150$$

$$u_1 = 75$$

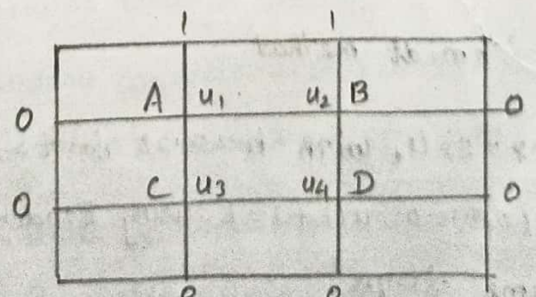
$$u_2 = 82.50$$

$$u_3 = 67.50$$

$$u_4 = u_1 = 75.$$

2. Solve the poisson equation $\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -36(x^3 + y^3 + 5)$. Subject to the condition $u = 0$ at $x = 0$, $x = 3$, $0 < y < 3$, $u = 0$ at $y = 0$ and $u = 1$ at $y = 3$, $0 < x < 3$. Find the solution taking $h = 1$ with a square mesh.

Solution:



We use formula

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh)$$

at each grid point, where

$$f(x,y) = -36(x^3+y^3+5),\ h=1$$

$$h^2 f(ih, jh) = f(i,j) = -36(i^3+j^3+5)$$

$$\therefore u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -36(i^2+j^2+5) \longrightarrow ①$$

We apply equation ① at A, B, C, D.

At A, $i=1, j=2$,

$$0+1+u_2+u_3-4u_1 = -36(1^3+2^3+5)$$

$$-4u_1 + u_2 u_3 = -505 \longrightarrow ②.$$

At B, $j=2, j=2$.

$$u_1+1+0+u_4-4u_2 = -36(2^3+2^3+5)$$

$$u_1 - 4u_2 + u_4 = -757 \longrightarrow ③$$

At C, $i=1, j=1$

$$\cancel{u_3+u_4+0+0-4} \quad 0+u_1+u_4+0-4u_3 = -36(1^3+1^3+5)$$

$$u_1 - 4u_3 + u_4 = -252 \longrightarrow ④.$$

At D, $i=2, j=1$

$$u_3+u_2+0+0-4u_4 = -36(2^2+1^3+5)$$

$$u_2+u_3-4u_4 = -504 \longrightarrow ⑤$$

Solving equations ②, ③, ④, ⑤ we get,

$$u_1 = 252.375$$

$$u_2 = 315.375$$

$$u_3 = 189.125$$

$$u_4 = 252.125$$

Bender-Schmidt method.

1. Solve $u_{xx} = 32u_t$ with $h = 0.25$ for $t > 0$, $0 < x < 1$ and ~~u(x,0)~~ $u(x,0) = u(0,t) = 0$, $u(1,t) = t$ using Bender-Schmidt formula, for 10 time steps.

Solution.

Given, $u_{xx} = 32u_t$

$a = 32$

Given $h = 0.25$

To use Bender-Schmidt formula, we require,

$$k = \frac{ah^2}{2} = \frac{32 \times (0.25)^2}{2} = 1$$

$\therefore$ Step size in x-direction is $h = 0.25$

Step size in t-direction is $k = 1$

Now we form the table.

j \ i	0 (i=0)	0.25 (i=1)	0.5 (i=2)	0.75 (i=3)	1 (i=4)
0 (j=0)	0	0	0	0	0
1 (j=1)	0	0	0	0	1
2 (j=2)	0	0	0	0.5	2
3 (j=3)	0	0	0.25	1	3
4 (j=4)	0	0.125	0.5	1625	4
5 (j=5)	0	0.25	0.875	2.25	5
6 (j=6)	0	0.4375	1.25	2.9375	6
7 (j=7)	0	0.625	1.6875	3.625	7
8 (j=8)	0	0.8438	2.125	4.3438	8
9 (j=9)	0	1.0625	2.5938	5.0625	9
10 (j=10)	0	1.2969	3.0625	5.7969	10

The initial condition $u(x, 0) = 0$ implies that the first row of the table (against $t = 0$) is zero.

The boundary condition $u(0, t) = 0$ implies that the first column of the table (below $x = 0$) is zero.

The boundary condition $u(1, t) = t$ gives us that last column of the table (below $x = 1$) as:

$u(1,0) = 0$, $u(1,1) = 1$, $u(1,2) = 2$, $u(1,3) = 3$,

$u(1,4) = 4$, $u(1,5) = 5$, $u(1,6) = 6$, $u(1,7) = 7$,

$u(1,8) = 8$, $u(1,9) = 9$, $u(1,10) = 10$.

We fill up the table using Bender - Schmidt formula.

$$u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]$$

2. Solve $32\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ for $x = 0\,(0.25)\,1$ and $t = 0\,(K)\,5$, by choosing an appropriate value of K, given that $u(x,0) = 0$, $0 \le x < 1$, $u(0,t) = 0$, $u(1,t) = 10 + t$.

Solution:

Given $a = 32$, $h = 0.25$

To apply Bender - Schmidt formula, we use:

$$K = \frac{ah^2}{2} = \frac{32}{2}(0.25)^2 = 1.$$

$u(0,t) = 0 \Rightarrow$ The first column is zero.

$u(x,0) = 0 \Rightarrow$ The first row is zero (expect last term)

$u(1,t) = 10 + t \Rightarrow u(1,0) = 10$, $u(1,1) = 11$

$u(1,2) = 12$, $u(1,3) = 13$, $u(1,4) = 14$, $u(1,5) = 15$.

The remaining values of the table are given by Bender-Schmidt formula:

$$u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]$$

t \ x	0	0.25	0.5	0.75	1
0	0	0	0	0	10
1	0	0	0	5	11
2	0	0	2.5	5.5	12
3	0	1.25	2.75	7.25	13
4	0	1.375	4.25	7.875	14
5	0	2.125	4.625	9.125	15

Difference Quotients Corresponding to partial Derivatives:

Consider a function $u(x,y)$ in two variables x and y. Let h and k be the step sizes respectively for x and y. If $x_i = x_0 + ih$ and $y_j = y_0 + jk$, we denote $u(x_i, y_j) = u_{ij}$.

Then the approximate values of the partial derivatives of $u(x,y)$ are,

$$\left(\frac{\partial u}{\partial x}\right)_{(x_i,y_i)} = \begin{cases} \dfrac{u_{i+1,j}-u_{i,j}}{h} & \text{(Forward difference quotient)} \\[2mm] \dfrac{u_{i,j}-u_{i-1,j}}{h} & \text{(Backward difference quotient)} \\[2mm] \dfrac{u_{i+1,j}-u_{i-1,j}}{2h} & \text{(central difference quotient.} \end{cases}$$

$$\left(\frac{\partial u}{\partial y}\right)_{(x_i,y_i)} = \begin{cases} \dfrac{u_{i,j+1}-u_{i-1,j}}{k} & \text{(Forward difference quotient)} \\[2mm] \dfrac{u_{i,j}-u_{i,j-1}}{k} & \text{(Backward difference quotient)} \\[2mm] \dfrac{u_{i,j+1}-u_{i,j-1}}{2k} & \text{(central difference quotient)} \end{cases}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{(x_i,y_i)} = \frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{(x_i,y_i)} = \frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{h^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{u_{i+1,j+1}-u_{i-1,j+1}-u_{i+1,j-1}+u_{i-1,j-1}}{4hk}.$$

Crank-Nicolson's method:

1. Solve by Crank-Nicolson's method $u_{xx} = u_t$ for $0 < x < 1$, $t > 0$, $u(0,t) = 0$, $u_1, t = 0$, $u(x,0) = 100(x - x^2)$. Compute u for one time step, $h = \frac{1}{4}$.

Solution:

Given $u_{xx} = u_t$, $0 < x < 1$, $t > 0$.

$\therefore a = 1,\ h = \frac{1}{4}$.

To use Crank-Nicolson's simplified formula, we need

$$\lambda = \frac{k}{ah^2} = 1$$

$$\frac{k}{1 \times (\frac{1}{4})^2} = 1$$

$$k = 16.$$

We use Crank-Nicholson's formula.

$$u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\right] \rightarrow ①$$

The given boundary and initial conditions are $u(0,t)=0$, $u(1,t)=0$, $u(x,0) = 100(x - x^2)$. These are marked in the table against $x=0$, $x=1$ and $t=0$ respectively.

$u(0,t)=0 \Rightarrow$ First column is zero.

$u(1,t) = 0 \Rightarrow$ Last column is zero.

$u(0,0) = 100(0-0) = 0$, $u(0.25, 0) = 100(0.25 - 0.25)^2 = 18.75$

$t \backslash x$	0	0.25	0.5	0.75	1
0	0	18.75	2.5	18.75	0
$\frac{1}{16}$	0	u_1	u_2	u_3	0

and so on give the values in the first row u_1, u_2, u_3.

Using equation ①, we get.

$$u_1 = \frac{1}{4}(0 + 25 + 0 + u_2)$$

$$u_2 = \frac{1}{4}(18.75 + 18.75 + u_1 + u_3)$$

$$u_3 = \frac{1}{4}(25 + 0 + u_2 + 0)$$

$$4u_1 - u_2 = 25 \longrightarrow (2)$$

$$-u_1 + 4u_2 - u_3 = 37.5 \longrightarrow (3)$$

$$-u_2 + 4u_3 = 25 \longrightarrow (4)$$

Solving equations (2), (3), (4), we get.

$$u_1 = 9.8214$$

$$u_2 = 14.2857$$

$$u_3 = 9.8214$$

2. Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $0 < x < 2$, $t > 0$, $u(0,t) = u(2,t) = 0$, $t > 0$ and $u(x,0) = \sin \frac{\pi x}{2}$, $0 \le x \le 2$ using $\Delta x = 0.5$ and $\Delta t = 0.25$ for two time steps by Crank-Nicolson implicit finite difference method.

Solution.

Given $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $0 < x < 2$, $t > 0$

$$\therefore a = 1, \; h = 0.25, \; k = 0.25$$

$$\lambda = \frac{k}{ah^2} = \frac{0.25}{1 \times (0.5)^2} = 1$$

Hence we can use the simplified Crank-Nicolson formula.

$$u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\right] \longrightarrow (1)$$

The given boundary and initial conditions are,

$$u(0,t) = 0$$

$$u(2,t) = 0$$

$$u(x,0) = \sin \frac{\pi x}{2}.$$

These are marked in the table against $x=0$, $x=2$ and $t=0$. The unknowns are represented by $u_1, u_2, u_3, u_4, u_5, u_6$.

t \ x	0	0.5	1	1.5	2
0	0	0.7071	1	0.7071	0
0.25	0	u_1	u_2	u_3	0
0.5	0	u_4	u_5	u_6	0

using equation ①, we get.

$$u_1 = \frac{1}{4}(0+1+0+u_2)$$

$$u_2 = \frac{1}{4}(0.7071+0.7071+u_1+u_3)$$

$$u_3 = \frac{1}{4}(1+0+u_2+0)$$

$$4u_1 - u_2 = 1 \longrightarrow ②$$

$$-u_1 + 4u_2 - u_3 = 1.4142 \longrightarrow ③$$

$$-u_2 + 4u_3 = 1 \longrightarrow ④$$

Solving equations ②, ③, ④ and we get,

$$u_1 = 0.3867,\ u_2 = 0.5469,\ u_3 = 0.3867.$$

Using equation ① for the third row in the table and using the values of u_1, u_2, u_3, we get,

$$u_4 = \frac{1}{4}(0+0.5469+0+u_5)$$

$$u_5 = \frac{1}{4}(0.3867+0.3867+u_4+u_6)$$

$$u_6 = \frac{1}{4}(0.5469+0+u_5+0)$$

$$4u_4 - u_5 = 0.5469 \longrightarrow ⑤$$

$$-u_4 + 4u_5 - u_6 = 0.7734 \longrightarrow ⑥$$

$$-u_5 + 4u_6 = 0.5469 \longrightarrow ⑦$$

Solving equations ④, ⑤, ⑥ we get

$$u_4 = 0.2115$$

$$u_5 = 0.5469$$

Scanned with CamScanner

$u_6 = 0.2115$

Thus the required solutions are,

$u_1 = 0.3867$, $u_2 = 0.5469$, $u_3 = 0.3867$

$u_4 = 0.2115$, $u_5 = 0.2991$, $u_6 = 0.2115$

Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $0 < x < 2$, $t > 0$, $u(0,t) = u(2,t) = 0$, $t > 0$ and $u(x,0) = \sin\frac{\pi x}{2}$, $0 \le x \le 2$ using $\Delta x = 0.5$ and $\Delta t = 0.25$ for two time steps by Crank-Nicholson implicit finite difference method.

Solution:

Given, $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $0 < x < 2$, $t > 0$

$a = 1$, $h = 0.25$, $K = 0.25$

$$\lambda = \frac{k}{ah^2} = \frac{0.25}{1 \times (0.5)^2} = 1.$$

Hence we can use the simplified Crank-Nicholson formula.

$$u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\right] \longrightarrow ①.$$

The given boundary and initial conditions are.

$$u(0,t) = 0,\ u(2,t) = 0,\ u(x,0) = \sin\frac{\pi x}{2}.$$

There are marked in the table against $x=0$, $x=2$ and $t=0$. The unknowns are represented by $u_1, u_2, u_3, u_4, u_5, u_6$.

t \ x	0	0.5	1	1.5	2.
0	0	0.7071	1	0.7071	0
0.25	0	u_1	u_2	u_3	0
0.5	0	u_4	u_5	u_6	0

using equation ①, we get,

$$u_1 = \frac{1}{4}(0 + 1 + 0 + u_2).$$

$u_2 = \frac{1}{4}(0.7071 + 0.7071 + u_1 + u_3)$

$u_3 = \frac{1}{4}(1 + 0 + u_2 + 0) \longrightarrow (2)$

i.e) $4u_1 - u_2 = 1 \longrightarrow (3)$

$-u_1 + 4u_2 - u_3 = 1.4142 \longrightarrow (4)$.

$-u_2 + 4u_3 = 1$.

Solving equations (2), (3), (4) and we get,

$u_1 = 0.3867, \; u_2 = 0.5469, \; u_3 = 0.3867$.

Using equation (1) for the third row in the table and using the values of u_1, u_2, u_3, we get,

$u_4 = \frac{1}{4}(0 + 0.5469 + 0 + u_5)$

$u_5 = \frac{1}{4}(0.3867 + 0.3867 + u_4 + u_6)$.

$u_6 = \frac{1}{4}(0.5469 + 0 + u_5 + 0)$.

i.e) $4u_4 - u_5 = 0.5469 \longrightarrow (5)$.

$-u_4 - 4u_5 - u_6 = 0.7734 \longrightarrow (6)$

$-u_5 + 4u_6 = 0.5469 \longrightarrow (7)$

Solving equations (4), (5), (6), we get,

$u_4 = 0.2115, \; u_5 = 0.2991, \; u_6 = 0.2115$.

Thus the required solutions are,

$u_1 = 0.3867, \; u_2 = 0.5469, \; u_3 = 0.3867$

$u_4 = 0.2115, \; u_5 = 0.2991, \; u_6 = 0.2115$.

Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, in $0 < x < 5, t > 0$ given that $u(x,0) = 20$, $u(0,t) = 0$, $u(5,t) = 100$. Compute u for one time step with $h = 1$, by Crank-Nickolson method.

Solution -

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 5, \; t > 0, \; h = 1.$$

$$a = 1, \; \lambda = \frac{k}{ah^2} = 1 \Rightarrow \frac{k}{1 \times (1)} = 1 \Rightarrow k = 1.$$

$$u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}\right]$$

t \ x	0	1	2	3	4	5
0	0	20	20	20	20	100
1	0	u_1	u_2	u_3	u_4	100.

$$u_1 = \frac{1}{4}[20 + u_2] \Rightarrow 4u_1 - u_2 = 20 \longrightarrow (1).$$

$$u_2 = \frac{1}{4}[40 + u_1 + u_3]$$

$$4u_2 - u_1 - u_3 = 40 \longrightarrow (2).$$

$$u_3 = \frac{1}{4}[40 + u_2 + u_4]$$

$$4u_3 - u_2 - u_4 = 40 \longrightarrow (3)$$

$$u_4 = \frac{1}{4}[220 + u_3]$$

$$4u_4 - u_3 = 220 \longrightarrow (4).$$

$$u_4 = \frac{220 + u_3}{4}.$$

Sub (3) $\Rightarrow 4u_3 - u_2 - \left(\frac{220 + u_3}{4}\right) = 40.$

$$16u_3 - 4u_2 - 220 - u_3 = 160.$$

From (1), $u_1 = \frac{u_2 + 20}{4}$, $15u_3 - 4u_2 = 380 \longrightarrow (5).$

(2) $\Rightarrow 4u_2 - \left(\frac{u_2 + 20}{4}\right) - u_3 = 40$

$$16u_2 - u_2 - 20 - 4u_3 = 160$$

$15u_2 - 4u_3 = 180.$

$4u_3 = 15u_3 - 180$

$u_3 = \dfrac{15u_2 - 180}{4} \rightarrow (6).$

$15\left(\dfrac{15u_2-180}{4}\right) - 4u_2 = 380.$

$15(15u_2 - 180) - 16u_2 = 1520.$

$225u_2 - 2700 - 16u_2 = 1520.$

$209u_2 = 4220.$

$u_2 = 20.191.$

$(6) \Rightarrow u_3 = 30.716$

$(4) \Rightarrow u_4 = 62.679$

$u_1 = 10.048.$

Source :

Text Books:

1. P.Kandasamy, V.Thilagavathy, K.Gunavathi : "Numerical Methods",S.Chand& Company Ltd,New Delhi, 2016.