4 Hazard Models

4-1 INTRODUCTION

In Chapter 3, we observed that the data obtained from failure tests can be analyzed to obtain reliability, probability of failure, hazard rate, and other necessary information. Obviously, the behavioural characteristics exhibited by one class of components differ from those exhibited by another class of components. In order to compare different behavioural characteristics and also to draw general conclusions from the behavioural patterns of similar components, a mathematical model representing the failure characteristics of the components becomes necessary. In this chapter, we start the discussion with several simple models and gradually move to the more involved problem of choosing a general model which fits all cases by the adjustment of constants. Our procedure will be as follows: First, we assume a function for the hazard rate $Z(t)$. This will be our hazard model. The suitability of this function depends on how well it agrees with observed results in practice. Next, we calculate reliability and failure density from Eqs. (3-22) and (3-25), which are reproduced here for convenience:

$$R(t) = \exp\left[-\int_0^t Z(\xi)\, d\xi\right], \tag{4-1}$$

$$f_d(t) = Z(t) \exp\left[-\int_0^t Z(\xi)\, d\xi\right]. \tag{4-2}$$

It was shown in Section 3-2 that the probability of failure $F(t)$ is

$$F(t) = 1 - R(t). \tag{4-3}$$

In Section 3-10, it was also shown that the probability of failure within time t is given by the area under the probability density curve $f_d(\xi)$ from $\xi = 0$ to $\xi = t$ [Eq. (3-19)]. This was called the cumulative distribution function. Since reliability $R(t) = 1 - F(t)$, the probability of a device functioning satisfactorily for at least t units of time will be the area under the probability density curve (PDC), from $\xi = t$ to $\xi = \infty$. This is because the total area under the PDC, from $\xi = 0$ to $\xi = \infty$, is equal to one.

If we are interested in determining the probability of failure in the interval from time t_1 to time t_2, we can first consider the probability of failure within time t_1, which is

$$F(t_1) = \int_0^{t_1} f_d(\xi)\, d\xi.$$

For failure within time t_2, the probability is

$$F(t_2) = \int_0^{t_2} f_d(\xi)\, d\xi.$$

Hence, for failure in the interval from time t_1 to time t_2, the probability is

$$\int_0^{t_2} f_d(\xi)\, d\xi - \int_0^{t_1} f_d(\xi)\, d\xi$$

$$= \int_{t_1}^{t_2} f_d(\xi)\, d\xi.$$

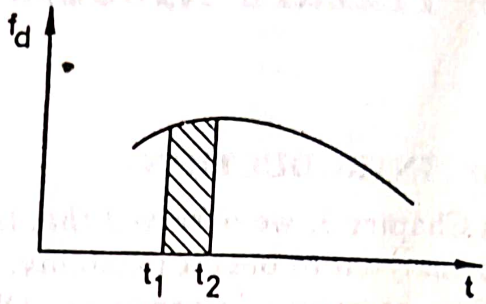

FIGURE 4-1

This represents the area under the PDC between t_1 and t_2 as shown in Fig. 4-1.

4-2 CONSTANT HAZARD

The simplest case that we can consider is that which has a constant hazard rate. In this, we assume that the failure rate remains constant with time. This corresponds to the middle zone in Fig. 3-5. The data obtained for a large number of tests indicate that a constant-hazard model is appropriate in many cases.

Let $Z(t) = \lambda$, a constant. The time integral is then given by

$$\int_0^t Z(\xi)\, d\xi = \int_0^t \lambda\, d\xi = \lambda t, \tag{4-4}$$

where ξ is a dummy variable. Hence, from Eq. (4-1),

$$R(t) = \exp\left[-\int_0^t Z(\xi)\, d\xi\right] = \exp[-\lambda t], \tag{4-5a}$$

$$F(t) = 1 - R(t) = 1 - e^{-\lambda t}. \tag{4-5b}$$

Similarly, from Eq. (4-2),

$$f_d(t) = Z(t) \exp\left[-\int_0^t Z(\xi)\, d\xi\right] = \lambda e^{-\lambda t}. \tag{4-6}$$

These four functions, $Z(t)$, $R(t)$, $F(t)$, and $f_d(t)$, are represented graphically in Fig. 4-2. As shown, the reliability at $t = 0$ is one and decreases exponentially with time. When t becomes very large, the reliability becomes negligibly small. The probability of failure is initially zero and it approaches one as t becomes large. Using Eq. (3-28), we can easily calculate the MTTF for a constant-hazard model. We have

$$\text{MTTF} = \int_0^\infty R(t)\, dt = \int_0^\infty e^{-\lambda t}\, dt = \left.\frac{-e^{-\lambda t}}{\lambda}\right|_0^\infty = \frac{1}{\lambda}. \tag{4-7}$$

Hence, for a constant-hazard model, the mean time to failure is simply the reciprocal of the hazard rate.

A constant-hazard model assumes that the parts do not deteriorate with time. The rate at which the random failures take place is assumed to be

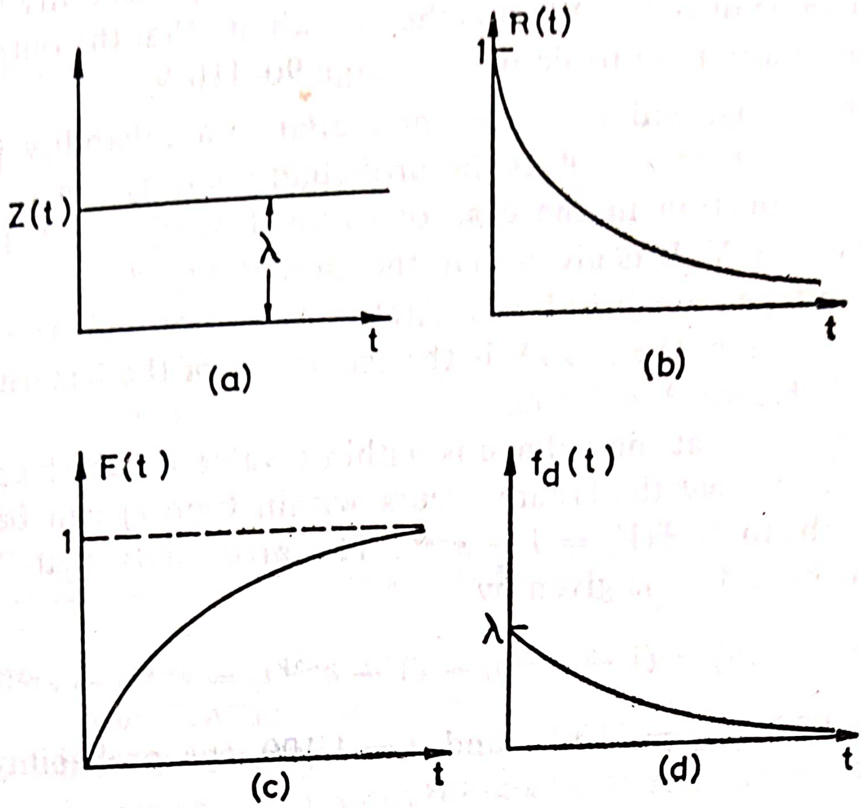

FIGURE 4-2

constant throughout the test period. However, there are cases in which parts or components wear out progressively with time, and consequently the failure rate increases, but this eventuality is not considered for the constant-hazard condition. Thus, if $\lambda = 0.1$ per hour, we can except 10 failures to occur in a population of 100 items during the first hour. The same number of failures will occur between the 150th hour and the 151st hour of operation in a population of 100 items that have survived 150 hours. A simple hazard model that admits deterioration with time (i.e., wear) will be considered in the next section.

The constant-hazard model is also known as the *exponential reliability case* and is quite important in reliability analysis. For this, the probability that the component will fail within time t is given by

$$F(t) = 1 - e^{-\lambda t}, \tag{4-8}$$

and the probability that the component will function satisfactorily for at least t units of time is given by

$$R(t) = e^{-\lambda t}. \tag{4-9}$$

Example 4-1 It is observed that the failure pattern of an electronic system follows an exponential distribution with mean time to failure of 1000 hours. What is the probability that the system failure occurs within 750 hours?

Since MTTF is 1000 hours, $1/\lambda = 1000$. From Eq. (4-5b), the probability of failure within 750 hours is

$$F(1000) = 1 - e^{-750/1000} = 1 - e^{-0.75} = 0.528.$$

Example 4-2 It is found that the random variations with respect to time in the output voltage of a particular system are exponentially distributed with a mean value of 100 V. What is the probability that the output voltage will be found at any time to lie in the range 90–110 V?

This will be recognized as a problem similar to a reliability problem if we identify the voltage output as the probability density function (i.e., the failure density function in the case of survival tests) and MTTF with a mean value of 100 V. It is given that the probability density function as regards the voltage, from Eq. (4-6), is $f_d(t) = \lambda e^{-\lambda t}$. For such an exponential distribution function, the MTTF is the reciprocal of the hazard rate, i.e., MTTF $= 1/\lambda$. Hence, $\lambda = 1/100$.

The probability that the voltage is within a value V (which corresponds to the probability that the failure occurs within time t) can be obtained from Eq. (4-5b) to be $F(V) = 1 - e^{-\lambda V}$. The probability that the voltage lies between V_1 and V_2 is given by

$$F(V_2) - F(V_1) = (1 - e^{-\lambda V_2}) - (1 - e^{-\lambda V_1}) = e^{-\lambda V_1} - e^{-\lambda V_2}.$$

With $V_1 = 90$ V, $V_2 = 110$ V, and $\lambda = 1/100$, the probability that the voltage is between these two values is

$$F(110) - F(90) = e^{-90\lambda} - e^{-100\lambda} = e^{-90/100} - e^{-110/100}$$

$$= e^{-0.9} - e^{-1.1} = 0.667 - 0.593 = 0.074.$$

4-3 LINEARLY-INCREASING HAZARD

As stated in Section 4-2, when there is wear or deterioration of parts or components, the failure rate increases with time. The simplest model that we can consider in this category is one in which the hazard increases linearly with time. This is shown in Fig. 4-3a.

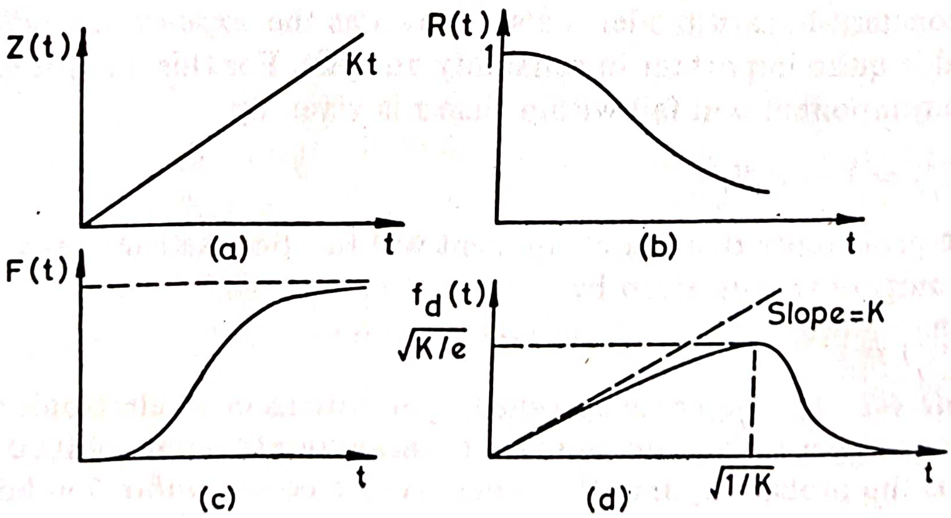

FIGURE 4-3

Let $Z(t) = Kt$, where K is a constant. Then, the time integral of this function is given by

$$\int_0^t Z(\xi)\, d\xi = \int_0^t K\xi\, d\xi = Kt^2/2. \tag{4-10}$$

From Eq. (4-1),

$$R(t) = \exp\left[-\int_0^t Z(\xi)\, d\xi\right] = \exp[-Kt^2/2]. \tag{4-11}$$

Therefore,

$$F(t) = 1 - R(t) = 1 - e^{-Kt^2/2}. \tag{4-12}$$

From Eq. (4-2),

$$f_d(t) = Z(t) \exp\left[-\int_0^t Z(\xi)\, d\xi\right] = Kte^{-Kt^2/2}. \tag{4-13}$$

The function $f_d(t)$ given by Eq. (4-13) is known as the *Rayleigh density function*. The four functions $Z(t)$, $R(t)$, $F(t)$, and $f_d(t)$ are shown in Fig. 4-3. It will be of interest to study the failure density curve. As shown in Fig. 4-3d, the curve has a slope K at time $t = 0$. The value of $f_d(t)$ reaches a maximum of $\sqrt{(K/e)}$ at time $t = \sqrt{(1/K)}$, and tends to zero as t becomes larger.

We can also calculate the MTTF when the hazard increases linearly, but the result is not as simple as that for the constant-hazard condition. In this case,

$$\text{MTTF} = \int_0^\infty R(t)\, dt = \int_0^\infty e^{-Kt^2/2}\, dt.$$

This integral can be obtained from a table of integrals in terms of a function called the *gamma function*. Thus,

$$\text{MTTF} = \int_0^\infty e^{-Kt^2/2}\, dt = \frac{\Gamma(1/2)}{2\sqrt{(K/2)}} = \sqrt{[\pi/(2K)]}.$$

Instead of the failure rate increasing with time due to wear and deterioration, we may have a situation in which the failure rate decreases with time. This can occur when untested components are put to use. Due to faulty assembly, weak parts, or such other factors, it is possible for the failure rate to be high initially and then to decrease with time. A model to fit this could be a linearly-decreasing hazard. We shall not discuss this in detail here.

In Section 4-4, we shall consider a model, called the *Weibull model*, which has a more general application than the models discussed in this section and in Section 4-2.

4-4 THE WEIBULL MODEL

There are many situations in which the failure rate cannot be approxi-

6 System Reliability

6-1 INTRODUCTION

So far, we have discussed reliability and probability of failure in respect of components or elements of a system. If the reliability factor or the probability of failure of the system is to be determined, we will find that it is very difficult to analyze the system in its entirety. In practice, the system is broken down to sub-systems and elements whose individual reliability factors can be estimated or determined. Depending on the manner in which these sub-systems and elements are connected to constitute the given system, the combinatorial rules of probability are applied to obtain the system reliability. In this chapter, we shall discuss methods to determine the system reliability from the reliability factors of the sub-systems and elements. The basic steps are as follows:

(a) First, the elements and sub-systems, which constitute the given system and whose individual reliability factors can be estimated, are identified. These will be called the units comprising the system.

(b) Next, the logical manner or configuration in which these units are connected to form the system is represented by a block diagram or a circuit diagram.

(c) The condition for the successful operation of the system is then determined, that is, it may be decided as to how the units should function. For example, should all the units be operative, or will it be sufficient for any one unit to function?

(d) Finally, the combinatorial rules of probability theory (i.e., addition rule, multiplication rule, and their combinations) are applied to arrive at the system reliability factor.

6-2 SERIES CONFIGURATION

The simplest combination of units that form a system is a series combination. This is also one of the most commonly used structures, and is shown in Fig. 6-1. In this case, the system consists of n units which are connected in series as shown. Let the successful operation of these individual units be represented by $X_1, X_2, \ldots, X_n$, and their respective probabilities by

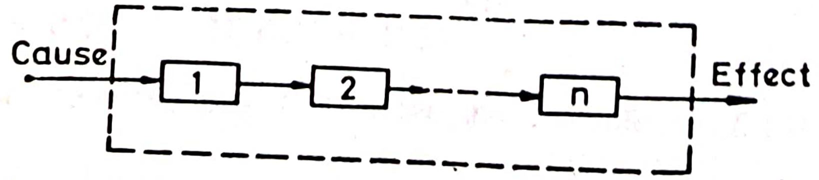

FIGURE 6-1

$P(X_1)$, $P(X_2)$, . . . , $P(X_n)$. For the successful operation of the system, it is necessary that all n units function satisfactorily. Hence, the probability of the simultaneous successful operation of all the units is $P(X_1$ and X_2 and . . . and $X_n)$. We shall assume that these units are not independent of one another, that is, the successful operation of unit 1 might affect the successful operation of all other units, and so on. This situation might occur, for example, when the heat dissipated by unit 1, which may be a resistor, affects the performance characteristics of units 2, 3, ... According to the multiplication rule stated in Section 5-7, the system reliability is given by

$$P(S) = P(X_1 \text{ and } X_2 \text{ and } \ldots \text{ and } X_n)$$
$$= P(X_1) \times P(X_2|X_1) \times P(X_3|X_1 \text{ and } X_2) \times \ldots$$
$$\times P(X_n|X_1 \text{ and } X_2 \text{ and } \ldots \text{ and } X_{n-1}). \qquad (6\text{-}1)$$

In this expression, $P(X_2|X_1)$ represents the probability of the successful operation of unit 2 under the condition that unit 1 operates successfully. Similarly, $P(X_n|X_1$ and X_2 and . . . and $X_{n-1})$ represents the probability of the successful operation of unit n under the condition that all the remaining units 1, 2, . . . , $n - 1$ are working successfully.

If the successful operation of each unit is independent of the successful operation of the remaining units, then events X_1, X_2, . . . , X_n are independent, and Eq. (6-1) becomes

$$P(S) = P(X_1) \times P(X_2) \times \ldots \times P(X_n). \qquad (6\text{-}2)$$

Example 6-1 An electronic equipment is operated by four dry cells, each giving 1.5 volts. The cells are connected in series. The probability of the successful operation of each cell under the given operating conditions is 0.90. Calculate the reliability of the power system.

The batteries are connected in series and it is assumed that the successful operation of one battery does not affect the operation of other batteries. Hence, the events are independent and

$$P(S) = 0.9 \times 0.9 \times 0.9 \times 0.9 = 0.656.$$

Example 6-2 In an hydraulic control system, the connecting linkage has a reliability factor of 0.98 and the valve which has to operate within a certain time limit has a reliability factor of 0.92. The pressure sensor which actuates the linkage has a reliability factor of 0.90. Assume that all the three elements, namely, the actuator, the linkage, and the hydraulic valve, are connected in series with independent reliability factors. What is the reliability of the control system?

Since the units are connected in series and have independent performance characteristics, the system reliability is

$$P(S) = 0.98 \times 0.92 \times 0.9 = 0.811.$$

These two examples bring out a very important point, namely, the reliability of a series system is always worse than the poorest component in the system. The fact that this is true for any series system can be shown easily. We have observed that the system reliability is given by

$$P(S) = P(X_1) \times P(X_2|X_1) \times P(X_3|X_2 \text{ and } X_1) \times \dots$$

$$\times P(X_n|X_1 \text{ and } X_2 \text{ and } \dots \text{ and } X_{n-1}).$$

Since probability, and therefore the reliability factor, is always less than or equal to one, every factor on the right-hand side will be less than or at the most equal to one. The reliability factor of the K-th unit is $P(X_K|X_1$ and X_2 and $\dots$ and $X_{K-1})$, which is also the probability that the K-th unit will function satisfactorily under the condition that the other units $1, 2, \dots, K-1$ operate successfully. Let the reliability factor of the K-th unit be the lowest. This lowest factor is to be multiplied by $(n-1)$ other factors which are all less than or equal to one. Hence, the product of all these factors can never exceed the value of the lowest factor.

Example 6-3 If the system consists of n identical units in series, and if each unit has a reliability factor p, determine the system reliability factor, assuming that all units function independently. Approximate this when the reliability factor is fairly high.

If q is the probability of failure of each unit, then $p = 1 - q$, and the system reliability is

$$P(S) = p \times p \times \dots \times p = p^n = (1 - q)^n.$$

If q is very small, this expression can be approximatedto $1 - nq$. Therefore,

$$P(S) \approx 1 - nq.$$

Example 6-4 A system has ten identical components connected in series. It is desired that the system reliability be 0.95. Determine how good each component should be.

Since the components are in series, the system reliability $P(S)$ is p^{10}, where p is the component reliability. Then,

$$P(S) = 0.95 = p^{10}.$$

Taking the logarithm on both sides, we get $\log 0.95 = 10 \log p$ or $p = 0.9949$. Hence, each unit must have a reliability factor of 0.9949 to give a system reliability factor of 0.95. As noted earlier, a series configuration for a system gives a reliability factor which is lower than that for any of the system components.

6-3 PARALLEL CONFIGURATION

Several systems exist in which successful operation depends on the satisfactory functioning of any one of their n sub-systems or elements. These are said to be connected in parallel. We can also have a system in which several signal paths perform the same operation, and the satisfactory performance of any one of these paths is sufficient to ensure the successful operation of the system. The elements for such a system are also said to be connected in parallel. A block diagram representing a parallel configuration is shown in Fig. 6-2. The reliability of the system can be calculated very easily by considering the conditions for system failure. Let $X_1, X_2, \ldots, X_n$ represent the successful operation of units $1, 2, \ldots, n$, respectively. Similarly, let $\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n$, respectively, represent their unsuccessful operation, i.e., the failure of the n units. If $P(X_1)$ is the probability of successful operation of unit 1, then $P(\bar{X}_1)$ is the probability of its failure. Further, we have already seen that $P(\bar{X}_1) = 1 - P(X_1)$.

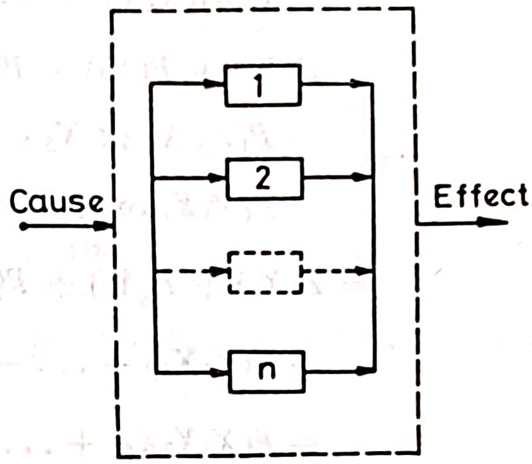

FIGURE 6-2

For the complete failure of the system, all n units have to fail simultaneously. If $P(\bar{S})$ is the probability of failure of the system, then

$$\begin{aligned} P(\bar{S}) &= P(\bar{X}_1 \text{ and } \bar{X}_2 \text{ and } \ldots \text{ and } \bar{X}_n) \\ &= P(\bar{X}_1) \times P(\bar{X}_2|\bar{X}_1) \times P(\bar{X}_3|\bar{X}_1 \text{ and } \bar{X}_2) \times \ldots \\ &\quad \times P(\bar{X}_n|\bar{X}_1 \text{ and } \bar{X}_2 \text{ and } \ldots \text{ and } \bar{X}_{n-1}). \end{aligned} \tag{6-3}$$

In this expression, $P(\bar{X}_3|\bar{X}_1 \text{ and } \bar{X}_2)$ represents the probability of failure of unit 3 under the condition that units 1 and 2 have failed. The other terms can also be interpreted in the same manner. If the unit failures are independent of one another, then

$$P(\bar{S}) = P(\bar{X}_1) \times P(\bar{X}_2) \times \ldots \times P(\bar{X}_n) = 1 - P(S).$$

Hence,

$$P(S) = 1 - [1 - P(X_1)] \times [1 - P(X_2)] \times \ldots \times [1 - P(X_n)]. \tag{6-4a}$$

If the n elements are identical and if the unit failures are independent of one another, then

$$\begin{aligned} P(S) &= 1 - P(\bar{S}) \\ &= 1 - [1 - P(X)]^n. \end{aligned} \tag{6-4b}$$

Instead of looking at the system from the failure point of view, we can observe it from the successful operation point of view and say that, for a successful operation of the system, at least one of the elements has to function successfully. Hence, if $P(X_i)$ is the probability of satisfactory functioning of element X_i, then the reliability of the system is

$$\begin{aligned} P(S) &= P(X_1 \text{ or } X_2 \text{ or } \ldots \text{ or } X_n) \\ &= P(X_1) + P(X_2 \text{ or } X_3 \text{ or } \ldots \text{ or } X_n) \\ &\quad - P[X_1(X_2 \text{ or } X_3 \text{ or } \ldots \text{ or } X_n)] \\ &= P(X_1) + P(X_2) + P(X_3 \text{ or } X_4 \ldots X_n) \\ &\quad - P[X_3(X_4 \text{ or } X_5 \text{ or } \ldots \text{ or } X_n)] \\ &\quad - P[X_1X_2 \text{ or } X_1X_3 \text{ or } \ldots \text{ or } X_1X_n] \\ &= P(X_1) + P(X_2) + P(X_3) + \ldots - P(X_1X_2) - P(X_1X_3) \\ &\quad - P(X_1X_4) - \ldots - P(X_2X_3) - P(X_2X_4) - \ldots + P(X_1X_2X_3) \\ &\quad + P(X_1X_2X_4) + \ldots + (-1)^{n-1}P(X_1X_2 \ldots X_n). \end{aligned} \tag{6-5}$$

Example 6-5 Consider a system consisting of three identical units connected in parallel. The unit reliability factor is 0.90. If the unit failures are independent of one another and if the successful operation of the system depends on the satisfactory performance of any one unit, determine the system reliability.

From Eq. (6-4b), we have

$$P(S) = 1 - (1 - 0.90)^3 = 1 - 0.1^3 = 0.999.$$

This reveals the important fact that a parallel configuration can greatly increase system reliability. With just three elements connected in parallel, it is possible to make the system almost failure-proof. This aspect will be discussed in greater detail in Chapter 7 which deals with reliability improvement.

Example 6-6 A parallel system is composed of ten identical independent components. If the system reliability $P(S)$ is to be 0.95, how poor can the components be?

Since the components are independent of one another, from Eq. (6-4b), we have

$$P(S) = 1 - [1 - P(X)]^{10} \quad \text{or} \quad 0.95 = 1 - [1 - P(X)]^{10}.$$

Therefore,

$$[1 - P(X)]^{10} = 1 - 0.95 = 0.05,$$

$$1 - P(X) = 0.7419 \quad \text{or} \quad P(X) = 0.2581.$$

The components can have a very low reliability factor of 0.2581 and still give the system a reliability factor as high as 0.95.

6-4 MIXED CONFIGURATIONS

After having discussed the two basic configurations, series and parallel, we can now study a few mixed configurations, that is, systems consisting of elements connected in series and parallel. For the sake of simplicity, we shall consider configurations consisting of one, two, three, and four elements. The elements could be dependent dissimilar units or independent identical units.

Let X_1, X_2, and X_3 represent the successful operation of units 1, 2, and 3, respectively, and let $P(X_1)$, $P(X_2)$, and $P(X_3)$ be the corresponding probability factors for successful operation. Similarly, $\bar{X}_1$, $\bar{X}_2$, and $\bar{X}_3$, respectively, represent failures of units 1, 2, and 3, the corresponding probability factors being $P(\bar{X}_1)$, $P(\bar{X}_2)$, and $P(\bar{X}_3)$. For identical independent units, $P(X)$ will be denoted by p. Several configurations of elements are considered as follows:

One-Element Model [Fig. 6-3]

$$P(S) = P(X_1) = p.$$

Two-Element Model [case 1 (Fig. 6-4)]

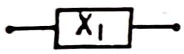

FIGURE 6-3

FIGURE 6-4

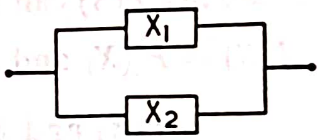

FIGURE 6-5

With dependent dissimilar units

$$P(S) = P(X_1 \text{ and } X_2) = P(X_1) \times P(X_2|X_1).$$

With independent identical units

$$P(S) = P(X_1) \times P(X_2)$$
$$= p^2.$$

Two-Element Model [case 2 (Fig. 6-5)]

With dependent dissimilar units

$$P(S) = P(X_1 \text{ or } X_2) = 1 - P(\bar{X}_1 \text{ and } \bar{X}_2).$$

With independent identical units

$$P(S) = 1 - P(\bar{X}_1) \times P(\bar{X}_2) = 1 - [1 - P(X_1)][1 - P(X_2)]$$
$$= 1 - (1 - p)^2 = 2p - p^2.$$

Three-Element Model [case 1 (Fig. 6-6)]

With dependent dissimilar units For the successful operation of the system, The failure of branch i is therefore given by

$$F_i = 1 - \prod_{j=1}^{n_i} P(X_{ij}).$$

The system fails when all branches fail, and hence the probability of failure is

$$F(S) = \prod_{i=1}^{k} [1 - \prod_{j=1}^{n_i} P(X_{ij})].$$

The reliability of the system is therefore the complement of the foregoing expression, i.e.,

$$R(S) = 1 - \prod_{i=1}^{k} [1 - \prod_{j=1}^{n_i} P(X_{ij})].$$

6-5 APPLICATION TO SPECIFIC HAZARD MODELS

It will be of interest to apply the formulas we have obtained for the series and parallel configurations to specific hazard models. We shall study two types of models: with a constant hazard, and with a linearly-increasing hazard. These were discussed earlier in Chapter 4. We shall study these for both series and parallel configurations.

Series Configuration

If the system consists of n items, all connected in series, the reliability of the system is given by Eq. (6-1) to be

$$R(t) = P(S)$$
$$= P(X_1) \times P(X_2|X_1) \times \ldots \times P(X_n|X_1 \text{ and } X_2 \text{ and } \ldots \text{ and } X_{n-1}).$$

If the n items are independent, then

$$R(t) = P(S) = P(X_1) \times P(X_2) \times P(X_3) \times \ldots \times P(X_n).$$

If each component exhibits a constant hazard rate, we get, for each component, a reliability factor in the form $\exp(-\lambda t)$ from Eq. (4-5a). The i-th element will have a reliability $\exp(-\lambda_i t)$. Hence, the reliability for the system will be

$$R(t) = P(S) = \exp(-\lambda_1 t) \times \exp(-\lambda_2 t) \times \ldots \times \exp(-\lambda_n t)$$
$$= \exp[-(\lambda_1 + \lambda_2 + \ldots + \lambda_n)t]$$
$$= \exp(-\sum_{i=1}^{n} \lambda_i t).$$

It is very important to note that this formula is applicable only when the n components are independent and connected in series.

If the components are governed by the rule for linearly-increasing hazard, then, from Eq. (4-11), the reliability factor for the i-th component is $\exp(-K_i t^2/2)$. Hence, for the system,

$$P(S) = R(t) = \exp(-K_1 t^2/2) \times \exp(-K_2 t^2/2) \times \ldots \times \exp(-K_n t^2/2)$$
$$= \exp[-t^2(K_1 + K_2 + \ldots + K_n)/2]$$
$$= \exp(-\sum_{i=1}^{n} K_i t^2/2).$$

We can also consider a third combination where p out of the n components of the system are governed by the constant-hazard rule, and the remaining $(n-p)$ components follow the rule for linearly-increasing hazard. For such a system, we have

$$P(S) = R(t) = \exp(-\sum_{i=1}^{p} \lambda_i t) \times \exp(-\sum_{i=p+1}^{n} K_i t^2/2).$$

Parallel Configuration

For a system with n components connected in parallel, the reliability is given by Eq. (6-3) to be

$$P(S) = R(t) = 1 - [P(\bar{X}_1) \times P(\bar{X}_2|\bar{X}_1) \times \ldots \times P(\bar{X}_n|\bar{X}_1 \text{ and } \bar{X}_2 \text{ and} \ldots \text{and } \bar{X}_{n-1})].$$

If the elements are independent, then

$$P(S) = R(t) = 1 - [P(\bar{X}_1) \times P(\bar{X}_2) \times \ldots \times P(\bar{X}_n)]$$
$$= 1 - [1 - P(X_1)] \times [1 - P(X_2)] \times \ldots \times [1 - P(X_n)].$$

If the components are governed by the constant-hazard rule, $P(X_i)$ for the i-th element is $\exp(-\lambda_i t)$. Hence,

$$P(S) = R(t) = 1 - [1 - \exp(-\lambda_1 t)] \times [1 - \exp(-\lambda_2 t)] \times \ldots \times [1 - \exp(-\lambda_n t)].$$

If the elements are independent and also identical, we have

$$P(S) = R(t) = 1 - [1 - \exp(-\lambda_1 t)]^n.$$

6-6 AN r-OUT-OF-n STRUCTURE

There are many situations where a system consisting of n components works satisfactorily when at least r of the n components are good. Three simple examples of the r-out-of-n system are:

(a) An eight-cylinder automobile engine in which the successful operation of at least six cylinders is enough for satisfactory performance of the automobile.

(b) A piece of stranded wire with n strands in which at least r strands are necessary to pass the required current.

(c) A shaft lift operated by four cables out of which at least two are necessary for safe operation.

We shall discuss the reliability of an r-out-of-n system when the n components are independent and identical. First, we shall obtain an expression for the probability of *exactly r successes* out of n identical independent trials. This can be obtained by considering the several ways of achieving r successes in n trials. One way is to have r consecutive successes followed by $n - r$ consecutive failures. Since each success and failure is independent, the probability of the assumed sequence is obviously $p^r(1 - p)^{n-r}$, where p is the probability of one success. But this is only one method of getting r successes. There are many other ways of doing this; in fact, the number of ways in which r successes can be obtained in n trials is equal to the number of ways in which a group of r things can be selected out of a group of n given things. Where the order of selection is not important, we have

$$\frac{n \times (n-1) \times (n-2) \times \ldots \times (n-r+1)}{r \times (r-1) \times \ldots \times 1} = \frac{n!}{(n-r)!\,r!} = \binom{n}{r}.$$

Hence, the total probability of getting r successes out of n trials is

$$\frac{n!}{(n-r)!\,r!} p^r(1-p)^{n-r} = \binom{n}{r} p^r(1-p)^{n-r}.$$

This is known as the *Bernoulli trial* or the *binomial experiment.*

Next, we shall try to obtain an expression for the probability of *at least r successes* out of n identical independent trials. This is easily obtained as follows:

the probability of exactly r successes $= \binom{n}{r} p^r(1-p)^{n-r}$

the probability of exactly $r + 1$ successes $= \binom{n}{r+1} p^{r+1}(1-p)^{n-r+1}$

⋮

the probability of exactly n successes $= p^n$

Hence, the probability of getting *at least* r successes is the sum of the above probabilities, that is,

$$\binom{n}{r} p^r(1-p)^{n-r} + \binom{n}{r+1} p^{r+1}(1-p)^{n-r+1} + \ldots + p^n$$

$$= \sum_{K=r}^{n} \binom{n}{K} p^K(1-p)^{n-K}.$$

Example 6-7 What is the probability of exactly 47 heads in 100 flips of a fair coin?

$$P(47 \text{ heads}) = \binom{100}{47} P(\text{heads})^{47} P(\text{tails})^{100-47}$$

$$= \frac{100!}{47!\,53!} (\tfrac{1}{2})^{47} (\tfrac{1}{2})^{53} = 0.0667 \text{ approximately.}$$

Example 6-8 A manufacturing process produces parts which are one per cent defective. Fifty of these parts are selected at random. What is the probability that there are two or less defective parts out of the fifty selected parts?

$$P(\text{exactly 2 defective}) = \binom{50}{2}(0.01)^2(0.99)^{48},$$

$$P(\text{exactly 1 defective}) = \binom{50}{1}(0.01)^1(0.99)^{49},$$

$$P(0 \text{ defective}) = (0.99)^{50}.$$

Therefore,

$$P(2 \text{ or less defective})$$

$$= \binom{50}{2}(0.01)^2(0.99)^{48} + \binom{50}{1}(0.01)(0.99)^{49} + (0.99)^{50}$$

$$= 1.598 \times (0.99)^{48}.$$

6-7 METHODS OF SOLVING COMPLEX SYSTEMS

When a system is formed from elements and units connected in parallel, series, or mixed configurations, a suitable method of calculating its reliability becomes necessary. There are several methods available for this purpose; in this section, we shall illustrate three such methods by their application to the solution of various problems. We shall begin with elementary cases and then move to complicated combinations. A revision of the probability rules given in Section 5-7 will be helpful.

Example 6-9 A system consists of three elements *a*, *b*, and *c*. The configuration of the system and the reliabilities of the elements are shown in Fig. 6-14. Determine the system reliability.

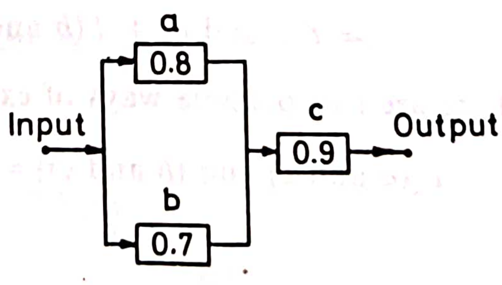

FIGURE 6-14

Method 1 (reduction to series elements) In this method, we systematically replace each parallel path by an equivalent single path, and ultimately reduce the given system to one consisting of only series elements. Accordingly, elements *a* and *b* in Fig. 6-14 are first replaced by an equivalent single element *d*. Using the

Example 6-11 Five elements (a, b, c, d, and f) of a system are connected as shown in Fig. 6-19, which also indicates the reliability of each element. Calculate the system reliability.

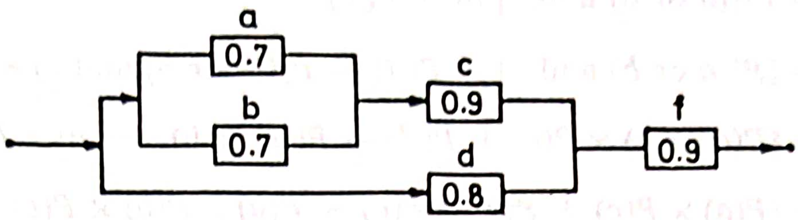

FIGURE 6-19

Method 1 (reduction to series elements) Elements a and b are replaced by an equivalent element g, the reliability of which is

$$P(g) = P(a \text{ or } b) = P(a) + P(b) - P(a \text{ and } b)$$
$$= P(a) + P(b) - P(a) \times P(b)$$
$$= 0.7 + 0.7 - 0.49 = 0.91.$$

Elements g and c, which are now in series, can be replaced by an equivalent element j whose reliability is

$$P(j) = P(g \text{ and } c) = P(g) \times P(c)$$
$$= (0.91)(0.9) = 0.819, \text{ say, } 0.82.$$

The system has now been modified to that shown in Fig. 6-20. Elements j

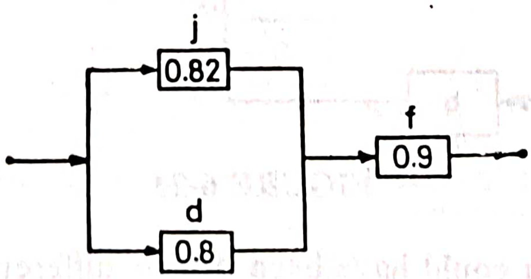

FIGURE 6-20

and d which are in parallel can be replaced by k such that

$$P(k) = P(j \text{ or } d) = P(j) + P(d) - P(j \text{ and } d)$$
$$= P(j) + P(d) - P(j) \times P(d)$$
$$= 0.82 + 0.80 - 0.656 = 0.964.$$

The system has now been reduced to one having only series elements k and f. Hence, the system reliability is

$$P(S) = P(k \text{ and } f) = P(k) \times P(f) = (0.964)(0.9) = 0.868.$$

Estimation of Mean Life with Complete Samples

Suppose n items are subjected to test and the test is terminated after all the items have failed. Let $(X_1, X_2, \ldots, X_n)$ be the random failure times and suppose the failure times are exponentially distributed with p.d.f. $f(x \mid \sigma) = (1/\sigma)\exp(-x/\sigma)$, $x, \sigma > 0$. Given the sample $(X_1, X_2, \ldots, X_n)$ we wish to estimate the mean life σ.

The likelihood function is given by

$$L(x_1, x_2, \ldots, x_n \mid \sigma) \equiv L = \frac{1}{\sigma^n}\exp\left(-\sum_1^n x_i/\sigma\right),$$

$$\frac{\partial}{\partial\sigma}\log L = -\frac{n}{\sigma} + \frac{\sum_1^n x_i}{\sigma^2}$$

The maximum likelihood estimator (MLE) of σ is the solution of the equation

$$\frac{\partial}{\partial\sigma}\log L = 0$$

Thus, $\hat{\sigma} = \bar{x}$ is the MLE of σ. Also $E(\hat{\sigma}) = E(\bar{X}) = \sigma$ and $\mathrm{Var}(\bar{X}) = \sigma^2/n$. Note that $\bar{x}$ is *unbiased* for σ. Consider the distribution of $\bar{X}$. Since $(X/\sigma) \sim \gamma(1)$, by a property of gamma distribution mentioned earlier, it follows that

$$\sum_1^n X_i/\sigma = \frac{n\bar{X}}{\sigma} \sim \gamma(n)$$

and we write

$$g(\bar{x} \mid \sigma) = \frac{1}{\Gamma(n)}\left(\frac{n\bar{x}}{\sigma}\right)^{n-1}\left(\frac{n}{\sigma}\right)\exp\left(-\frac{n\bar{x}}{\sigma}\right)$$

$$= \frac{\left(\frac{n}{\sigma}\right)^n}{\Gamma(n)}\bar{x}^{n-1}\exp\left(-\frac{n\bar{x}}{\sigma}\right) \quad \bar{x} > 0 \tag{6}$$

We now consider the minimum variance unbiased estimator of σ and indicate the proof of the fact that $\hat{\sigma} = \bar{x}$ is also the uniformly minimum variance unbiased estimator (UMVUE) of σ. In general, the search for UMVUE is limited to the functions of the complete sufficient statistic when it exists. The sufficiency of any statistic can be checked by the factorizibility criterion, but the completeness is much more difficult to establish. In the single parameter exponential distribution, the sufficiency of $\bar{x}$ is easy to check. Consider the ratio

$$\frac{L(x_1, x_2, \ldots, x_n \mid \sigma)}{g(\bar{x} \mid \sigma)}$$

which in this case reduces to

$$\frac{\Gamma(n)}{n\left(\sum_1^n x_i\right)^n} \quad \text{for} \quad x_i > 0,\ i = 1, 2, \ldots, n$$

This ratio is independent of σ and is a function of observations alone. Hence $\bar{x}$ is sufficient. For the proof of completeness of $\bar{x}$, we refer to Lehman and Scheffé (1955). Thus $\bar{x}$ is UMVUE of σ.

In this particular case, both methods of estimation lead to the same estimator. But we will see in the next section that this is not so when we are estimating the reliability $R(t) = \exp(-t/\sigma)$, which is the probability of survival for at least time t.

Reliability Estimation

We use the term 'reliable' in various contexts in everyday life, such as a reliable friend, a reliable service station, reliable news, etc. As an abstract concept it means something or someone we may depend upon or count on. In life testing research we are more concerned with a quantitative measure of the reliability of an item or a device we are interested in.

The reliability of a unit (or a system) is defined as the probability that it will perform satisfactorily at least for a specified period of time without a major breakdown. If X is the lifetime of the unit, the reliability of the unit at time t is given by

$$R(t) = P(X \geqslant t) = 1 - F(t)$$

where F is the d.f. of the failure time X.

Suppose a manufacturer wants to promote a new brand of light bulb in the market. A random sample of 10 bulbs is put to test and their failure times recorded. Suppose the bulbs failed after 125, 189, 210, 351, 465, 580, 630, 760, 810 and 870 hours. Looking at the data, a prospective buyer asks, "If I buy a new bulb of this brand, what is the probability that it will survive at least 600 hours?"

If we do not assume any particular distribution of the failure times, this probability may be estimated by the ratio

$$[R(t)]_{t=600} = \frac{\text{number of bulbs surviving} \geqslant 600 \text{ hours}}{\text{number of bulbs initially exposed to test}} = \frac{4}{10} = 0.4$$

If we assume that the failure time distribution is exponential with mean life σ, by definition

$$R(t) = \int_t^\infty \frac{1}{\sigma} \exp(-x/\sigma)\, dx = \exp(-t/\sigma)$$

Since σ is unknown, we have to estimate $R(t)$. First consider the MLE. We have already seen that the MLE of σ is given by $\hat{\sigma} = \bar{x}$. Now MLE has an important property that if $\hat{\theta}$ is the MLE of θ, then $\tau(\hat{\theta})$ is the MLE of $\tau(\theta)$ provided $\tau(\theta)$ is a fairly well behaved function of θ. In particular say, $\tau(\theta)$ is a monotone differentiable function of θ (see Zehna, 1966). Thus we have $\hat{R}(t) = \exp(-t/\bar{x})$. From the failure times of 10 bulbs, we obtain $\bar{x} = 499$, and hence the MLE of reliability at time $t = 600$ hours is given by

$$\hat{R}(600) = \exp\left(-\frac{600}{499}\right) = \exp(-1.2024) = 0.3006$$

called 'failure-censored' samples. Failure-censored sampling is almost mandatory in dealing with high cost sophisticated items such as colour television tubes.

Another factor that affects the life-testing experiment is the amount of time required to obtain the complete sample. To limit this factor, we may put n items to test and terminate the experiment at a pre-assigned time t_0. The samples obtained from such an experiment are called 'time-censored' samples. Time-censored sampling is almost essential in dealing with life-testing experiments in which the cost of experiments increases heavily with time.

In the failure-censored case data consist of the life times of the r items that failed (say $x_{(1)} < x_{(2)} < \ldots < x_{(r)}$) and the fact that $(n - r)$ items have survived beyond $x_{(r)}$. In the time-censored case data consist of the life times of items that failed before the time t_0, say $x_{(1)}\, x_{(2)} < \ldots < x_{(m)}$, assuming that m items failed before t_0 and the fact that $(n - m)$ items have survived beyond t_0. In the time-censored case t_0, the time of termination, is fixed while m, the number of items that failed before t_0, is a random variable. In the failure-censored case the situation is reverse in that r, the number of items that failed, is fixed, while $x_{(r)}$, the time at which the experiment is terminated, is a random variable. One may, of course, consider a combination of both time and failure censoring where one terminates the experiment when either a pre-assigned number r of items have failed or the experiment has run up to time t_0, whichever comes first. This approach is useful when the items involved are high cost ones and the time element in obtaining the observations is also important. For example, consider a new type of rotary engine. This is a high cost item and would consume fuel and require trained technicians as observers. On the other hand, consider polyethylene bags which are being tested for their breaking strength. In this case one would hardly recommend either time or failure-censored sampling.

We will now consider the estimation of parameters and reliability functions for one and two-parameter exponential distributions under different types of censoring commonly used in life-testing experiments.

Failure-censored Samples TYPE II Censoring

Let us first consider the case where the items that failed are not replaced.

We consider the single parameter exponential distribution first. As stated earlier, the data consist of failure times $x_{(1)} < x_{(2)} < \ldots < x_{(r)}$ of r items that failed and the fact that $(n - r)$ items survived until $x_{(r)}$. The likelihood of the sample is given by

$$L\{x_{(1)}, x_{(2)}, \ldots, x_{(r)} \mid \sigma\} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{\sigma^r} \exp\left\{-\frac{\sum_{1}^{r} x_{(i)} + (n-r)x_{(r)}}{\sigma}\right\} \tag{11}$$

This can be obtained from the following considerations.

For the complete sample case, the joint p.d.f. of $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ is given by

$$g\{x_{(1)}, x_{(2)}, \ldots, x_{(n)}\} = \frac{n!}{\sigma^n} \exp\left(\frac{\sum_1^n x_{(i)}}{\sigma}\right), \; 0 < x_{(1)} < x_{(2)} \ldots < x_{(n)} \quad (12)$$

In the failure-censored case we terminate the experiment at $x_{(r)}$ and so we integrate out (12) with respect to $x_{(r+1)}, x_{(r+2)}, \ldots, x_{(n)}$ over the region $x_{(r)} < x_{(r+1)} < x_{(r+2)} < \ldots < x_{(n)}$ which immediately leads to (11).

The MLE of σ is easily shown to be $\hat{\sigma}_r = \left(\sum_1^r x_{(i)} + (n-r)x_{(r)}\right)\Big/r.$

Using the transformation $z_i = (n-i+1)\{x_{(i)} - x_{(i-1)}\}, \; i = 1, 2, \ldots, r$ with $x_{(0)} = 0$, $\hat{\sigma}_r = \left(\sum_1^r z_i\right)\Big/r$. We have shown earlier that $(Z_1, Z_2, \ldots, Z_r)$ are i.i.d. as $f(z/\sigma) = \frac{1}{\sigma}\exp(-z/\sigma), \; z, \sigma > 0$.

Hence $\frac{r\hat{\sigma}_r}{\sigma} \sim \gamma(r)$, $E\left(\frac{r\hat{\sigma}_r}{\sigma}\right) = \text{Var}\left(\frac{r\hat{\sigma}_r}{\sigma}\right) = r$. Thus $E(\hat{\sigma}_r) = \sigma$ and $\text{Var}(\hat{\sigma}_r) = \frac{\sigma^2}{r}$ and the p.d.f. of $\hat{\sigma}_r$ is given by

$$g(\hat{\sigma}_r \mid \sigma) = \frac{1}{\Gamma(r)}\left(\frac{r}{\sigma}\right)^r (\hat{\sigma}_r)^{r-1} \exp\left(-\frac{r\hat{\sigma}_r}{\sigma}\right), \; 0 < \hat{\sigma}_r < \infty \quad (13)$$

Consider the ratio

$$\frac{L\{x_{(1)}, x_{(2)}, \ldots, x_{(r)} \mid \sigma\}}{g(\hat{\sigma}_r \mid \sigma)} = \frac{\Gamma(r)n(n-1)(n-2)\ldots(n-r+1)}{(r)^r \hat{\sigma}_r^{r-1}}$$

which is independent of the unknown parameter σ and thus $\hat{\sigma}_r$ is sufficient for σ. Also, it is known to be complete (Tukey, 1949; Smith, 1957). Since $E(\hat{\sigma}_r) = \sigma$, it follows, therefore, that $\hat{\sigma}_r$ is also the UMVUE of σ. The MLE of the reliability function $R(t \mid \sigma)$ is given by

$$\hat{R}(t) = \exp(-t/\hat{\sigma}_r)$$

Following the same line of derivation as in the complete sample case, we obtain

$$g(x_i \mid \hat{\sigma}_r) = \frac{\Gamma(r)}{\Gamma(r-1)r\hat{\sigma}_r}\left(1 - \frac{x_i}{r\hat{\sigma}_r}\right)^{r-2}, \; 0 < x_i < r\hat{\sigma}_r,$$

where x_i is the failure time of any of the r units that failed. The UMVUE of reliability, $\tilde{R}(t)$ is given by

$$\begin{aligned} R(t) &= P(X_i > t \mid \hat{\sigma}_r) \\ &= \frac{\Gamma(r)}{\Gamma(r-1)r\hat{\sigma}_r}\int_t^{r\hat{\sigma}_r}\left(1 - \frac{x_i}{r\hat{\sigma}_r}\right)^{r-2} dx_i \\ &= \left(1 - \frac{t}{r\hat{\sigma}_r}\right)^{r-1}, \; t < r\hat{\sigma}_r \end{aligned}$$

and

$$= 0, \; t \geq r\hat{\sigma}_r$$

Next, consider the case where the items that fail are immediately replaced and the test terminates after the rth failure. Here the likelihood function is given by

$$L\{x_{(1)}, x_{(2)}, \ldots, x_{(r)} \mid \sigma\} = \frac{n^r}{\sigma^r}\left[\exp\left\{-\frac{x_{(1)}}{\sigma}\right\}\right]\left[\exp\left\{-\frac{x_{(2)}-x_{(1)}}{\sigma}\right\}\right]\ldots$$

$$\left[\exp\left\{-\frac{x_{(r)}-x_{(r-1)}}{\sigma}\right\}\right]\left[\int_{x_{(r)}}^{\infty}\frac{1}{\sigma}\exp\left(-\frac{x}{\sigma}\right)dx\right]^{n-1}$$

$$= \left(\frac{n}{\sigma}\right)^r \exp\left\{-\frac{nx_{(r)}}{\sigma}\right\},\ 0 < x_{(1)} < x_{(2)} \ldots < x_{(r)} < \infty$$

The MLE $\sigma_r^* = \frac{nx_{(r)}}{r}$. Also

$$\frac{r\sigma_r^*}{\sigma} = \frac{n}{\sigma}\cdot[x_{(1)} + \{x_{(2)} - x_{(1)}\} + \{x_{(3)} - x_{(2)}\} + \{x_{(r)} - x_{(r-1)}\}] \sim \gamma(r)$$

Thus, in the experiment of either type (with or without replacement) the MLE of σ is unbiased, has the same variance and has the same distribution. $\frac{L\{x_{(1)}, x_{(2)}, \ldots, x_{(r)} \mid \sigma\}}{g(\sigma_r^* \mid \sigma)} = \frac{n\Gamma(r)}{rx_{(r)}^{r-1}}$ is independent of σ and thus σ_r^* is sufficient. It is also known to be complete. Hence σ_r^* is the MLE as well as UMVUE of σ. The MLE of reliability function $R(t \mid \sigma)$ is given by $R^*(t) = \exp(-t/\sigma_r^*)$. Since σ_r^* is complete and sufficient, if we can find a function $h\{\sigma_r^* \mid t\}$ such that $E\{h(\sigma_r^* \mid t)\} = \exp(-t/\sigma)$, then the UMVUE of reliability is given by $h(\sigma_r^* \mid t)$.

Consider the function

$$h(\sigma_r^* \mid t) = \left(1 - \frac{t}{r\sigma_r^*}\right)^{r-1},\ t < r\sigma_r^*$$

and

$$= 0,\ t \geqslant r\sigma_r^*$$

From (13),

$$g(\sigma_r^* \mid \sigma) = \frac{1}{\Gamma(r)}\left(\frac{r}{\sigma}\right)^r(\sigma_r^*)^{r-1}\exp\left(-\frac{r\sigma_r^*}{\sigma}\right),\ 0 < \sigma_r^* < \infty$$

$$E\{h(\sigma_r^* \mid t)\} = \frac{1}{\Gamma(r)}\left(\frac{r}{\sigma}\right)^r\int_{t/r}^{\infty}\left(1 - \frac{t}{r\sigma_r^*}\right)^{r-1}(\sigma_r^*)^{r-1}\exp\left(-\frac{r\sigma_r^*}{\sigma}\right)d\sigma_r^*$$

$$= \frac{r}{\Gamma(r)\sigma^r}\int_{t/r}^{\infty}(r\sigma_r^* - t)^{r-1}\exp\left(-\frac{r\sigma_r^*}{\sigma}\right)d\sigma_r^*$$

$$= \frac{1}{\Gamma(r)\sigma^r}\int_0^{\infty}y^{r-1}\exp\left\{-\frac{1}{\sigma}(y+t)\right\}dy$$

$$= \exp(-t/\sigma)$$

This shows that $h(\sigma_r^* \mid t)$ is the UMVUE of $R(t \mid \sigma)$.

Example 1.2 Sixty items were placed on test and the test was terminated

$$\hat{\sigma}_r = \frac{y_r}{r} = \frac{14{,}000}{15} = 933 \text{ hours}$$

The UMVUE's are

$$\tilde{\mu} = x_{(1)} - \frac{y_r}{n(r-1)} = 650 - \frac{14{,}000}{30 \times 14} = 617 \text{ hours}$$

$$\tilde{\sigma}_r = \frac{y_r}{r-1} = \frac{14{,}000}{14} = 1000 \text{ hours}$$

Based on the first r ordered failure times, the MLE of reliability is given by $\hat{R}(t \mid \mu, \sigma) = \exp\left\{-\frac{1}{\hat{\sigma}_r}(t - \hat{\mu})\right\}$ and the UMVUE of $R(t \mid \mu, \sigma)$ is given by

$$\tilde{R}(t \mid \mu, \sigma) = \frac{n-1}{n}\left\{1 - \frac{t - x_{(1)}}{y_r}\right\}^{r-2}, \quad x_{(1)} < t < x_{(1)} + y_r$$

$$= 1, \quad x_{(1)} > t$$

and

$$= 0, \quad t > x_{(1)} + y_r$$

(Laurent, 1963)

EXAMPLE 1.4 Given the data in Example 1.3, obtain the MLE and UMVUE of the reliability at $t = 1000$ hours. We have $\hat{\sigma}_r = 933$, $\hat{\mu} = 650$. Hence

$$\hat{R}(1000 \mid \mu, \sigma) = \exp\left\{-\frac{1000 - 650}{933}\right\} = 0.6873$$

and

$$\tilde{R}(1000 \mid \mu, \sigma) = \frac{29}{30}\left\{1 - \frac{1000 - 650}{14{,}000}\right\}^{13} = 0.6956$$

Time-censored Samples TYPE I Censoring

The time-censored samples arise when we terminate the life-testing experiment at a pre-assigned time t_0. As mentioned earlier, here the number of items that failed before time t_0 is a random variable which we denote by M. Let $p(t_0)$ be the probability of failure before time t_0. Then M has a binomial distribution

$$P(M = m) = \binom{n}{m} p^m q^{n-m}, \quad m = 0, 1, 2, \ldots, n$$

where $p \equiv p(t_0) = 1 - \exp(-t_0/\sigma)$ and $q = 1 - p$. Suppose the items that failed are *not-replaced*. The data consist of the life times $x_{(1)} < x_{(2)} < \ldots < x_{(m)}$ of m items that failed before t_0 and $(n - m)$ items that survived beyond t_0. The likelihood of the sample is given by

$$L\{x_{(1)}, x_{(2)}, \ldots, x_{(m)}, m \mid \sigma\} = \exp\left(-\frac{nt_0}{\sigma}\right) \quad \text{for } m = 0$$

and

$$= \frac{n!}{m!\,(n-m)!}\frac{1}{\sigma^m}\exp\left\{-\frac{\sum_{i=1}^{m} x_{(i)} + (n-m)t_0}{\sigma}\right\}$$

for $m = 1, 2, \ldots, n$.

This follows from the following considerations. The likelihood for $m = 0$ is obvious. For $m > 0$ consider the conditional p.d.f. of the failure time, given that the item has failed before time t_0. This is given by

$$h(x \mid \sigma) = \frac{\frac{1}{\sigma}\exp\left(-\frac{x}{\sigma}\right)}{1 - \exp(-t_0/\sigma)}, \quad 0 < x < t_0$$
$$= 0 \quad \text{otherwise}$$

Thus, the joint p.d.f. of $x_{(1)}, x_{(2)}, \ldots, x_{(m)}$ is given by

$$g\{x_{(1)}, x_{(2)}, \ldots, x_{(m)} \mid \sigma\} = \frac{m!}{\sigma^m} \frac{\exp\left\{-\sum_{i=1}^{m} x_{(i)}/\sigma\right\}}{\left\{1 - \exp\left(-\frac{t_0}{\sigma}\right)\right\}^m}$$

The likelihood of the sample is the joint p.d.f. of $\{x_{(1)}, x_{(2)}, \ldots, x_{(m)}\}$ and m. Hence

$$L\{x_{(1)}, x_{(2)}, \ldots, x_{(m)}, m \mid \sigma\} = g\{x_{(1)}, x_{(2)}, \ldots, x_{(m)} \mid \sigma\}\binom{n}{m} p^m q^{n-m}$$
$$= \frac{n!}{(n-m)!} \frac{1}{\sigma^m} \exp\left[-\frac{\sum_{i=1}^{m} x_{(i)} + (n-m)t_0}{\sigma}\right]$$

For $m > 0$, $L\{x_{(1)}, x_{(2)}, \ldots, x_{(m)}, m \mid \sigma)$ is maximized for

$$\hat{\sigma} = \frac{\sum_{i=1}^{m} x_{(i)} + (n-m)t_0}{m}$$

However, for $m = 0$, the likelihood $= \exp(-nt_0/\sigma)$ and is a strictly increasing function of σ. So the MLE should be taken to be $\hat{\sigma} = \infty$ if $m = 0$. But then the random variable $\hat{\sigma}$ is improper in that it assumes the value $+\infty$ with positive probability and, therefore, neither the mean nor the variance would exist. We, however, note that $P(M = 0) = \exp(-nt_0/\sigma)$ would be quite small for a fairly large value of nt_0/σ. This can be achieved by having n sufficiently large and t_0/σ not too small. Thus, in practice for the time-censored life-testing experiment, we will start with fairly large number of items n and would select termination time t_0 such that t_0 is not too small as compared to σ. Since σ is unknown, one cannot guarantee that $m > 0$ for a given choice of n and t_0. There is always a positive probability however small, that we obtain $m = 0$. In such a case we follow the recommendation of Bartholomew (1957) that we take $\hat{\sigma} = nt_0$. Thus the MLE σ is given by

$$\left.\begin{aligned} \hat{\sigma} &= \frac{\sum_{1}^{m} x_{(i)} + (n-m)t_0}{m}, & m > 0 \\ &= nt_0 \quad \text{for} & m = 0 \end{aligned}\right\} \tag{16}$$

The distribution theory of $\hat{\sigma}$ in the time-censored case is quite complicated. This is mainly due to the fact that m is a random variable and the numerator of (16) is a sum of random number of random variables. Bartholomew (1957) has obtained approximations to the bias and variance of $\hat{\sigma}$. The asymptotic (large sample) bias and variance are given by

$$\text{Bias}(\hat{\sigma}) \simeq \frac{tq}{np^2} \text{ and } \text{Var}(\hat{\sigma}) \simeq \frac{\sigma^2}{np}$$

The MLE of the reliability is, of course, given by $\hat{R}(t) = \exp\left(-\frac{t}{\hat{\sigma}}\right)$. Not much is known about the bias and the variance of $\hat{R}(t)$. The case of the UMVUE appears to be much more hopeless and to the best of knowledge of the authors, the problem of obtaining the UMVUE of σ for the time-censored case remains unsolved. This situation illustrates the principle that sometimes very (apparently) minor modifications in the experimental set up can lead to quite difficult problems in estimating the parameters.

Estimator based on n and m (*Bartholomew*, 1963)
Consider only the number of failures during $[0, t_0]$ and ignore the times when failures occurred. We have the likelihood

$$L = \binom{n}{m} p^m q^{n-m}, \quad m = 0, 1, 2, \ldots, n;$$

$$= \binom{n}{m}\left\{1 - \exp\left(-\frac{t_0}{\sigma}\right)\right\}^m \exp\left\{-(n-m)\frac{t_0}{\sigma}\right\}$$

$$\frac{\partial}{\partial\sigma}\log L = \frac{t_0}{\sigma^2}\left[n - \frac{m}{1 - \exp(-t_0/\sigma)}\right] - E\left(\frac{\partial^2}{\partial\sigma^2}\log L\right)$$

and

$$= \frac{t_0^2 n \exp(-t_0/\sigma)}{\sigma^4\{1 - \exp(-t_0/\sigma)\}} \quad \text{for large } n$$

Thus, the MLE of σ and the asymptotic variance of the estimator are given by

$$\hat{\sigma}' = \frac{-t_0}{\log(1 - m/n)} \tag{17}$$

and

$$\text{Var}(\hat{\sigma}') = \frac{\sigma^2 p}{nq(\log q)^2} \tag{18}$$

The estimator (17) is recommended for $0.2 < m/n < 0.8$. The bias in this estimator is infinite since the probability that $m = n$ is positive (Gnedenko, Belyayev and Solovyev, 1969).

The limiting relative efficiency of $\hat{\sigma}'$ against $\hat{\sigma}$ in (16)

$$= \lim_{n\to\infty} \frac{\text{Var}(\hat{\sigma})}{\text{Var}(\hat{\sigma}')}$$

$$= \frac{q}{p^2}\{\log(1-p)\}^2$$

$$= (1-p)\left(1 + p + \frac{11p^2}{12} + \ldots\right)$$

$$= 1 - \frac{p^2}{12} \text{ if } p \text{ is small}$$