9·8. EXPONENTIAL DISTRIBUTION

UNIT – III Continuation

Definition. *A r.v. X is said to have an exponential distribution with parameter $\theta > 0$, if its p.d.f. is given by :*

$$f(x, \theta) = \begin{cases} \theta e^{-\theta x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases} \qquad \ldots (9 \cdot 26)$$

The cumulative distribution function $F(x)$ is given by

$$F(x) = \int_0^x f(u)\, du = \theta \int_0^x \exp(-\theta u)\, du$$

$$F(x) = \begin{cases} 1 - \exp(-\theta x), \geq 0 \\ 0, \text{ otherwise} \end{cases} \qquad \ldots (9 \cdot 26\, a)$$

9·8·1. Moment Generating Function of Exponential Distribution

$$M_X(t) = E(e^{tX}) = \theta \int_0^\infty e^{tx} e^{-\theta x}\, dx = \theta \int_0^\infty \exp\{-(\theta - t)x\}\, dx$$

$$= \frac{\theta}{(\theta - t)} = \left(1 - \frac{t}{\theta}\right)^{-1} = \sum_{r=0}^{\infty} \left(\frac{t}{\theta}\right)^r, \theta > t$$

$$\therefore \qquad \mu_r' = E(X^r) = \text{Coefficient of } \frac{t^r}{r!} \text{ in } M_X(t) = \frac{r!}{\theta^r}; r = 1, 2, \ldots$$

$$\Rightarrow \qquad \text{Mean} = \mu_1' = \frac{1}{\theta} \text{ and variance} = \mu_2 = \mu_2' - \mu_1'^2 = \frac{2}{\theta^2} - \frac{1}{\theta^2} = \frac{1}{\theta^2}$$

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Theorem. *If $X_1, X_2, \ldots, X_n$ are independent r.v.'s, X_i having an exponential distribution with parameter θ_i ; $i = 1, 2, \ldots, n$; then $Z = \min\ (X_1, X_2, \ldots, X_n)$ has exponential distribution with parameter $\sum_{i=1}^{n} \theta_i$.*

Proof.

$$G_Z(z) = P(Z \le z) = 1 - P(Z > z) = 1 - P[\min(X_1, X_2, \ldots, X_n) > z]$$

$$= 1 - P(X_i > z ; i = 1, 2, \ldots, n) = 1 - \prod_{i=1}^{n} P(X_i > z)$$

($\because X_1, X_2, \ldots, X_n$ are independent)

$$= 1 - \prod_{i=1}^{n} [1 - P(X_i \le z)] = 1 - \prod_{i=1}^{n} [1 - F_{X_i}(z)]$$

(where F is the distribution function of X_i).

$$= 1 - \prod_{i=1}^{n} \left[1 - \left(1 - e^{-\theta_i z}\right)\right] = \begin{cases} 1 - \exp\left\{\left(-\sum_{i=1}^{n} \theta_i\right) z\right\}, & z > 0 \\ 0, \text{otherwise} \end{cases}$$

$$\therefore \quad g_Z(z) = \frac{d}{dz}(G(z)) = \begin{cases} \left(\sum_{i=1}^{n} \theta_i\right) \exp\left\{\left(-\sum_{i=1}^{n} \theta_i\right) z\right\}, & z > 0 \\ 0, \text{ otherwise} \end{cases}$$

$\Rightarrow \quad Z = \min(X_1, X_2, \ldots X_n)$ is an exponential variate with *parameter* $\sum_{i=1}^{n} \theta_i$.

Cor. If X_i ; $i = 1, 2, \ldots, n$ are identically distributed, following exponential distribution with parameter θ, then $Z = \min(X_1, X_2, \ldots, X_n)$ is also exponentially distributed with parameter $n\,\theta$.

... of Exponential Distribution. See Example 9.48, page 9.75.

Definition. *A continuous r.v. X is said to have standard Laplace (double exponential) distribution if its p.d.f. is given by :*

$$f(x) = \frac{1}{2} \exp. (-|x|); -\infty < x < \infty. \qquad \dots (9.27)$$

9.9.1. Characteristic Function of Standard Laplace Distribution.

$$\varphi_X(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{itx} e^{-|x|}\, dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} \cos tx \cdot e^{-|x|}\, dx + i\int_{-\infty}^{\infty} \sin tx\, e^{-|x|}\, dx\right]$$

$$= \frac{1}{2}\cdot 2\int_0^{\infty} \cos tx\,.\, e^{-|x|}\, dx,$$

since the integrands in the first and second integrals are even and odd function of x respectively.

$$\therefore \quad \varphi_X(t) = \int_0^{\infty} e^{-x} \cos tx\, dx = 1 - t^2\int_0^{\infty} e^{-x} \cos tx\, dx \quad \text{(On integration by parts)}$$

$$= 1 - t^2\, \varphi_X(t)$$

$$\Rightarrow \quad \varphi_X(t) = \frac{1}{1+t^2} = (1+t^2)^{-1} \qquad \dots (9.27a)$$

$$K_X(t) = \log \Phi_X(t) = -\log(1+t^2) = -\left[t^2 - \frac{t^4}{2} + \frac{t^6}{3} \dots\dots\right]$$

$$= (it)^2 + \frac{(it)^4}{2} + \frac{(it)^6}{3} + \dots$$

Hence $\kappa_1 = \kappa_3 = 0$; $\kappa_2 = 2$; $\kappa_4 = 4 \times 3 = 12$

$\therefore$ Mean $= \kappa_1 = 0$; Variance $= \mu_2 = \kappa_2 = 2$

$\mu_3 = \kappa_3 = 0$; $\mu_4 = \kappa_4 + 3k_2^2 = 12 + 12 = 24$

$\Rightarrow$ $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = 0$; $\beta_2 = \dfrac{\mu_4}{\mu_2^2} = 6$

Remark. Mean deviation about mean for standard Laplace distribution is 1. (Try it).

9·9·2. Two Parameter Laplace Distribution.

Definition. *A continuous r.v. X is said to have a double exponential (Laplace) distribution with two parameters λ and μ if its p.d.f. is given by*

$$f(x, \mu, \lambda) = \frac{1}{2\lambda} \exp(-|x-\mu|/\lambda) \; ; -\infty < x < \infty, \lambda > 0 \qquad \text{(9.27)}$$

We write $X \sim \text{Lap}(\lambda, \mu)$.

Let $\quad Y = \dfrac{x-\mu}{\lambda} \quad \Rightarrow \quad X = \mu + \lambda Y \qquad (*)$

The *p.d.f.* $g(.)$ of Y is given by :

$$g(y) = f(x)\left|\frac{dx}{dy}\right| = \frac{1}{2\lambda} \exp.(-|y|).\lambda \qquad \text{[From (*)]}$$

$$= \frac{1}{2} \exp(-|y|) \; ; -\infty < y < \infty,$$

which is the *p.d.f.* of standard Laplace distribution

Hence, $\quad$ if $X \sim Lap(\lambda, \mu)$, *then* $Y = \dfrac{X-\mu}{\lambda} \sim Lap(1, 0) \qquad \ldots (9.27a)$

i.e., Y has standard Laplace distribution.

9·9·3. Characteristic Function of Two Parameter Laplace Distribution.

If $X \sim \text{Lap}(\lambda, \mu)$, then

$$\Phi_X(t) = E(e^{itX}) = E[e^{it(\mu + \lambda Y)}], \qquad \text{where } Y = \frac{X-\mu}{\lambda} \sim \text{Lap}(1, 0)$$

$$= e^{it\mu}. E(e^{i.\lambda t.Y}) = e^{it\mu}. \phi_Y(\lambda t)$$

$$= \frac{e^{it\mu}}{1 + \lambda^2 t^2} \qquad \ldots (9.27c)$$

$$\left[\because Y \text{ is standard Laplace variate with } \phi(t) = \frac{1}{1+t^2}\right]$$

9·9·4. Moments of Two Parameter Laplace Distribution. If $X \sim \text{Lap}(\lambda, \mu)$, the rth moment about origin is given by :

$$\mu_r' = E(X^r) = \frac{1}{2\lambda} \int_{-\infty}^{\infty} x^r \exp\left(\frac{-|x-\mu|}{\lambda}\right) dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} (z\lambda + \mu)^r \exp(-|z|)\, dz, \qquad \left(z = \frac{x-\mu}{\lambda}\right)$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \left[\sum_{k=0}^{r} \binom{r}{k} (z\lambda)^k \mu^{r-k}\right] \exp(-|z|)\, dz$$

$$= \frac{1}{2} \sum_{k=0}^{r} \left[\binom{r}{k} \lambda^k \mu^{r-k} \int_{-\infty}^{\infty} z^k \exp(-|z|) dz\right]$$

9.12. CAUCHY DISTRIBUTION

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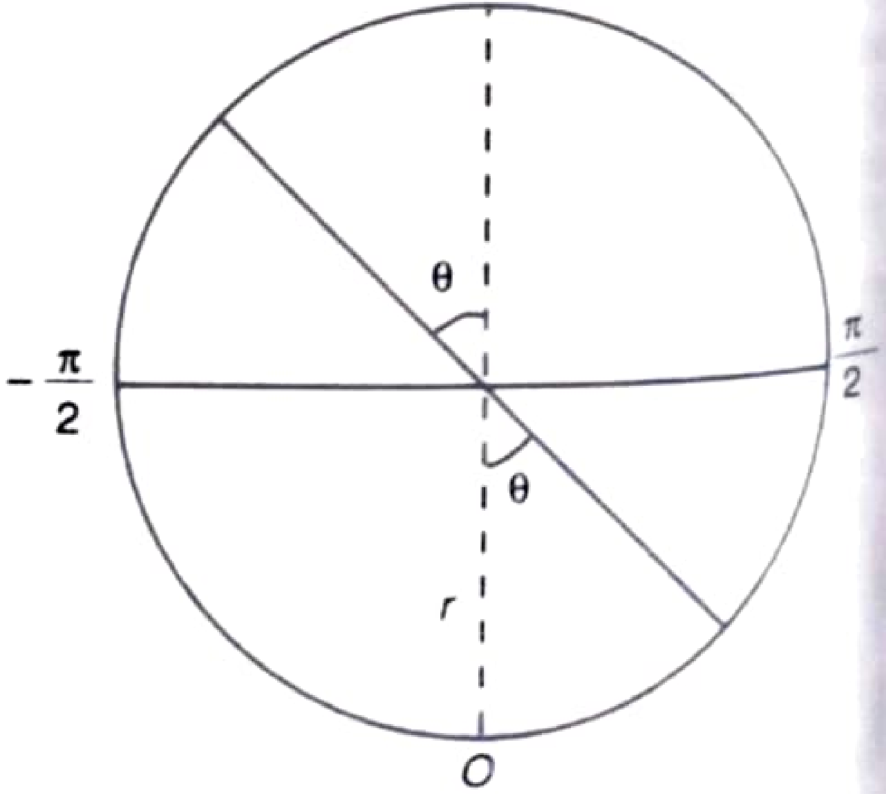
Let us consider a roulette wheel in which the probability of the pointer stopping at any part of the circumference is constant. In other words, the probability that any value of θ lies in the interval $[-\pi/2, \pi/2]$ is constant and consequently θ is a rectangular variate in the range $[-\pi/2, \pi/2]$ with probability differential given given by :



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$$dP(\theta) = \begin{cases} (1/\pi)\, d\theta, & -\pi/2 \le \theta \le \pi/2 \\ 0, & \text{otherwise} \end{cases} \quad \ldots (9{\cdot}29h)$$

Let us now transform to variable X by the substitution : $x = r \tan \theta \Rightarrow dx = r \sec^2 \theta \, d\theta$

Since, $-\pi/2 \le \theta \le \pi/2$, the range for X is from $-\infty$ to ∞. Thus the probability differential of X becomes :

$$dF(x) = \frac{1}{\pi} \cdot \frac{dx}{r \sec^2 \theta} = \frac{1}{\pi} \cdot \frac{dx}{[r\{1 + (x^2/r^2)\}]} = \frac{r}{\pi} \cdot \frac{dx}{r^2 + x^2}; -\infty < x < \infty$$

In particular if we take $r = 1$, we get : $f(x) = \frac{1}{\pi} \cdot \frac{1}{1 + x^2}, -\infty < x < \infty$

This is the p.d.f. of a standard Cauchy variate and we write $X \sim C(1, 0)$.

Definition. *A random variable X is said to have a standard Cauchy distribution if its p.d.f. is given by :*

$$f_X(x) = \frac{1}{\pi(1 + x^2)}, -\infty < x < \infty \quad \ldots (9{\cdot}30)$$

and X is termed as standard Cauchy variate.

More generally, Cauchy distribution with parameters λ and μ has the p.d.f.,

$$g_Y(y) = \frac{\lambda}{\pi[\lambda^2 + (y - \mu)^2]}, -\infty < y < \infty; \lambda > 0 \quad \ldots (9{\cdot}30a)$$

and we write $X \sim C(\lambda, \mu)$

But putting $X = (Y - \mu)/\lambda$ in (9·30*a*), we get (9·30).

Hence if $Y \sim C(\lambda, \mu)$, then $X = (Y - \mu)/\lambda \sim C(1, 0)$ ... (9·30*b*)

9·12·1. Characteristic Function of (Standard) Cauchy Distribution. If X is a standard Cauchy variate then

$$\varphi_X(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{itx}}{1 + x^2} dx \quad \ldots (*)$$

To evaluate (*) consider standard Laplace distribution $f_1(z) = \frac{1}{2} e^{-|z|}, -\infty < z < \infty$.

Then $\varphi_1(t) = \phi_Z(t) = E(e^{itZ}) = \frac{1}{1 + t^2}$.

Since $\varphi_1(t)$ is absolutely integrable in $(-\infty, \infty)$, we have by Inversion theorem

$$\frac{1}{2} e^{-|z|} = f_1(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itz} \varphi_1(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-itz}}{1 + t^2} dt$$

$$\Rightarrow \quad e^{-|z|} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{itz}}{1 + t^2} dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{itz}}{1 + t^2} dt \quad [\text{Changing } t \text{ to } -t]$$

On interchanging t and z, we have $e^{-|t|} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{itz}}{1 + z^2} dz$ ... (**)

From (*) and (**), we get $\varphi_X(t) = e^{-|t|}$ ... (9·31)

Remarks 1. If Y is a Cauchy variate with parameters λ and μ, then

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$$X = \frac{Y-\mu}{\lambda} \sim C(1, 0), \quad \Rightarrow \quad Y = \mu + \lambda X$$

$$\therefore \quad \varphi_Y(t) = E(e^{itY}) = e^{i\mu t} E(e^{it\lambda X}) = e^{i\mu t} \varphi_X(t\lambda)$$

$$= e^{i\mu t - \lambda |t|}, \ \lambda > 0$$

7 [Using (9·31)] ... (9·31*a*)

2. Additive Property of Cauchy Distribution. *If X_1 and X_2 are independent Cauchy variates with parameters (λ_1, μ_1) and (λ_1, μ_2) respectively, then $X_1 + X_2$ is a Cauchy variate with parameters $(\lambda_1 + \lambda_2, \mu_1 + \mu_2)$.*

Proof.

$$\varphi_{X_j}(t) = \exp\{i\mu_j t - \lambda_j |t|\}, (j = 1, 2) \qquad \text{[From (9·31}a\text{)]}$$

$$\therefore \quad \varphi_{X_1 + X_2}(t) = \varphi_{X_1}(t)\, \varphi_{X_2}(t) \qquad \text{(Since } X_1, X_2 \text{ are independent)}$$

$$= \exp\left[it(\mu_1 + \mu_2) - (\lambda_1 + \lambda_2)|t|\right]$$

and the result follows by uniqueness theorem of characteristic functions.

3. Since $\varphi'_X(t)$ in (9·31) [where (′) denotes differentiation w.r. to t] does not exist at $t = 0$, the mean of the Cauchy distribution does not exist.

4. Let $X_1, X_2, \ldots, X_n$ be a sample of n independent observations from a standard Cauchy distribution and define $\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$. Then

$$\varphi_{\overline{X}}(t) = \varphi_{\Sigma X_i}(t/n) = \prod_{j=1}^{n} [\varphi_{X_j}(t/n)] = [\varphi_{X_j}(t/n)]^n \qquad \text{(since } X_i\text{'s are } i.i.d.\text{)}$$

$$= [e^{-|t/n|}]^n = e^{-|t|} = \varphi_X(t)$$

Hence by uniqueness theorem of characteristic functions, we have :

"*The arithmetic mean $\overline{X}$ of sample $X_1, X_2, \ldots, X_n$ of independent observations from a standard Cauchy distribution is also a standard Cauchy variate.* In other words, the arithmetic mean of a random sample of any size yields exactly as much information as a single determination of X."

The implies that the sample mean $\overline{X}_n$ of random sample of size n, as an estimate of population mean does not improve with increasing n, which contradicts in Weak Law of Large Numbers (WLLN).

9·12·2. Moments of Cauchy Distribution

$$E(Y) = \int_{-\infty}^{\infty} y f(y)\, dy = \frac{\lambda}{\pi} \int_{-\infty}^{\infty} \frac{y}{\lambda^2 + (y-\mu)^2} dy = \frac{\lambda}{\pi} \int_{-\infty}^{\infty} \frac{(y-\mu)+\mu}{\lambda^2 + (y-\mu)^2} dy$$

$$= \mu \frac{\lambda}{\pi} \int_{-\infty}^{\infty} \frac{dy}{\lambda^2 + (y-\mu)^2} + \frac{\lambda}{\pi} \int_{-\infty}^{\infty} \frac{(y-\mu)}{\lambda^2 + (y-\mu)^2} dy$$

$$= \mu \cdot 1 + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{z}{\lambda^2 + z^2} dz$$

Although the integral $\int_{-\infty}^{\infty} \frac{z}{\lambda^2 + z^2} dz$, is not completely convergent, *i.e.*, $\lim_{\substack{n \to \infty \\ n' \to \infty}} \int_{-n}^{n'} \frac{z}{\lambda^2 + z^2} dz$, does not exist, its principal value viz., ... $\int^{n}$ ... z ... dz exists

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obviously, the probability curve is symmetrical about the point $x = \mu$. Hence for this distribution, the mean, median mode coincide at the point $x = \mu$.

$$\mu_2 = E(Y - \mu)^2 = \int_{-\infty}^{\infty} (y - \mu)^2 f(y)\, dy = \frac{\lambda}{\pi} \int_{-\infty}^{\infty} \frac{(y - \mu)^2}{\lambda^2 + (y - \mu)^2}\, dy,$$

which does not exist since the integral is not convergent. Thus, in general, for the Cauchy's distribution the moments μ_r, $(r \geq 2)$ do not exist.

Remark. The role of Cauchy distribution in statistical theory often lies in providing counter examples, *e.g.*, it is often quoted as a distribution for which moments do not exist. It also provides an example to show that $\varphi_{X+Y}(t) = \varphi_X(t)\, \varphi_Y(t)$ does *not* imply that X and Y are independent.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a standard Cauchy distribution. Let $\bar{X} = \sum_{i=1}^{n} X_i / n$. Since $E(X_i)$ does not exist ($\because$ mean of a Cauchy distribution does not exist),

Normal distribution: 9

Definition

A r.v x is said to have a normal dist., with parameters μ (mean) and σ^2 (variance), if its density func., is given by the prob., law

$$f(x:\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$$

(or)

$$f(x:\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}. \qquad \begin{array}{l} -\infty < x < \infty \\ -\infty < \mu < \infty \\ \sigma > 0 \end{array}$$

characteristics of normal dist.,:

(or) Properties of Normal dist., :

(1) The normal dist., is a symmetrical dist, & the graph of the normal dist., is bell shaped (ie) The graph of the dist., is symmetrical about $x = \mu$.

(2) The curve has a single peak point (ie) the dist., is unimodal.

(3) The tails of the normal dist., extended indefinitely and never touch the horizontal axis.

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $z = \frac{x-\mu}{\sigma}$; $dz = \frac{dx}{\sigma}$, $dx = \sigma\, dz$

$x - \mu = z\sigma$

$$\therefore V(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (z\sigma)^2 e^{-z^2/2} \sigma\, dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-z^2/2} dz$$

Let $z^2/2 = v$, $z^2 = 2v$, $2z\frac{dz}{2} = dv$, $z\,dz = dv$

$dz = \frac{dv}{z}$

$$\therefore V(x) = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2v\, e^{-v} \frac{dv}{\sqrt{2v}}$$

$= \frac{dv}{\sqrt{2v}}$

$$= \frac{\sigma^2}{\sqrt{2}\sqrt{\pi}} \int_{-\infty}^{\infty} 2v\, e^{-v} \frac{dv}{\sqrt{2}\sqrt{v}}$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} v\, e^{-v} v^{-1/2} dv = \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} v^{1/2} e^{-v} dv$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} v^{3/2-1} e^{-v} dv$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\tfrac{3}{2} \qquad \because \int_0^{\infty} e^{-x} x^{\lambda-1} dx = \Gamma\lambda$$

$\Gamma\lambda = (\lambda-1)\Gamma\lambda-1$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \left(\frac{3}{2}-1\right) \Gamma\tfrac{5}{2}-1$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2} \Gamma\tfrac{1}{2} = \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2}\sqrt{\pi} = \sigma^2$$

$V(x) = \sigma^2$

9 Derive median of the Normal dist.

$$\int_{-\infty}^{Md} f(x)\, dx = \int_{Md}^{\infty} f(x)\, dx = \frac{1}{2}$$

$$\Rightarrow \int_{-\infty}^{Md} f(x)\, dx = \frac{1}{2}$$

2. If $X \sim N(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma}$ is a std. Normal variate with $E(Z) = 0$ and $Var(Z) = 1$ and we write $Z \sim N(0,1)$

3. The p.d.f of std. Normal variate is given by,

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty.$$

Result 1

$$\phi(-z) = 1 - \phi(z)$$

Proof

$$\phi(-z) = P(Z \le -z) = P(Z \ge z) \quad \text{(By symmetry)}$$
$$= 1 - P(Z \le z)$$
$$= 1 - \phi(z)$$

Result 2

$$P(a \le X \le b) = \phi\left(\frac{b-\mu}{\sigma}\right) - \phi\left(\frac{a-\mu}{\sigma}\right) \text{ where } X \sim N(\mu, \sigma^2)$$

Proof:

$$P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right); \quad \left[Z = \frac{X-\mu}{\sigma}\right]$$
$$= P\left(Z \le \frac{b-\mu}{\sigma}\right) - P\left(Z \le \frac{a-\mu}{\sigma}\right)$$
$$= \phi\left(\frac{b-\mu}{\sigma}\right) - \phi\left(\frac{a-\mu}{\sigma}\right)$$

Prove that the total area under the Normal curve is unity (ie) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Proof:

$$\int f(x)\,dx = \int \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

let $z = \frac{x-\mu}{\sigma} \Rightarrow dz = \frac{dx}{\sigma}$, $dx = \sigma\,dz$

$$= \int \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} \sigma\,dz$$

$$= \frac{1}{\ } \int^{\infty} e^{-z^2/2}\,dz$$

Now we have to prove that

$$\int_{-\infty}^{\infty} e^{-z^2/2}\,dz = \sqrt{2\pi}$$

we can write

$$\int_{-\infty}^{\infty} e^{-z^2/2}\,dz = 2\int_{0}^{\infty} e^{-z^2/2}\,dz$$

let $z^2/2 = v$, $z^2 = 2v$, $z = \sqrt{2v}$

$v = z^2/2$, $dv = \frac{2z\,dz}{2} \Rightarrow dv = z\,dz$ & $\therefore \frac{dv}{z} = dz$

$\Rightarrow \frac{dv}{\sqrt{2v}} = dz$

$$\therefore 2\int_{0}^{\infty} e^{-z^2/2}\,dz = 2\int_{0}^{\infty} e^{-v}\frac{dv}{\sqrt{2v}}$$

$$\sqrt{2}\cdot\sqrt{2}\int_{0}^{\infty} e^{-v}\,v^{-1/2}\frac{dv}{\sqrt{2}} = \sqrt{2}\int_{0}^{\infty} e^{-v}\,v^{-1/2}\,dv$$

$$\sqrt{2}\int_{0}^{\infty} e^{-v}\,v^{1/2-1}\,dv = \sqrt{2}\,\Gamma\tfrac{1}{2} = \sqrt{2}\sqrt{\pi} = \sqrt{2\pi}$$

$\therefore$ Required area $= \frac{1}{\sqrt{2\pi}}\cdot\sqrt{2\pi} = 1$ $\left\{ \because \int_{0}^{\infty} e^{-x}x^{\lambda-1}\,dx = \Gamma\lambda \text{ \& } \Gamma\tfrac{1}{2} = \sqrt{\pi} \right\}$

Prove that $\int_{-\infty}^{\infty} e^{-\theta^2/2}\,d\theta = \sqrt{2\pi}$

Proof:

$$\int_{-\infty}^{\infty} e^{-\theta^2/2}\,d\theta = 2\int_{0}^{\infty} e^{-\theta^2/2}\,d\theta$$

Let $\theta^2/2 = v$; $\theta^2 = 2v$; $\theta = \sqrt{2v}$

$\frac{2\theta\,d\theta}{2} = dv$; $\theta\,d\theta = dv$; $d\theta = \frac{dv}{\theta} = \frac{dv}{\sqrt{2v}}$

$$\therefore 2\int_{0}^{\infty} e^{-\theta^2/2}\,d\theta = 2\int_{0}^{\infty} e^{-v}\frac{dv}{\sqrt{2v}}$$

$$= \sqrt{2}\cdot\sqrt{2}\int_{0}^{\infty} e^{-v}\frac{dv}{\sqrt{2}}\,v^{-1/2}$$

$= \sqrt{2}$ ...

$= \sqrt{2} \int_0^\infty e^{-v} v^{1/2 - 1} dv$

$= \sqrt{2}\, \Gamma \tfrac{1}{2} = \sqrt{2}\sqrt{\pi} = \sqrt{2\pi}$ $\quad \because \int_0^\infty e^{-x} x^{n-1} dx = \Gamma n$

Derive mean and variance of the Normal dist.:

Mean :

$\mu_1' = E(x) = \int x f(x)\, dx$

$= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

Let $z = \frac{x-\mu}{\sigma}$; $dz = \frac{dx}{\sigma}$, $dx = \sigma\, dz$

$x = \mu + \sigma z$

$\therefore E(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (z\sigma + \mu) e^{-z^2/2} \sigma\, dz$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z\sigma e^{-z^2/2} dz +$

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu e^{-z^2/2} dz$

$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-z^2/2} dz + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz$

$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-z^2/2} dz + \mu$ $\quad$ odd func,

$f(z) = z e^{-z^2/2}$

$f(-z) = -z e^{-z^2/2}$

$= -f(z)$

$= \frac{\sigma}{\sqrt{2\pi}} \times 0 + \mu$

$= \mu$

$\because \int_{-\infty}^{\infty} z e^{-z^2/2} dz$ is an odd func

$\int$ odd func $= 0$

$\Rightarrow E(x) = \text{Mean} = \mu$

Variance :

$V(x) = E[x - E(x)]^2 = E[x-\mu]^2$

$= \int (x-\mu)^2 f(x)\, dx$

(3) the normal curve approaches the x axis asymptotically on either side of the origin.

(4) Mean, Median and mode of the dist. Coincide.

(5) The normal dist. is a two parameter prob. dist., : The parameters mean & SD (μ, σ) completely determine the dist..

(6) Linear combination of individual normal variate is also a normal variate.

(7) Coeff. of skewness, $\beta_1 = 0$

Kurtosis, $\beta_2 = 3$

(8). Q.D : M.D : SD : : 10 : 12 : 15

(9) Area property:

$P(\mu - \sigma < x < \mu + \sigma) = 0.6826$

$P(\mu - 2\sigma < x < \mu + 2\sigma) = 0.9544$

$P(\mu - 3\sigma < x < \mu + 3\sigma) = 0.9973$

(10) All odd moments vanish and even moments are given by

$\mu_{2n} = 1 \cdot 3 \cdot 5 \ldots (2n-1)\sigma^{2n}$

(11) Since f(x) being the prob, can never be negative, no portion of the curve lies below the x-axis.

Remarks:

1. A random variable X with mean μ and variance σ^2 & foll., the normal law is expressed by $X \sim N(\mu, \sigma^2)$

$$\Rightarrow \int_{-\infty}^{Md} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

let $z = \frac{x-\mu}{\sigma} \Rightarrow dz = \frac{dx}{\sigma}$, $dx = \sigma\, dz$

Range $x = -\infty \Rightarrow z = -\infty$

$x = Md \Rightarrow z = \frac{Md-\mu}{\sigma}$

$$\int_{-\infty}^{\frac{Md-\mu}{\sigma}} \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2} \sigma\, dz = \frac{1}{2}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{Md-\mu}{\sigma} = t_1} e^{-z^2/2} dz = \frac{1}{2}$$

and also $\int_{\frac{Md-\mu}{\sigma} = t_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \frac{1}{2}$

From the table of Normal prob., we get,

$$\int_0^{\infty} f(x)\, dx = \int_{-\infty}^{0} f(x)\, dx = \frac{1}{2}$$

$\Rightarrow t_1 = 0$ $\frac{Md-\mu}{\sigma} = 0$; $Md - \mu = 0$, $Md = \mu$

$\therefore$ Median $= \mu$

Desire mode of the Normal dist.:

It is the value of x which maximise the prob. func. . The mode is obtained by the relation.

$$\frac{d}{dx} f(x) = 0 \quad \& \quad \frac{d^2}{dx^2} f(x) < 0$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left[\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right]$$

$$-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \quad 2\left(\frac{x-\mu}{\sigma}\right).$$

$\Rightarrow \frac{x-\mu}{\sigma} = 0$ (ie) $x - \mu = 0 \Rightarrow x = \mu$

$\therefore$ Mode $= \mu$

Desire

Moment Generating function of Normal distribution:

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $z = \frac{x-\mu}{\sigma}$, $dz = \frac{dx}{\sigma}$, $dx = \sigma\, dz$,

$z\sigma = x - \mu \;:\; \mu + z\sigma = x$

Range: $x = -\infty$ to ∞ ; $z = -\infty$ to ∞

$$M_x(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu + z\sigma)} e^{-z^2/2} \sigma\, dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz\sigma} e^{-z^2/2}\, dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z^2 - 2tz\sigma)}\, dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z^2 - 2tz\sigma + t^2\sigma^2 - t^2\sigma^2)}\, dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z - t\sigma)^2 + \frac{t^2\sigma^2}{2}}\, dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} e^{t^2\sigma^2/2} \int_{-\infty}^{\infty} e^{-1/2(z - t\sigma)^2}\, dz$$

$$= \frac{e^{t\mu + \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z - t\sigma)^2}\, dz$$

Put $z - t\sigma = \theta$ $\quad$ Range
$dz = d\theta$ $\quad$ $\theta = -\infty$ to ∞ as $z = -\infty$ to ∞

$$M_x(t) = \frac{e^{t\mu + t^2\sigma^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2\,\theta^2}\, d\theta$$ 17

$$= e^{t\mu + t^2\sigma^2/2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\theta^2/2}\, d\theta\right]$$

$$= e^{t\mu + \frac{t^2\sigma^2}{2}} \qquad \because \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\theta^2/2}\, d\theta$$

$$\therefore M_x(t) = e^{t\mu + \frac{t^2\sigma^2}{2}} \qquad = \frac{1}{\sqrt{2\pi}}\sqrt{2\pi} = 1$$

Derive characteristics function of Normal dist.

$$\phi_x(t) = E(e^{itx}) = \int_{-\infty}^{\infty} e^{itx}\frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $z = \frac{x-\mu}{\sigma}$, $dz = \frac{dx}{\sigma}$, $dx = \sigma\, dz$.

$z\sigma = x-\mu$; $x = \mu + z\sigma$

$$\phi_x(t) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{it(\mu+z\sigma)}\, e^{-z^2/2}\,\sigma\, dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{it\mu}\int_{-\infty}^{\infty} e^{-1/2(z^2 - 2itz\sigma)}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{it\mu}\int_{-\infty}^{\infty} e^{-1/2\left[z^2 - 2itz\sigma + (it\sigma)^2 - (it\sigma)^2\right]}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{it\mu}\, e^{(it\sigma)^2/2}\int_{-\infty}^{\infty} e^{-1/2(z - it\sigma)^2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{it\mu + \frac{(it\sigma)^2}{2}}\int_{-\infty}^{\infty} e^{-1/2(z-it\sigma)^2}\, dz$$

Put $z - it\sigma = \theta$, $dz = d\theta$.

Range $\theta = -\infty$ to ∞ as $z = -\infty$ to ∞

$$\phi_x(t) = \frac{1}{\sqrt{2\pi}} e^{it\mu + \frac{(it\sigma)^2}{2}}\int_{-\infty}^{\infty} e^{-\theta^2/2}\, d\theta$$

$$= e^{it\mu + \frac{(it\sigma)^2}{2}}\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\theta^2/2}\, d\theta$$

$$= e^{itu + \frac{(it\sigma)^2}{2}}$$ 18

$$\phi_x(t) = e^{it\mu - \frac{t^2\sigma^2}{2}}$$

$\because i^2 = -1$, If $\mu = 0$ & $\sigma^2 = 1$

$$\phi_x(t) = e^{it.0 - t^2 1/2}$$

$$\phi_x(t) = e^{-t^2/2}$$; which is the char, func, of std., Normal variate

Prove Additive Property of Normal dist.,:

Statement:

The sum of independent normal variate is also a normal variate.

Proof:

Let us consider two variables x_1 and x_2.

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x_1 and x_2 be the two indep. normal variates with means μ_1 and μ_2 and variance σ_1^2 & σ_2^2. then x_1+x_2 is also a normal variate with mean $\mu_1+\mu_2$ and variance $\sigma_1^2+\sigma_2^2$.

we know that $M_X(t) = e^{t\mu + t^2\sigma^2/2}$

$$M_{x_1}(t) = e^{t\mu_1 + t^2\sigma_1^2/2} \quad \& \quad M_{x_2}(t) = e^{t\mu_2 + t^2\sigma_2^2/2}$$

Also we know that

$$M_{x_1+x_2}(t) = M_{x_1}(t) \cdot M_{x_2}(t)$$

$$= e^{t\mu_1 + t^2\sigma_1^2/2} \cdot e^{t\mu_2 + t^2\sigma_2^2/2}$$

$$= e^{t(\mu_1+\mu_2) + 1/2\, t^2(\sigma_1^2+\sigma_2^2)}$$

Thus x_1+x_2 is also a normal variate with mean $\mu_1+\mu_2$ and variance $\sigma_1^2+\sigma_2^2$

Recurrence relation for moments:

Prove that all the odd order moments vanish.

Proof:

Consider the rth order moments about mean

$$\mu_r = E(x-\mu)^r = \int_{-\infty}^{\infty} (x-\mu)^r \cdot f(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^r \cdot \frac{1}{\sigma\sqrt{2\pi}}\, e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

let $z = \frac{x-\mu}{\sigma}$, $dz = \frac{dx}{\sigma}$, $dx = \sigma\, dz$.

$z\sigma = x-\mu$.

$$\mu_r = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (z\sigma)^r \cdot e^{-1/2\, z^2}\, \sigma\, dz$$

$$= \frac{\sigma^r}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^r\, e^{-z^2/2}\, dz.$$

Give the importance of Normal dist.

Normal dist., plays a very important role in Statistical theory because of the following reasons:

(i) Most of the dist.,s occurring in practise eg., Binomial, Poisson, Hypergeometric dist., etc., can be approximated by normal dist., . Moreover, many of the sampling dist.,s ~~etc.~~, can ~~be approximated~~ by Normal dist., eg., Student's 't', Snedecor's F, χ^2 dist., etc., tend to normality for large samples.

(ii) Even if a variable is not normally distributed, it can sometimes be brought to normal form by simple transformation of variable. For ex., if the dist. of x is skewed the dist., of $\sqrt{x}$ might come out to be normal.

(iii) If $x \sim N(\mu, \sigma^2)$, then

$$P(\mu - 3\sigma < x < \mu + 3\sigma) = 0.9973$$

$$\Rightarrow P(-3 < z < 3) = 0.9973$$

$$\Rightarrow P(|z| < 3) = 0.9973$$

$$\Rightarrow P(|z| > 3) = 0.0027$$

This property of the normal dist., forms the basis of entire Large Sample theory

(iv) Many of the dist.,s of sample statistic (e.g. the dist. of sample mean, sample variance etc) tend to normality for large samples and as such they can best be studied with the help of the normal curves.

(v) The entire theory of small sample tests viz., t, F, χ^2 tests etc., is based on the fundamental assumption that the parent popu, from which the samples have been drawn follow normal dist..

(vi) Theory of normal curves can be applied to the graduation of the curves which are not normal

(vii) Normal dist., finds large applications in Statistical Quality Control in industry for setting control limits.

$$\Rightarrow \qquad \mu_{2r+2} = \sigma^2 \mu_{2r} + \sigma^3 \frac{d\mu_{2r}}{d\sigma}.$$

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9·2·15. Log-normal Distribution. The positive *r.v.* X is said to have a log-normal distribution if $\log_e X$ is normally distributed.

Let $Y = \log_e X \sim N(\mu, \sigma^2)$. For $x > 0$,

$$F_X(x) = P(X \le x) = P(\log_e X \le \log_e x) = P(Y \le \log_e x)$$

(Since log X is monotonic increasing function.)

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\log x} \exp\{-(y-\mu)^2/2\sigma^2\}\, dy \qquad [\text{Since } Y \sim N(\mu, \sigma^2)]$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_0^x \exp\{-(\log u - \mu)^2/2\sigma^2\} \frac{du}{u}, \qquad (y = \log u)$$

For $x \le 0$, $F_X(x) = P(X \le x) = 0$, because X is a positive *r.v.*

Let us define

$$f_X(u) = \begin{cases} \dfrac{1}{u\sigma\sqrt{2\pi}} \cdot \exp\{-(\log u - \mu)^2/2\sigma^2\}, & u > 0 \\ 0, & u \le 0 \end{cases} \quad \ldots (9.17)$$

Then $F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$, for every x and hence $f(x)$ defined in (9·17) is a *p.d.f.* of X.

Remark. If $X \sim N(\mu, \sigma^2)$, then $Y = e^X$, is called a log-normal random variable, since its logarithm $\log Y = X$, is a normal *r.v.*

Moments. The *r*th moment about origin is given by :

$$\begin{aligned} \mu_r' &= E(X^r) = E(e^{rY}) && [\because Y = \log X \Rightarrow X = e^Y] \\ &= M_Y(r) && (\textit{m.g.f. of } Y,\ r \textit{ being the parameter}) \\ &= \exp(\mu r + \tfrac{1}{2} r^2 \sigma^2) && [\because Y \sim N(\mu, \sigma^2)] \quad \ldots (9.18) \end{aligned}$$

Remarks 1. In particular if we take $\mu = \log\alpha$, $\alpha > 0$, *i.e.*, $\log X \sim N(\log\alpha, \sigma^2)$, then

$$\mu_r' = E(X^r) = \exp\{r.\log\alpha + \tfrac{1}{2} r^2\sigma^2\} = \alpha^r.\exp\{r^2\sigma^2/2\} \quad \ldots (9.18a)$$

$\therefore$ Mean $= \mu_1' = \alpha e^{\sigma^2/2}$ and $\mu_2 = \mu_2' - \mu_1'^2 = \alpha^2 e^{\sigma^2}(e^{\sigma^2} - 1)$

2. Log normal distribution arises in problems of economics, biology, geology, and reliability theory. In particular, it arises in the study of dimensions of particles under pulverisation.

3. If $X_1, X_2, \ldots, X_n$ is a set of independently identically distributed random variables such that mean of each $\log X_i$ is μ and its variance is σ^2, then the product $X_1 . X_2 \ldots X_n$ is asymptotically distributed according to logarithmic normal distribution and with mean μ and variance $n\sigma^2$.

9·11. LOGISTIC DISTRIBUTION

Definition. *A continuous r.v. X is said to have a Logistic distribution with parameters α and β, if its distribution function is of the form :*

$$F_X(x) = [1 + \exp\{-(x - \alpha)/\beta\}]^{-1}, \ \beta > 0 \quad \ldots (9{\cdot}29)$$

$$= \frac{1}{2}\left[1 + \tanh\left\{\frac{1}{2}(x - \alpha)/\beta\right\}\right]; \ \beta > 0 \quad \ldots (9{\cdot}29a)$$

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The *p.d.f.* of Logistic distribution with parameters α and β (> 0) is given by :

$$f(x) = \frac{d}{dx}[F(x)]$$

$$= \frac{1}{\beta}\left[1 + \exp\{-(x-\alpha)/\beta\}\right]^{-2} \left[\exp\{-(x-\alpha)/\beta\}\right] \qquad \ldots (9.29b)$$

$$= \frac{1}{4\beta} \operatorname{sech}^2\left\{\frac{1}{2}(x-\alpha)/\beta\right\} \qquad \ldots (9.29c)$$

The *p.d.f.* of standard Logistic variate $Y = (X - \alpha)/\beta$, is given by :

$$g_Y(y) = f(x) \cdot \left|\frac{dx}{dy}\right| = e^{-y}(1 + e^{-y})^{-2} ; -\infty < y < \infty \qquad \ldots (9.29d)$$

$$= \frac{1}{4} \operatorname{sech}^2\left(\frac{1}{2} y\right) ; -\infty < y < \infty \qquad \ldots (9.29e)$$

The distribution function of Y is : $\quad G_Y(y) = (1 + e^{-y})^{-1} ; -\infty < y < \infty \qquad \ldots (9.29f)$

Logistic distribution is extensively used as growth function in population and demographic studies and in time series analysis. Theoretically, Logistic distribution can be obtained as :

(*i*) The limiting distribution (as $n \to \infty$) of the standardised mid-range, (average of the smallest and the largest sample observations), in random samples of size n.

(*ii*) A mixture of extreme value distributions.

9·11·1. Moment Generating Function of Logistic Distribution. The *m.g.f.* of standard Logistic variate Y is given by :

$$M_Y(t) = E(e^{tY}) = \int_{-\infty}^{\infty} e^{ty} \cdot g(y)\, dy = \int_{-\infty}^{\infty} e^{ty} \cdot e^{-y}(1 + e^{-y})^{-2}\, dy$$

$$= \int_{-\infty}^{\infty} e^{ty} e^{-y} \left(\frac{1 + e^y}{e^y}\right)^{-2} dy = \int_{-\infty}^{\infty} e^{ty} \cdot e^{y}(1 + e^y)^{-2}\, dy$$

Put $\quad z = (1 + e^y)^{-1} \quad \Rightarrow \quad e^y = \frac{1}{z} - 1 = \frac{1-z}{z}$

$$\therefore \quad M_Y(t) = \int_1^0 \left(\frac{1-z}{z}\right)^t \cdot (-dz) = \int_1^0 z^{-t}(1-z)^t\, dz = \beta(1-t, 1+t),\ 1 - t > 0$$

$$= \Gamma(1-t)\,\Gamma(1+t)/\Gamma(2) = \Gamma(1-t)\,\Gamma(1+t)$$

$$= \pi t \operatorname{cosec} \pi t ;\ t < 1$$

$$= 1 + \frac{\pi^2 t^2}{6} + \frac{7}{360}\pi^4 t^4 + \ldots \qquad \text{(See Remark 2 below)} \qquad \ldots (9.29g)$$

$\therefore \quad E(Y) = \text{Coefficient of } t \text{ in } (9.29g) = 0 \quad \Rightarrow \quad \text{Mean} = 0$

$\therefore \quad \mu_2 = E(Y^2) = \text{Coefficient of } \frac{t^2}{2!} \text{ in } (9.29g) = \frac{\pi^2}{3}; \qquad \mu_3 = E(Y^3) = 0$

$\mu_4 = E(Y^4) = \text{Coefficient of } \frac{t^4}{4!} \text{ in } (9.29g) = \frac{7}{15}\pi^4$

Hence for standard Logistic distribution :

Mean $= 0$, Variance $= \mu_2 = \frac{\pi^2}{3}$; $\quad \beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0$, $\quad \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{7 \times 9}{15} = 4.2$

9·10. WEIBUL DISTRIBUTION 27

A random variable X has a Weibul distribution with three parameters c (> 0), α (> 0) and μ if the $r.v.$ $Y = \left(\dfrac{X-\mu}{\sigma}\right)^c$ ... (9·28)

has the exponential distribution with $p.d.f.$

$$p_Y(y) = e^{-y}, \; y > 0. \qquad \ldots (9{\cdot}28a)$$

Definition. *A continuous r.v. X has a Weibul distribution with parameters c (> 0), a (> 0), and μ if its p.d.f. is :*

$$f(x\,;c,\alpha,\mu) = \frac{c}{\alpha}\left(\frac{x-\mu}{\alpha}\right)^{c-1} exp\left\{-\left(\frac{x-\mu}{\alpha}\right)^{c}\right\};\; x>\mu,\, c>0 \qquad \ldots (9{\cdot}28b)$$

The standard Weibul distribution is obtained on taking $\alpha = 1$ and $\mu = 0$, so that the $p.d.f.$ of *standard Weibul distribution* which depends only on a single parameter c is given by :

$$p_X(x) = c\,x^{c-1}.\exp(-x^c)\,;\; x>0,\; c>0 \qquad \ldots (9{\cdot}28c)$$

9·10·1. Moments of Standard Weibul Distribution. For standard Weibul distribution, ($\alpha = 1$, $\mu = 0$), from (9·28), we get $Y = X^c$ which has the exponential distribution (9·28a). We have

$$\mu'_r = E(X^r) = E(Y^{1/c})^r = E(Y^{r/c}) = \int_0^\infty e^{-y}\,y^{r/c}\,dy \quad \Rightarrow \quad \mu'_r = \Gamma\left(\frac{r}{c}+1\right)$$

$$\therefore \quad \text{Mean} = E(X) = \Gamma\left(\frac{1}{c}+1\right)$$

$$\text{and} \quad \text{Var}(X) = E(X^2) - [E(X)]^2 = \Gamma\left(\frac{2}{c}+1\right) - \left[\Gamma\left(\frac{1}{c}+1\right)\right]^2$$

Similarly, we can obtain expressions for higher order moments and hence for β_1 and β_2. For large c, the mean is approximated by :

$$E(X) \simeq 1 - \frac{\gamma}{c} + \frac{1}{2c^2}\left(\frac{\pi^2}{6} + \gamma^2\right) = 1 - 0.57722\, c^{-1} + 0.98905\, c^{-2},$$

where $\gamma = 0.57722$ is Euler's constant.

The distribution is named after Waloddi Weibul, a Swedish physicist, who used it in 1939 to represent the distribution of the breaking strength of materials. Kao, J.H.K. (1958-59) advocated the use of this distribution in reliability studies and quality control work. It is also used as a tolerance distribution in the analysis of quantum response data.

9·10·2. Characterisation of Weibul Distribution. Dubey, S.D. (1968) has obtained the following result :

"Let X_i $(i = 1, 2, \ldots, n)$ be i.i.d. random variables. Then min $(X_1, X_2 \ldots, X_n)$ has a Weibul distribution if and only if the common distribution of X_i's is a Weibul distribution."

Proof. Let X_i, $(i = 1, 2, \ldots, n)$ be *i.i.d. r.v's* . each with Weibul distribution (9·28*b*) and let $Y = \min(X_1, X_2, \ldots, X_n)$. Then

$$P(Y > y) = P\{\min(X_1, X_2, \ldots, X_n) > y\} = P\left(\bigcap_{i=1}^{n} X_i > y\right)$$

$$= \prod_{i=1}^{n} P(X_i > y) = \left[P(X_i > y)\right]^n, \quad \text{(since } X_i\text{'s are } i.i.d.\ r.v.\text{'s)} \qquad (*)$$

Now
$$P(X_i > y) = \int_y^\infty c\,\alpha^{-1}\left(\frac{x-\mu}{\alpha}\right)^{c-1} \exp\left\{-\left(\frac{x-\mu}{\alpha}\right)^c\right\} dx$$

$$= \int_{[(y-\mu/\alpha)]^c}^\infty e^{-t}\, dt, \qquad t = \left(\frac{x-\mu}{\alpha}\right)^c$$

$$= \exp\left\{-\left(\frac{y-\mu}{\alpha}\right)^c\right\}$$

Substituting in (*), we get

$$P(Y > y) = \left[\exp\left\{-\left(\frac{y-\mu}{\alpha}\right)^c\right\}\right]^n = \exp\left\{-n\left(\frac{y-\mu}{\alpha}\right)^c\right\} = \exp\left[-\left\{\frac{n^{1/c}(y-\mu)}{\alpha}\right\}^c\right]$$

This implies that Y has the same Weibul distribution as X_i's with the difference that the parameter α is replaced by $\alpha\, n^{-1/c}$.