1. Binomial Distribution.

Def: A random variable X is said to follow binomial distribution if it assumes only non-negative values and its probability mass function is given by:

$$P(X=x) = p(x) = \binom{n}{x} p^x q^{n-x}, \quad x = 0,1,2,\ldots,n; \quad q = 1-p.$$

Physical conditions for Binomial distribution:

i) Each trial results in two exhaustive and mutually disjoint outcomes, termed as success and failure.

ii) The number of trials 'n' is finite.

iii) The trials are independent of each other.

iv) The probability of success 'p' is constant for each trial.

v) Binomial distribution is defined under the above experimental conditions.

Moments.

$$\mu_1' = E(X) = \sum x \binom{n}{x} p^x q^{n-x} = \sum x \cdot \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= np \sum \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} q^{n-x}$$

$$= np \sum \binom{n-1}{x-1} p^{x-1} q^{n-x}$$

$$= np (q+p)^{n-1} = np. \quad \text{—— (1)}$$

$$\mu_2' = E(X^2) = \sum x^2 \binom{n}{x} p^x q^{n-x}$$

$$= \sum \{x(x-1)+x\} \binom{n}{x} p^x q^{n-x}$$

$$= \sum x(x-1) \frac{n!}{x!(n-x)!} p^x q^{n-x} + np$$

$$= n(n-1)p^2 \sum \frac{(n-2)!}{(x-2)!(n-x)!} p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 (q+p)^{n-2} + np \quad \text{—— (2)}$$

$$= n(n-1)p^2 + np$$

$$\mu_2 = \text{variance} = \mu_2' - \mu_1'^2$$

$$= n(n-1)p^2 + np - n^2p^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np - np^2 = np(1-p) = npq.$$

$$\mu_3' = \sum x^3 \binom{n}{x} p^x q^{n-x} = \sum \{x(x-1)(x-2) + 3x(x-1) + x\} \binom{n}{x} p^x q^{n-x}$$

$$= \sum x(x-1)(x-2) \binom{n}{x} p^x q^{n-x} + 3\sum x(x-1)p(x) + \sum x \cdot p(x)$$

$$= \sum x(x-1)(x-2) \frac{n!}{x!(n-x)!} p^x q^{n-x} + 3n(n-1)p^2 + np$$

$$= n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np. \quad \text{—— (3)}$$

$\mu_4' = E(x^4) = \Sigma x^4 \binom{n}{x} p^x q^{n-x}$

$= \Sigma \{x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x\} p(x)$

on simplification

$= n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np.$

Central moments are

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

Substituting and simplifying, we get

$= npq(q-p)$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$= npq(1 + 3(n-2)pq)$ (on simplification)

$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{n^2p^2q^2(q-p)^2}{n^3p^3q^3} = \frac{(1-2p)^2}{npq}.$

$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{npq[1+3(n-2)pq]}{n^2p^2q^2} = 3 + 6pq$

$\gamma_1 = \sqrt{\beta_1} = \frac{1-2p}{\sqrt{npq}}$. $\gamma_2 = \beta_2 - 3 = \frac{1-6pq}{npq}$

Recurrence relation for the moments.

By def $\mu_r = E[X - E(X)]^r = \Sigma (x-np)^r \binom{n}{x} p^x q^{n-x}$

Differentiating w.r.t. p, we get

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n} \binom{n}{x}\left[-nr(x-np)^{r-1}p^x q^{n-x} + (x-np)^r\{xp^{x-1}q^{n-x} - (n-x)p^x q^{n-x-1}\}\right]$$

$$= -nr\sum_{x=0}^{n} \binom{n}{x}(x-np)^{r-1}p^x q^{n-x} + \sum_{x=0}^{n}\binom{n}{x}(x-np)^r p^x q^{n-x}\left(\frac{x}{p} - \frac{n-x}{q}\right)$$

$$= -nr\Sigma(x-np)^{r-1}p(x) + \sum_{x=0}^{n}(x-np)^r p(x) \cdot \frac{x-np}{pq}$$

$$= -nr\Sigma(x-np)^{r-1}p(x) + \frac{1}{pq}\Sigma(x-np)^{r+1}p(x)$$

$$= -nr\mu_{r-1} + \frac{1}{pq}\mu_{r+1}$$

$$\Rightarrow \mu_{r+1} = pq\left[nr\mu_{r-1} + \frac{d\mu_r}{dp}\right] \quad \text{— (A)}$$

In (A), if we substitute $r = 1, 2$ and 3, we get μ_2, μ_3 and μ_4.

Mean Deviation about Mean

$$M.D = \sum_{x=0}^{n} |x-np|\, p(x) = \sum_{x=0}^{n} |x-np| \binom{n}{x} p^x q^{n-x}.$$

$$= \sum_{x=0}^{np} -(x-np)\binom{n}{x} p^x q^{n-x} + \sum_{x=np}^{n} (x-np)\binom{n}{x} p^x q^{n-x}$$

$$= 2\sum_{x=np}^{n} (x-np)\binom{n}{x} p^x q^{n-x}$$

$$\left\{\text{since } \sum_{x=0}^{n} x\binom{n}{x} p^x q^{n-x} = np \text{ and } \sum (x-np)\binom{n}{x} p^x q^{n-x} = 0\right\}$$

$$= 2\sum_{\mu}^{n}\left\{[xq-(n-x)p]\binom{n}{x} p^x q^{n-x}\right\}$$

$$= 2\sum_{\mu}^{n}\left[\frac{n!}{(x-1)!\,(n-x)!} p^x q^{n-x+1} - \frac{n!}{x!\,(n-x-1)!} p^{x+1} q^{n-x}\right]$$

$$= 2\sum t_{x-1} - t_x \quad \text{where } t_x = \frac{n!}{x!\,(n-x-1)!} p^{x+1} q^{n-x}.$$

$$= 2\left(t_{\mu-1} - t_n\right) = 2t_{\mu-1}$$

This is obtained by summing over x and using $t_n = 0$

$$\therefore M.D = 2t_{\mu-1} = 2\cdot\frac{n!}{(\mu-1)!\,(n-\mu)!} p^{\mu} q^{n-\mu+1}$$

$$= 2npq\binom{n-1}{\mu-1} p^{\mu-1} q^{n-\mu}.$$

Mode: Binomial distribution has two modes.

Case i) when $(n+1)p$ is not an integer.
Mode is integral part of $(n+1)p$

ii) when $(n+1)p$ is integer.
Let $m = (n+1)p$. There are two modes m and $m-1$.

Moment Generating function. Let $X \sim B(n,p)$

$$M_X(t) = E(e^{tx}) = \sum e^{tx}\binom{n}{x} p^x q^{n-x} = (q+pe^t)^n.$$

Note: MGF about mean of binomial distribution.

$$E[e^{t(x-np)}] = e^{-tnp}\cdot E[e^{tx}] = e^{-tnp}(q+pe^t)^n$$

$$= \left(qe^{-pt} + pe^{tq}\right)^n$$

$$= \left[1 + \binom{n}{1}\left\{\frac{t^2}{2!}\cdot pq + \frac{t^3}{3!}pq(q-p) + \frac{t^4}{4!}pq(1-3pq)+\cdots\right\} + \binom{n}{2}\left\{\frac{t^2}{2!}pq + \frac{t^3}{3!}pq(q-p)+\cdots\right\}^2 + \cdots\right\}$$

now $\mu_2 =$ coeff of $\frac{t^2}{2!} = npq$

$\mu_3 =$ " " of $\frac{t^3}{3!} = npq(q-p)$

$\mu_4 =$ " " $\frac{t^4}{4!} = 3n^2p^2q^2 + npq(1-6pq)$.

Additive Property.

Let $X_1 \sim B(n_1, p_1)$ and $X_2 \sim B(n_2, p_2)$ and X_1 and X_2 are independent. $X_1 + X_2$ is not a binomial variate.

Let $X_1 \sim B(n_1, p)$ and $X_2 \sim B(n_2, p)$ and X_1 and X_2 are independent. Then $X_1 + X_2$ is a binomial variate.

Proof : Assignment.

Characteristic Function.

$$\phi_X(t) = E(e^{itx}) = \Sigma e^{itx} \binom{n}{x} p^x q^{n-x}$$
$$= (q + pe^{it})^n$$

Cumulants.

$$K_X(t) = \log M_X(t)$$
$$= \log (q + pe^t)^n$$
$$= n \log (q + pe^t)$$
$$= n\left[p\left(1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots\right) - \frac{p^2}{2}\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots\right)\right.$$
$$\left. + \frac{p^3}{3}\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots\right) - \frac{p^4}{4}\left(t + \frac{t^2}{2!} + \dots\right)\right]$$

Mean $= K_1 =$ Coeff of t in $K_X(t) = np$

$\mu_2 = K_2 =$ " $\frac{t^2}{2!}$ $= npq$

$\mu_3 = K_3 =$ " $\frac{t^3}{3!}$ $= npq(q-p)$

$K_4 =$ $\frac{t^4}{4!}$ $= npq[1 - 6p(1-p)] = npq(1-6pq)$

$\mu_4 = K_4 + 3K_2^2 = npq[1 + 3pq(n-2)]$

Recurrence Relation for Cumulants of B. D

$$K_{r+1} = pq \frac{dK_r}{dp}$$

Proof : Assignment.

Recurrence Relation for the Probabilities of Binomial Distribution.

$$\frac{p(x+1)}{p(x)} = \frac{\binom{n}{x+1} p^{x+1} q^{n-x+1}}{\binom{n}{x} p^x q^{n-x}} = \frac{n-x}{x+1} \cdot \frac{p}{q}$$

$$\Rightarrow \quad p(x+1) = \left\{\frac{n-x}{x+1} \cdot \frac{p}{q}\right\} p(x).$$

2. Poisson Distribution:

A random variable X is said to follow Poisson distribution if it assumes only non-negative values and its probability mass function is given by

$$P(X=x) = p(x,\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad \lambda>0,\ x=0,1,2,\dots$$

Note: 1. $\sum_{x=0}^{\infty} P(X=x) = e^{-\lambda}\sum \frac{\lambda^{x}}{x!} = e^{-\lambda}.e^{\lambda} = 1$

2. $F(x) = P[X \le x] = \sum_{r=0}^{x} p(r) = \sum \frac{e^{-\lambda}\lambda^{r}}{r!} = e^{-\lambda}\sum\frac{\lambda^{r}}{r!}$

Instances where Poisson Distribution is applied.

1. Number of deaths from a disease.
2. Number of suicides reported in a particular city.
3. The number of defective material in a packing manufactured by a good concern.
4. The number of faulty blades in a packet of 100.
5. The number of air accidents in some unit of time.
6. Number of printing mistakes at each page of the book.
7. The number of telephone calls received at a particular exchange in some unit of time.
8. The number of cars passing a crossing per minute during the busy hours of a day.
9. The emission of radioactive particles.

Poisson Process

Let X_i be the number of telephone calls received in time interval 't' on a telephone switch board. Consider the following experimental conditions:

i) The probability of getting a call in small time interval $(t, t+dt)$ is λdt, where λ is a positive constant and dt denotes a small in time 't'.

ii) The probability of getting more than one call on this time interval is very small. That is of the order of $(dt)^2$, such that $\lim_{dt\to 0}\frac{O(dt)^2}{dt}=0$

iii) The probability of any particular call in the time interval $(t, t+dt)$ is independent of the actual time t and also of all previous calls.

Under these conditions, it can be shown that the probability of getting x calls in time 't', say $P_x(t)$ is given by

$$P_x(t) = e^{-\lambda t}\frac{(\lambda t)^{x}}{x!}, \quad x=0,1,2,\dots\infty$$

Proof: Assignment/ Seminar.

Poisson Distribution as Limiting form of Binomial Distribution.

Poisson distribution is a limiting case of the binomial distribution under the following conditions:

i) n, the number of trials is indefinitely large. i.e $n \to \infty$

ii) p, the constant probability of success for each trial is indefinitely small, i.e $p \to 0$.

iii) $np = \lambda$ (say) is finite

Under the above conditions

$$\lim_{n\to\infty} b(x; n, p) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2 \dots \infty$$

Proof: Seminar / Assignment.

Moments:

$$\mu_1' = E(x) = \Sigma x\, p(x,\lambda) = \Sigma x \cdot \frac{e^{-\lambda}\lambda^x}{x!} = \lambda e^{-\lambda} \sum_{x=1} \frac{\lambda^{x-1}}{(x-1)!} = \lambda . e^{-\lambda} . e^{\lambda} = \lambda \quad (1)$$

$\mu_1' =$ Mean $= \lambda$

$$\mu_2' = E(x^2) = E[x(x-1) + x] = \Sigma x(x-1) p(x) + \Sigma x p(x)$$

$$= \lambda^2 e^{-\lambda} \sum_{x=2} \frac{\lambda^{x-2}}{(x-2)!} + \lambda$$

$$= \lambda^2 . e^{-\lambda} . e^{\lambda} + \lambda = \lambda^2 + \lambda \quad (2)$$

$$\mu_3' = E(x^3) = \Sigma x^3 p(x) = \Sigma [x(x-1)(x-2) + 3x(x-1) + x] p(x)$$

$$= \lambda^3 . e^{-\lambda} \sum_{x=3} \frac{\lambda^{x-3}}{(x-3)!} + 3\lambda^2 + \lambda \quad \text{from (1) \& (2)}$$

$$= \lambda^3 . e^{-\lambda} . e^{\lambda} + 3\lambda^2 + \lambda = \lambda^3 + 3\lambda^2 + \lambda \quad (3)$$

$$\mu_4' = E(x^4) = \Sigma x^4 p(x) = \Sigma \{x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x\} p(x)$$

$$= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$$

$\mu_2 =$ Variance $= \mu_2' - \mu_1'^2 = \lambda$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = (\lambda^3 + 3\lambda^2 + \lambda) - 3\lambda(\lambda^2 + \lambda) + 2\lambda^3 = \lambda$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$= 3\lambda^2 + \lambda$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{1}{\lambda} \quad \text{and} \quad \gamma_1 = \sqrt{\beta_1} = \frac{1}{\sqrt{\lambda}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1}{\lambda} \quad \text{and} \quad \gamma_2 = \beta_2 - 3 = \frac{1}{\lambda}.$$

Recurrence relation for the moments of P.D.

$$\mu_r = E[X - E(X)]^r = \sum_{x=0}^{\infty} (x-\lambda)^r p(x,\lambda) = \sum_{x=0}^{\infty} (x-\lambda)^r \frac{e^{-\lambda}\lambda^x}{x!}$$

Differentiating w.r.t. λ,

$$\frac{d\mu_r}{d\lambda} = \sum_{x=0}^{\infty} r(x-\lambda)^{r-1}(-1)\frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty} \frac{(x-\lambda)^r}{x!}\left[x\lambda^{x-1} e^{-\lambda} - \lambda^x e^{-\lambda}\right]$$

$$= -r\sum_{r=0} (x-\lambda)^{r-1}\frac{e^{-\lambda}\lambda^x}{x!} + \sum \frac{(x-\lambda)^r}{x!}\left[\lambda x^{x-1} e^{-\lambda}(x-\lambda)\right]$$

$$= -r\sum (x-\lambda)^{r-1}\frac{e^{-\lambda}\lambda^x}{x!} + \frac{1}{\lambda}\sum (x-\lambda)^{r+1} \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= -r\mu_{r-1} + \frac{\mu_{r+1}}{\lambda}$$

$$\Rightarrow \mu_{r+1} = r\lambda\mu_{r-1} + \lambda\frac{d\mu_r}{d\lambda}$$

Moment Generating Function.

$$M_X(t) = E(e^{tX}) = \sum e^{tx} \cdot \frac{e^{-\lambda}\lambda^x}{x!} = e^{\lambda(e^t - 1)}$$

Characteristic Function.

$$\phi_X(t) = E(e^{itX}) = \sum e^{itx}\frac{e^{-\lambda}\lambda^x}{x!} = e^{\lambda(e^{it}-1)}$$

Cumulants

$$K_X(t) = \log M_X(t)$$

$$= \log e^{\lambda(e^t - 1)}$$

$$= \lambda(e^t - 1)$$

$$= \lambda\left[\cancel{1} + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots - \cancel{1}\right]$$

$$= \lambda\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots\right]$$

From the above, using the relationship between moments and cumulants, we have

Mean: $k_1 = \lambda$, $\mu_2 = k_2 = \lambda$, $\mu_3 = k_3 = \lambda$ and $\mu_4 = \lambda + 3\lambda^2$

$$\beta_1 = \frac{1}{\lambda}, \quad \beta = \frac{1}{\lambda} + 3,$$

Additive Property.

Let $x_1, x_2, \dots x_n$ are n independent Poisson variables with parameters $\lambda_1, \lambda_2 \dots \lambda_n$, then $\sum_{i=1}^{n} x_i$ is a Poisson variate with parameter $\sum_{i=1}^{n} \lambda_i$.

3. Negative Binomial distribution.

Suppose we have a succession of n Bernoulli trials. We assume that i) the trials are independent ii) the probability of success 'p' in a trial remains constant from trial to trial.

Let $f(x; r, p)$ denote the probability that there are x failures preceding the rth success in $x+r$ trials. Now, the last trial must be success, whose probability is p. For the remaining $(x+r-1)$ trials we must have $(r-1)$ successes whose probability is given by binomial probability law by the expression: $\binom{x+r-1}{r-1} p^{r-1} q^{x}$

$\therefore$ by compound probability theorem, $f(x; r, p)$ is given by the product of these two probabilities:

ie $$f(x; r, p) = \binom{x+r-1}{r-1} p^{r-1} q^{x} \cdot p$$

$$= \binom{x+r-1}{r-1} p^{r} q^{x}$$

Def: A random variable X is said to follow a negative binomial distribution with parameters r and p if its probability mass function is given by

$$p(x) = \binom{x+r-1}{r-1} p^{r} q^{x} \quad ; \quad x = 0, 1, 2 \ldots \quad (1)$$

Remarks:

i) $$\sum_{x=0}^{\infty} p(x) = p^{r} \sum_{x=0}^{\infty} \binom{-r}{x} (-q)^{x} = p^{r} \cdot (1-q)^{-r} = 1 \quad (2)$$

$\therefore$ $p(x)$ represents the probability function.

ii) If $p = \frac{1}{Q}$ and $q = \frac{P}{Q}$ so that $Q - P = 1$ ($\because p+q=1$), then

$$p(x) = \left\{ \binom{-r}{x} Q^{-r} \left(-\frac{P}{Q}\right)^{x} ; \quad x = 0, 1, 2 \ldots \quad (3)\right.$$

This is the general term in the negative binomial expansion $(Q-P)^{-r}$.

Some Important deductions.

i) Pascal's distribution.

The negative binomial distribution given in (2) when regarded as one having two parameters p and r is known as Pascal's distribution.

ii) Geometric distribution.

If we take $r = 1$ in (1), we have

$$p(x) = q^x p \quad ; \quad x = 0, 1, 2 \ldots$$

which is the probability function of geometric distribution.

iii) Polya's distribution.

If we take $r = \frac{1}{\beta}$, $p = \frac{1}{1+\beta\mu}$, $q = 1 - p = \frac{\beta\mu}{1-\beta\mu}$ in (1)

and using

$$\binom{x+r-1}{r-1} = (-1)^x \binom{-r}{x}, \text{ we get}$$

$$p(x) = \frac{r(r+1)(r+2)\cdots(r+x-1)}{x!} p^r q^x$$

$$= \frac{(1+\beta)(1+2\beta)\cdots[(1+\beta(x-1)]}{x!} \left(\frac{1}{1+\beta\mu}\right)^{1/\beta} \left(\frac{\mu}{1+\beta\mu}\right)^x ;$$

$$x = 0, 1, 2 \ldots \quad (4)$$

This is known as Polya's distribution with two parameters β and μ.

iv) Second Form of Geometric distribution.

Taking $\beta = 1$ in Polya's distribution (8.22c),

we get

$$p(x) = \left(\frac{1}{1+\mu}\right)\left(\frac{\mu}{1+\mu}\right)^x ; \quad x = 0, 1, 2 \ldots \quad (5)$$

which is geometric distribution with $p = \frac{1}{1+\mu}$, $q = 1 - p = \frac{\mu}{1+\mu}$.

Moment Generating Function.

$$M_x(t) = E(e^{tx}) = \Sigma e^{tx} p(x)$$

$$= \Sigma \binom{-r}{x} Q^{-r} \left(-\frac{Pe^t}{Q}\right)^x$$

$$= (Q - Pe^t)^{-r}$$

$$\mu_1' = \text{Mean} = \left.\frac{d M_x(t)}{dt}\right|_{t=0}$$

$$= \left[r(-Pe^t)(Q - Pe^t)^{-r-1} \right]_{t=0}$$

$$= rP$$

$$\mu_2' = \frac{d^2 M(t)}{dt^2}\Big|_{t=0}$$

$$= \left[rPe^t (Q-Pe^t)^{-r-1} + (-r-1) rPe^t (Q-Pe^t)^{-r-2} (-Pe^t) \right]_{t=0}$$

$$= rP + r(r+1)P^2$$

$$\therefore \mu_2 = \mu_2' - \mu_1'^2 = r(r+1)P^2 + rP - r^2P^2$$

$$= rPQ.$$

Note: As $Q > 1$, $rP < rPQ$, i.e. Mean < variance, which is a distinguishing feature of the negative binomial distribution.

Cumulants.

$$K_X(t) = \log M_X(t)$$

$$= -r \log (Q - Pe^t)$$

$$= -r \log \left[Q - P\left(1 + t + \frac{t^2}{2!} + \dots\right)\right]$$

$$= -r \log \left[1 - P\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots\right)\right]$$

Using the relationship between central moments and cumulants, we get

$$\text{Mean} = k_1 = rPQ$$

$$\mu_3 = rPQ(Q+P)$$

$$k_4 = rPQ(1 + 6PQ)$$

$$\mu_4 = rPQ[1 + 3PQ(r+2)]$$

Since $Q = \frac{1}{p}$, $P = qQ = q/p$, we have in terms of p and q

$$\text{Mean} = \frac{rq}{p}, \quad \text{variance} = \mu_2 = \frac{rq}{p^2}, \quad \mu_3 = \frac{rq(1+q)}{p^3}$$

$$\mu_4 = \frac{rq[p^2 + 3q(r+2)]}{p^4}$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(1+q)^2}{rq}, \quad \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{p^2 + 3q(r+2)}{rq}$$

$$\gamma_1 = \sqrt{\beta_1} = (1+q)/\sqrt{rq} \quad \text{and} \quad \gamma_2 = \beta_2 - 3 = (p^2 + 6q)/rq$$

Poisson Distribution as a Limiting case of the Negative Binomial distribution.

Negative binomial distribution tends to Poisson distribution as $P \to 0$, $r \to \infty$ such that $rp = \lambda$ (Finite.

Proof: Assignment.

Probability Generating Function

Let X be a random variable following negative binomial distribution, then

$$P_X(s) = E(s^X) = \sum_{x=0}^{\infty} s^x p(x) = \sum_{x=0}^{\infty} \binom{-r}{x} p^r (-qs)^x$$

$$= p^r (1-qs)^{-r} = [p/(1-qs)]^r$$

4. Geometric Distribution.

Def: A random variable X is said to have a geometric distribution if it assumes only non-negative values and its probability mass function is given by

$$p(x) = q^x p \quad ; \quad x = 0, 1, 2 \ldots$$
$$0 < p \le 1 ; \quad q = 1-p$$

Lack of Memory Property

The geometric distribution is said to lack memory in a certain sense. Suppose an event E can occur at one of the times $t = 0, 1, 2 \ldots$ and the occurrence (waiting) time X has geometric distribution with parameter p.

Thus $P(X = t) = q^t p ; \quad t = 0, 1, 2 \ldots$

Suppose we know that the event E has not occurred before k, i.e. $X \geq k$. Let $Y = X - k$. Then Y is the amount of additional time needed for E to occur. We can show that

$$P(Y = t / X \geq k) = P(X = t) = pq^t$$

which implies that the additional time to wait has the same distribution as initial time to wait.

Since the distribution does not depend upon k, it, in a sense, 'lacks memory' of how much we shifted the time origin. If 'B' were waiting for the event E and is relieved by 'C' immediately before time k, then the waiting time distribution of 'C' is the same as that of 'B'.

Proof: We have

$$P(X \geq r) = \sum_{s=r}^{\infty} pq^s = p(q^r + q^{r+1} + q^{r+2} + \dots) = \frac{pq^r}{(1-q)} = q^r \quad \text{---} (1)$$

$$P(Y \geq t / X \geq k) = \frac{P(Y \geq t \cap X \geq k)}{P(X \geq k)} = \frac{P(X-k \geq t \cap X \geq k)}{P(X \geq k)} \quad \because Y = X-k$$

$$= \frac{P(X \geq k+t)}{P(X \geq k)} = \frac{q^{t+k}}{q^k} = q^t \quad \text{frm (1)} \quad \text{---} (2)$$

$$\therefore P(Y=t / X \geq k) = P(Y \geq t / X \geq k) - P(Y \geq t+1 / X \geq k)$$

$$= q^t - q^{t+1} = q^t(1-q) = pq^t = P(X=t)$$

Moments

$$\text{Mean} = \mu_1' = E(X) = \sum x \cdot p(x) = \sum_{x=1}^{\infty} x \cdot pq^x = pq \sum_{x=1}^{\infty} x q^{x-1}$$

$$= pq(1-q)^{-2} = \frac{q}{p}$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= E(X(X-1)) + E(X) - [E(X)]^2$$

$$E(X(X-1)) = \sum x(x-1) p(x) = \sum x(x-1) pq^x$$

$$= 2pq^2 \sum_{x=2}^{\infty} \left[\frac{x(x-1)}{2 \times 1} q^{x-2}\right] = 2pq^2(1-q)^{-3} = \frac{2q^2}{p^2}$$

$$V(X) = \mu_2 = \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2} = \frac{q^2}{p^2} + \frac{q}{p} = \frac{q}{p^2}$$

Moment Generating function.

$$M_X(t) = E(e^{tx}) = \sum_x e^{tx} \cdot q^x \cdot t$$

$$= p\sum_{x=0} (q \cdot e^t)^x$$

$$= p(1-qe^t)^{-1} = \frac{p}{1-qe^t}$$

$$\mu_1' = \frac{d}{dt} M(t) \Big|_{t=0} = \frac{q}{p}$$

$$\mu_2' = \frac{d^2 M_X(t)}{dt^2} \Big|_{t=0} = \frac{q}{p} + \frac{2q^2}{p^2}$$

$$\mu_2 = \mu_2' - \mu_1'^2 = \frac{q}{p} + \frac{2q^2}{p^2} - \frac{q^2}{p^2} = qp^{-2}$$