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UNIT-IV

4.1 Convergence of sequence of random variables & Mode of convergence

Convergence in Probability:

A sequence's of r.v's $\{X_n\}$ is said to be converge to X in probability denoted, $X_n \xrightarrow{P} X$, if for every $\varepsilon > 0$, as $n \to \infty$

$P[|X_n - X| \geq \varepsilon] \longrightarrow 0$

Equivalent. $X_n \xrightarrow{P} X$; if for every $\epsilon > 0$ as $n \to \infty$. $P[|X_n - X|] < \epsilon \longrightarrow 0$ ——— (1)

This concept plays an important role in statistics consistency of estimators weak law of large numbers are examples.

Here, it mean that the difference between X_n and X is likely to be small with large probability.

<u>Necessary and sufficient conditions for convergence in probability</u>

$X_n \xrightarrow{P} 0$ iff $E\left(\frac{|X_n|}{1+|X_n|}\right) \longrightarrow 0$ as $n \to 0$

<u>Proof:</u>

For any X_n, the random variable $\frac{|X_n|}{1+|X_n|}$ is bounded by unity.

$\therefore$ Take $g(X_n) = \frac{|X_n|}{1+|X_n|}$, then $\sup g(X_n) = 1$

$\therefore$ From the basic inequality

$$\frac{E(g(x)) - g(a)}{\text{as } \sup g(x)} \leq P[|x| \geq a] \leq \frac{E\, g(x)}{g(a)}$$

$$\frac{E|X_n|}{1+|X_n|} - \frac{\varepsilon}{1+\varepsilon} \leq P[|X_n| \geq \varepsilon] \leq \frac{E\left[\frac{|X_n|}{1+|X_n|}\right]}{\frac{\varepsilon}{1+\varepsilon}} \quad \text{—— (1)}$$

From the R.H.S of (1) we see that

$$E\left(\frac{|X_n|}{1+|X_n|}\right) \longrightarrow 0 \Rightarrow P[|X_n| \geq \varepsilon] \longrightarrow 0$$

$$\Rightarrow X_n \xrightarrow{P} 0.$$

This is sufficient condition for convergence in probability necessary condition

consider L.H.S of (I)

$P[|x_n| \geq \epsilon] \longrightarrow 0 \Rightarrow E\left(\frac{|x_n|}{1+|x_n|}\right) - \frac{\epsilon}{1+\epsilon} \longrightarrow 0$

Since ϵ is arbitrary and $\frac{|x_n|}{1+|x_n|}$ is a non-negative random variable its expectation is greater than or equal to.

i.e., $E\left(\frac{|x_n|}{1+|x_n|}\right) \geq 0$.

Hence $E\left(\frac{|x_n|}{1+|x_n|}\right) \longrightarrow 0$ as $n \to \infty$

This is the necessary condition.

Convergence in Probability Implies Mutual convergence:

i.e., $x_n \xrightarrow{P} x \quad x_n - x_m \xrightarrow{P} 0$

Proof:

Since $x_n \xrightarrow{P} x$, $x < \infty$

consider, $|x_n - x_m| = |x_n - x + x - x_m|$

$\leq |x_n - x| + |x - x_m|$

Now,

$[|x_n - x_m| \geq \epsilon] \subseteq [|x_n - x| > \epsilon/2] + [|x - x_m| \geq \epsilon/2]$

$\Rightarrow P[|x_n - x_m| \geq \epsilon] \subset P[|x_n - x| > \epsilon/2] + P[|x - x_m| > \epsilon/2] \longrightarrow 0$

as $x_n \xrightarrow{P} x$ as $n, m \to \infty$ each term of the R.H.S goes to zero.

$\Rightarrow P[|x_n - x_m| \geq \epsilon] \longrightarrow 0$

$\Rightarrow x_n - x_m \xrightarrow{P} 0$

Thus convergence probability implies mutual convergence

Hence the proof.

Almost Surely Convergence:

The sequence of $\{x_n\}$ is said to converge to x a.s. [almost surely(or) strongly (or) in probability] $x_n(\omega) \to x(\omega)$ $\forall\ \omega$ except those belongs to the null set N i.e. $x_n \xrightarrow{a.s.} x \Rightarrow x_n(\omega) \longrightarrow x(\omega)$ $\forall\ \omega \in N^c$

$\longrightarrow P(\omega) = 0$

$\Rightarrow P[\omega : x_n(\omega) \longrightarrow x(\omega)] = 1$

Here almost sure convergence is unique

i.e., $x_n \xrightarrow{a.s.} x$, $x_n \xrightarrow{a.s.} x' \Longrightarrow x = x'$ a.s

Proof:

$x_n \xrightarrow{a.s.} x$ iff $P\left[\bigcup_{k=n}^{\infty} \{\omega ; |x_n(\omega) - x(\omega)| \geq \epsilon\}\right] \to 0$

consider $|x - x'| = |x - x_k + x_k - x'|$

$\leq |x - x_k| + |x_k - x'|$

$\Rightarrow [|x - x'| \geq \epsilon] \subset [|x_k - x| \geq \epsilon/2] \cup [|x_k - x'| \geq \epsilon/2]$

$\subset \bigcup_{k=n}^{\infty} [\omega : |x_k(\omega)| > \epsilon/2] \cup$

$\bigcup_{k=n}^{\infty} [\omega : |x_k(\omega) - x'(\omega)| \geq \epsilon/2]$

$\Rightarrow P[|x - x'| \geq \epsilon] \leq P\left[\bigcup_{k=n}^{\infty} \{\omega ; |x_k(\omega)| \geq \epsilon/2\}\right] +$

$P\left[\bigcup_{k=n}^{\infty} \omega : |x_k(\omega) - x'(\omega)| \geq \epsilon/2\right]$

Since, $x_n \to x$ and $x_n \to x'$ each term in R.H.S goes to 0

$\Rightarrow P[|x-x'| \geq \epsilon_1] = 0$

$\Rightarrow x = x'$ almost surely

Theorem:

Almost Sure (a.s.) convergence implies convergence in Probability

$x_n \xrightarrow{a.s} x \Rightarrow x_n \xrightarrow{P} x$

Proof:

Consider. $\epsilon_1 > 1/r$. then

$[|x_n - x| > 1/r] \supset [|x_n - x| > \epsilon]$

$\Rightarrow \{\bigcup_{n=m}^{\infty} [|x_n - x| \geq 1/r]\} \supset [|x_n - x| > \epsilon]$

$\Rightarrow \bigcap_{n=m}^{\infty} [|x_n - x| < \frac{1}{r}] \supset [|x_n - x| \leq \epsilon_1]$

$\Rightarrow P[\bigcap_{n=m}^{\infty} |x_n - x| < \frac{1}{r}] \geq P[|x_n - x| < \epsilon_1]$

We are given $x_n \xrightarrow{a.s} x$ this implies

$P[\bigcap_{n=m}^{\infty} |x_n - x| < \frac{1}{r}] \to 1$

$\Rightarrow x_n \xrightarrow{P} x$

Hence the proof

Convergence in distribution:

Let $\{F_n(x)\}$ be a sequence of distribution functions, not necessary of random variables.

Let $F_n(x) \to F(x)$ as $n \to \infty$. (i.e.) $F_n(x)$ converges to $F(x)$ as $n \to \infty$.

Then the sequence $\{F_n(x)\}$ is said to be converge weekly to F.

Convergence in r^{th} mean:

A Square of r.v. $\{x_n\}$ is said to converge to x r^{th} mean denoted by $x_n \xrightarrow{r} x$, if $E|x_n - x|^r \to 0$ as $n \to \infty$ for $r=2$ $|x_n - x|^2$ is called convergence in mean square or quadratic mean.

Theorem:

$x_n \xrightarrow{r} x \Rightarrow E|x_n|^r \to E|x|^r$

Proof:

Case (i) $r \geq 1$

Consider C.R. inequality.

$E[|x+y|^r \leq C_r E|x|^r + C_r |x. E(y)|^r$

for $r \leq 1$

$\leq E|x|^r + E|y|^r \quad \because C_r = 1$

Put $x_n - x$ in the place of x and x in the place of y.

$\therefore E|x_n - x + x|^r \leq E|x_n - x|^r + E|x_n|^r$

$\Rightarrow E|x_n|^r \leq E|x_n - x|^r + E|x|^r$

$\Rightarrow E|x_n - x|^r \geq E|x_n|^r - E|x|^r$

But $x_n \xrightarrow{r} x \Rightarrow E|x_n - x|^r \to 0$

$E|x_n|^r - E|x|^r = 0$

$\Rightarrow E|x_n|^r = E|x|^r$

Case (ii) $r > 1$.

Consider minkowski inequality $r > 1$

$$E^{1/r}|X+Y|^r \le E^{1/r}|X|^r + E^{1/r}|Y|^r$$

Put $X = X_n - X$ & $Y = X$

$$E^{1/r}|X_n|^r \le E^{1/r}|X_n - X|^r + E^{1/r}|X|^r \quad \text{——— (1)}$$

$$E^{1/r}|X_n - X|^r \ge E^{1/r}|X_n|^r - E^{1/r}|X|^r \quad \text{——— (2)}$$

$\therefore$ $E^{1/r}|X_n - X|^r \ge E^{1/r}|X_n|^r - E^{1/r}|X|^r$

$$E^{1/r}|X_n - X|^r \longrightarrow 0$$

$$\Rightarrow E^{1/r}|X_n|^r - E^{1/r}|X|^r = 0$$

$$\Rightarrow E^{1/r}|X_n|^r = E^{1/r}|X|^r$$

Moment Generating Function

Moment Generating Function. The moment generating function (m.g.f.) of a random variable X (about origin) having the probability function $f(x)$ is given by

$$(6\cdot54)\ldots \quad \begin{cases} M_X(t) = E(e^{tX}) = \int_x e^{tx} f(x)\,dx, & \text{(for continuous probability distribution)} \\ \quad = \sum_x e^{tx} f(x), & \text{(for discrete probability distribution)} \end{cases}$$

the integration or summation being extended to the entire range of x, t being the real parameter and it is being assumed that the right-hand side of (6·54) is absolutely convergent for some positive number h such that $-h < t < h$. Thus

$$M_X(t) = E(e^{tX}) = E\left[1 + tX + \frac{t^2X^2}{2!} + \ldots + \frac{t^rX^r}{r!} + \ldots\right]$$

$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \ldots + \frac{t^r}{r!}E(X^r) + \ldots$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \ldots + \frac{t^r}{r!}\mu_r' + \ldots \qquad \ldots(6\cdot55)$$

where $\mu_r' = E(X^r) = \int x^r f(x)\,dx$, *for continuous distribution*

$\quad = \sum_x x^r p(x)$, *for discrete distribution*,

is the rth moment of X about origin. Thus we see that the coefficient of $\frac{t^r}{r!}$ in $M_X(t)$ gives μ_r' (above origin). Since $M_X(t)$ generates moments, it is known as moment generating function.

Differentiating (6·55) w.r.t. t and then putting $t = 0$, we get

$$\left[\frac{d^r}{dt^r}\{M_X(t)\}\right]_{t=0} = \left[\frac{\mu_r'}{r!}\cdot r! + \mu'_{r+1}t + \mu'_{r+2}\cdot\frac{t^2}{2!} + \ldots\right]_{t=0}$$

$$\Rightarrow \qquad \mu_r' = \left[\frac{d^r}{dt^r}\{M_X(t)\}\right]_{t=0} \qquad \ldots(6\cdot56)$$

In general, the moment generating function of X about the point $X = a$ is defined as

$$M_X(t)\ (\text{about } X = a) = E[e^{t(X-a)}]$$

$$= E\left\{1 + t(X-a) + \frac{t^2}{2!}(X-a)^2 + \ldots + \frac{t^r}{r!}(X-a)^r + \ldots\right\}$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \ldots + \frac{t^r}{r!}\mu_r' + \ldots \qquad \ldots(6\cdot57)$$

where $\mu_r' = E\{(X-a)^r\}$, is the rth moment about the point $X = a$.

4.2 Characteristic Function: Definition and Properties

Characteristic Function. In some cases m.g.f. does not exist, since the integral $\int_{-\infty}^{\infty} e^{tx} f(x)\,dx$ or the series $\sum_x e^{tx} p(x)$ does not converge absolutely for real values of t for some distributions. For example, for the continuous probability distribution

$$dF(x) = C \frac{1}{(1+x^2)^m}\,dx\ ;\ m > 1, -\infty < x < \infty,$$

the m.g.f. does not exist, since the integral

$$M_X(t) = C \int_{-\infty}^{\infty} e^{tx} \frac{1}{(1+x^2)^m}\,dx,$$

does not converge absolutely for finite positive values of m because the function e^{tx} dominates the function x^{2m} so that $e^{tx}/x^{2m} \to \infty$ as $x \to \infty$.

Again, for the discrete probability distribution

$$\left.\begin{aligned} f(x) &= \frac{6}{\pi^2 x^2}\ ;\ x = 1, 2, 3, \ldots \\ &= 0, \text{ elsewhere} \end{aligned}\right\}$$

$$M_X(t) = \sum_x e^{tx} f(x) = \frac{6}{\pi^2} \sum_{x=1}^{\infty} \left(\frac{e^{tx}}{x^2} \right)$$

The series is not convergent (by D'Alembert's Ratio Test) for $t > 0$. Thus there does not exist a positive number h such that $M_X(t)$ exists for $-h < t < h$. Hence $M_X(t)$ does not exist in this case also.

A more serviceable function than the m.g.f. is what is known as characteristic function and is defined as

$$\phi_X(t) = E(e^{itX}) = \int e^{itx} f(x)\,dx \qquad \textit{(for continuous probability distributions)}$$

$$= \sum_x e^{itx} f(x) \qquad \textit{(for discrete probability distributions)} \qquad \ldots(6{\cdot}64)$$

If $F_X(x)$ is the distsribution function of a continuous random variable X, then

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{itx}\,dF(x) \qquad \ldots(6{\cdot}64a)$$

Obviously $\phi(t)$ is a complex valued function of real variable t. It may be noted that

$$|\phi(t)| = \left| \int e^{itx} f(x)\,dx \right| \le \int |e^{itx}|\, f(x)\,dx = \int f(x)\,dx = 1,$$

since $\quad |e^{itx}| = |\cos tx + i \sin tx|^{1/2} = (\cos^2 tx + \sin^2 tx)^{1/2} = 1$

Since $|\phi(t)| \le 1$, characteristic function $\phi_X(t)$ always exists.

Yet another advantage of characteristic function lies in the fact that it uniquely determines the distribution function, *i.e.*, if the characteristic function of a distribution is given, the distribution can be uniquely determined by the theorem, known as the Uniqueness Theorem of Characteristic Functions

Properties of Characteristic Functions. For all real 't', we have

(i) $\phi(0) = \int_{-\infty}^{\infty} dF(x) = 1$ ...(6·64b)

(ii) $|\phi(t)| \le 1 = \phi(0)$ ...(6·64c)

(iii) $\phi(t)$ is continuous everywhere, *i.e.*, $\phi(t)$ is a continuous function of t in $(-\infty, \infty)$. Rather $\phi(t)$ is uniformly continuous in 't'.

Proof. For $h \ne 0$, $|\phi_X(t+h) - \phi_X(t)| = \left| \int_{-\infty}^{\infty} \left[e^{i(t+h)x} - e^{itx} \right] dF(x) \right|$

$$\le \int_{-\infty}^{\infty} \left| e^{itx}(e^{ihx} - 1) \right| dF(x)$$

$$= \int_{-\infty}^{\infty} \left| e^{ihx} - 1 \right| dF(x) \qquad \ldots(*)$$

The last integral does not depend on t. If it tends to zero as $h \to 0$ then $\phi_X(t)$ is uniformly continuous in 't'.

Now $\left| e^{ihx} - 1 \right| \le \left| e^{ihx} \right| + 1 = 2$

$$\therefore \quad \int_{-\infty}^{\infty} \left| e^{ihx} - 1 \right| dF(x) \le 2 \int_{-\infty}^{\infty} dF(x) = 2.$$

Hence by Dominated Convergence Theorem (D.C.T.), taking the limit inside the integral sign in (*), we get

$$\lim_{h \to 0} \left| \phi_X(t+h) - \phi_X(t) \right| \le \int_{-\infty}^{\infty} \lim_{h \to 0} \left| e^{ihx} - 1 \right| dF(x) = 0$$

$$\Rightarrow \lim_{h \to 0} \phi_X(t+h) = \phi_X(t) \ \forall\, t.$$

Hence $\phi_X(t)$ is uniformly continuous in 't'.

(iv) $\phi_X(-t)$ and $\phi_X(t)$ are conjugate functions, *i.e.*, $\phi_X(-t) = \overline{\phi_X(t)}$ where $\bar{a}$ is the complex conjugate of a.

Proof.
$$\phi_X(t) = E(e^{itX}) = E[\cos tX + i \sin tX]$$

$$\Rightarrow \overline{\phi_X(t)} = E[\cos tX - i \sin tX]$$
$$= E[\cos(-t)X + i \sin(-t)X]$$
$$= E(e^{-itX}) = \phi_X(-t).$$

Theorems on Characteristic Function.

Theorem *If the distribution function of a r.v. X is symmetrical about zero, i.e.,*

$$1 - F(x) = F(-x) \quad \Rightarrow \quad f(-x) = f(x),$$

then $\phi_X(t)$ is real valued and even function of t.

Proof. By definition we have

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx = \int_{-\infty}^{\infty} e^{-ity} f(-y)\, dy \qquad (x = -y)$$

$$= \int_{-\infty}^{\infty} e^{-ity} f(y)\, dy \qquad [\because f(-y) = f(y)]$$

$$= \phi_X(-t) \qquad \ldots(*)$$

$\Rightarrow$ $\phi_X(t)$ is an even function of t.

Also $\overline{\phi_X(t)} = \phi_X(-t)$ [c.f. Property (iv) § 6·12·1]

$\therefore$ $\overline{\phi_X(t)} = \phi_X(-t) = \phi_X(t)$ (From *)

Hence $\phi_X(t)$ is a real valued function of t.

Theorem *If X is some random variable with characteristic function $\phi_X(t)$, and if $\mu_r' = E(X^r)$ exists, then*

$$\mu_r' = (-i)^r \left[\frac{\partial^r}{\partial t^r} \phi(t) \right]_{t=0}$$

Proof. $\phi(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx$

Differentiating (under the integral sign) 'r' times w.r.t. t, we get

$$\frac{\partial^r}{\partial t^r} \phi(t) = \int_{-\infty}^{\infty} (ix)^r \cdot e^{itx} f(x)\, dx = (i)^r \int_{-\infty}^{\infty} x^r e^{itx} f(x)\, dx$$

$$\therefore \left[\frac{\partial^r}{\partial t^r} \phi(t) \right]_{t=0} = (i)^r \left[\int_{-\infty}^{\infty} x^r e^{itx} f(x)\, dx \right]_{t=0}$$

$$= (i)^r \int_{-\infty}^{\infty} x^r f(x)\, dx = i^r E(X^r) = i^r \mu_r'$$

Hence $\mu_r' = \left(\frac{1}{i}\right)^r \left[\frac{\partial^r}{\partial t^r} \phi(t) \right]_{t=0} = (-i)^r \left[\frac{\partial^r}{\partial t^r} \phi(t) \right]_{t=0}$

Theorem $\phi_{cX}(t) = \phi_X(ct)$, *$c$, being a constant.*

Theorem *If X_1 and X_2 are independent random variables, then*

$$\phi_{X_1+X_2}(t) = \phi_{X_1}(t)\ \phi_{X_2}(t) \qquad \text{...}(*)$$

More, generally for independent random variables X_i; $i = 1, 2, ..., n$, we have

$$\phi_{X_1+X_2+\dots+X_n}(t) = \phi_{X_1}(t)\ \phi_{X_2}(t)\ \dots \phi_{X_n}(t)$$

Important Remark. Converse of $(*)$ is not true, *i.e.*,

$\phi_{X_1+X_2}(t) = \phi_{X_1}(t)\ \phi_{X_2}(t)$ does not imply that X_1 and X_2 are independent.

For example, let X_1 be a standard Cauchy variate with p.d.f.

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$$

Then $\phi_{X_1}(t) = e^{-|t|}$ (c.f.

Let $X_2 \equiv X_1$, *i.e.*, $P(X_1 = X_2) = 1$. ...(**)

Then $\phi_{X_2}(t) = e^{-|t|}$

Now $\phi_{X_1+X_2}(t) = \phi_{2X_1}(t) = \phi_{X_1}(2t) = e^{-2|t|}$

$$= \phi_{X_1}(t)\ \phi_{X_2}(t)$$

i.e., $(*)$ is satisfied but obviously X_1 and X_2 are not independent, because of (**).

Theorem *Effect of Change of Origin and Scale on Characteristic Function. If $U = \frac{X-a}{h}$, a and h being constants, then*

$$\phi_U(t) = e^{-iat/h}\ \phi_X(t/h)$$

In particular if we take $a = E(X) = \mu$ (say) and $h = \sigma_X = \sigma$ then the characteristic function of the standard variate

$$Z = \frac{X - E(X)}{\sigma_X} = \frac{X-\mu}{\sigma},$$

is given by $\phi_Z(t) = e^{-i\mu t/\sigma}\ \phi_X(t/\sigma)$

Characteristic function of Binomial distribution

$$C_x(t) = E(e^{itx})$$
$$= \sum_{x=0}^{n} e^{itx} p(x)$$
$$= \sum_{x=0}^{n} e^{itx} \binom{n}{x} p^x q^{n-x}$$
$$= \sum_{x=0}^{n} (pe^{it})^x \binom{n}{x} q^{n-x}$$
$$= [q + pe^{it}]^n$$

Characteristic function of Poisson distribution

$$C_x(t) = E[e^{itx}]$$
$$= \sum_{x=0}^{n} e^{itx} p(x)$$
$$= \sum_{x=0}^{n} e^{itx} \frac{e^{-\lambda}\lambda^x}{x!}$$
$$= e^{-\lambda} \sum_{x=0}^{n} \frac{e^{itx}\lambda^x}{x!}$$
$$= e^{-\lambda} e^{\lambda it}$$

Normal Distribution

Characteristic function of Normal Distribution

(i) $x \sim N(0, \sigma^2)$

$$C_x(t) = E[e^{itx}]$$
$$= \int_{-\infty}^{\infty} e^{itx} f(x)\, dx.$$

The normal p.d.f is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$C_x(t) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{itx - 1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{itx - 1/2 \frac{x^2}{\sigma}} dx$$

put $z = x/\sigma$

$dz = dx/\sigma$

$\sigma dz = dx.$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{itx - 1/2\, x^2/\sigma} dx.$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{it\sigma z - 1/2 z^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-1/2 z^2 + it\sigma} dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-1/2(z^2 - 2i + 2\sigma)} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2\left[z^2 - 2i + 2\sigma + (i + \sigma)^2 - (it\sigma)^2\right]} dz$$

$$= \frac{e^{-t^2\sigma^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2 (z - it\sigma)^2} dz$$

$$= \frac{e^{-t^2 \sigma^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} du$$

$$= \frac{e^{-t^2\sigma^2/2}}{\sqrt{2\pi}} \sqrt{2\pi} \quad \left[\int e^{-a^2 x^2} dx = \sqrt{\pi}/a\right]$$

$$= e^{-t^2\sigma^2/2}.$$

(ii) $X \sim N(\mu, \sigma^2)$

$$C_x(t) = E\left[e^{itx}\right]$$

$$= \int_{-\infty}^{\infty} e^{itx} f(x) \, dx.$$

The normal p.d.f is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$C_x(t) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{\sqrt{2\pi}\sigma} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{itx} \, ^{1/2}\left(\frac{x-\mu}{\sigma}\right)^2 dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\left[it(z\sigma+\mu) - 1/2(z^2)\right]} \sigma \, dz.$$

Put $z = x-\mu/\sigma$

$x = z\sigma - \mu$

$\sigma dz = dx.$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{itz\sigma} e^{it\mu} e^{-1/2 z^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{it\mu} \int_{-\infty}^{\infty} e^{itz\sigma} e^{-1/2 z^2} dz$$

$$= \frac{e^{it\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2} (z^2 - 2i + 2\sigma) \, dz$$

$$= \frac{e^{it\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2} \left(z^2 - 2i + 2\sigma + (it\sigma)^2 - (i+\sigma^2)\right] dz$$

$$= \frac{e^{i+\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2}\left(z - (it\sigma)^2 \quad 1/2 \, (i+\sigma)^2 \, dz.\right.$$

$$= \frac{e^{i+\mu} \cdot e^{-i+\sigma/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2}\left(z - i + \sigma\right)^2 dz$$

$$= \frac{e^{i+\mu} - \frac{i^2+\sigma^2}{2}}{2\pi} \times \sqrt{2\pi}$$

$$= e^{i+\mu} - i^2 + \sigma^2/2$$

4.3 Inversion Theorem

Inversion Theorem Lemma. *If $(a-h, a+h)$ is the continuity interval of the distribution function $F(x)$, then*

$$F(a+h)-F(a-h)=\lim_{T\to\infty}\frac{1}{\pi}\int_{-T}^{T}\frac{\sin ht}{t}e^{-ita}\phi(t)\,dt,$$

$\phi(t)$ being the characteristic function of the distribution.

Corollary. If $\phi(t)$ is absolutely integrable over R^1, *i.e.*, if

$$\int_{-\infty}^{\infty}|\phi(t)|\,dt<\infty,$$

then the derivative of $F(x)$ exists, which is bounded, continuous on R^1 and is given by

$$f(x)=F'(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}\phi(t)\,dt,$$

for every $x\in R^1$.

Proof. In the above lemma replacing a by x and on dividing by $2h$, we have

$$\frac{F(x+h)-F(x-h)}{2h}=\frac{1}{2\pi}\cdot\lim_{T\to\infty}\int_{-T}^{T}\frac{\sin ht}{ht}e^{itx}\phi(t)\,dt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin ht}{ht}e^{-itx}\phi(t)\,dt$$

$$\therefore\ \lim_{h\to 0}\frac{F(x+h)-F(x-h)}{2h}=\frac{1}{2\pi}\lim_{h\to 0}\int_{-\infty}^{\infty}\frac{\sin ht}{ht}\cdot e^{-itx}\phi(t)\,dt$$

Since $$\int_{-\infty}^{\infty}|\phi(t)|\,dt<\infty,$$

the integrand on the right hand side is bounded by an integrable function and hence by Dominated Convergence Theorem, we get

$$\lim_{h\to 0}\frac{F(x+h)-F(x-h)}{2h}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{h\to 0}\left(\frac{\sin ht}{ht}\right)\cdot e^{-itx}\phi(t)\,dt$$

By mean value theorem of differential calculus, we have

$$\lim_{h\to 0}\frac{F(x+h)-F(x-h)}{2h}=F'(x)=f(x),$$

where $f(\cdot)$ is the p.d.f. corresponding to $\phi(t)$. Thus

$$f(x)=F'(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}\phi(t)\,dt,$$

4.4 Uniqueness Theorem

Uniqueness Theorem of Characteristic Functions. *Characteristic function uniquely determines the distribution, i.e., a necessary and sufficient condition for two distributions with p.d.f.'s $f_1(\cdot)$ and $f_2(\cdot)$ to be identical is that their characteristic functions $\phi_1(t)$ and $\phi_2(t)$ are identical.*

Proof. If $f_1(\cdot)=f_2(\cdot)$, then from the definition of characteristic function, we get

$$\phi_1(t)=\int_{-\infty}^{\infty}e^{itx}f_1(x)\,dx=\int_{-\infty}^{\infty}e^{itx}f_2(x)\,dx=\phi_2(t)$$

Conversely if $\phi_1(t)=\phi_2(t)$, then from corollary to Theorem 6·26, we get

$$f_1(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}\phi_1(t)\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}\phi_2(t)\,dt=f_2(x)$$

Theorem *Necessary and sufficient condition for the random variables X_1 and X_2 to be independent is that their joint characteristic function is equal to the product of their individual characteristic functions, i.e.,*

$$\phi_{X_1,X_2}(t_1, t_2) = \phi_{X_1}(t_1)\,\phi_{X_2}(t_2) \qquad \ldots(*)$$

Proof. (*i*) *Condition is Necessary.* If X_1 and X_2 are independent then we have to show that (*) holds. By def.,

$$\begin{aligned}\phi_{X_1,X_2}(t_2, t_2) &= E\left(e^{it_1X_1 + it_2X_2}\right) = E\left(e^{it_1X_1}\cdot e^{it_2X_2}\right)\\ &= E\left(e^{it_1X_1}\right)E\left(e^{it_2X_2}\right) \quad (\because X_1, X_2 \text{ are independent})\\ &= \phi_{X_1}(t_1)\,\phi_{X_2}(t_2),\end{aligned}$$

as required.

(ii) Condition is sufficient. We have to show that if (*) holds, then X_1 and X_2 are independent.

Let $f_{X_1,X_2}(x_1, x_2)$ be the joint p.d.f. of X_1 and X_2 and $f_1(x_1)$ and $f_2(x_2)$ be the marginal p.d.f.'s of X_1 and X_2 respectively. Then by definition (for continuous r.v.'s), we get

$$\phi_{X_1}(t_1) = \int_{-\infty}^{\infty} e^{it_1x_1} f_1(x_1)\,dx_1$$

$$\phi_{X_2}(t_2) = \int_{-\infty}^{\infty} e^{it_2x_2} f_2(x_2)\,dx_2$$

$$\therefore \quad \phi_{X_1}(t_1)\,\phi_{X_2}(t_2) = \left[\int_{-\infty}^{\infty} e^{it_1x_1} f_1(x_1)\,dx_1\right]\left[\int_{-\infty}^{\infty} e^{it_2x_2} f_2(x_2)\,dx_2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(t_1x_1 + t_2x_2)} f_1(x_1)\,f_2(x_2)\,dx_1\,dx_2 \qquad \ldots(**)$$

by Fubini's theorem, since the integrand is bounded by an integrable function.

Also by def^n

$$\phi_{X_1,X_2}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(t_1x_1 + t_2x_2)} f(x_1, x_2)\,dx_1\,dx_2$$

If (*) holds, we get from (**)

$$\phi_{X_1,X_2}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(t_1x_1 + t_2x_2)} f_1(x_1)\,f_2(x_2)\,dx_1\,dx_2$$

Hence by uniqueness theorem of characteristic functions, we get

$$f(x_1, x_2) = f_1(x_1)\,f_2(x_2),$$

which implies that X_1 and X_2 are independent.

Problem:

For the given p.d.f $f(x) = \frac{1}{2} e^{-|x|}$, $-\infty < x < \infty$, find its characteristic function

$$C_x(t) = E\left[e^{itx}\right]$$

$$= \sum_{-\infty}^{\infty} e^{itx} f(x)\,dx$$

$$= \sum_{-\infty}^{\infty} e^{itx} \frac{1}{2} e^{-|x|} dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{0} e^{itx} e^{x}\,dx + \int_{0}^{\infty} e^{itx} e^{-x}\,dx\right]$$

$$= \frac{1}{2} \int_{-\infty}^{0} e^{x(1+it)}\,dx + \int_{0}^{\infty} e^{x(it-1)}\,dx.$$

$$= \frac{1}{2}\left[\int_{-\infty}^{0} e^{x(1+it)}\,dx\right] + \frac{1}{2}\int_{0}^{\infty} e^{-x(1-it)}\,dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{0} \frac{e^{(1+it)x}}{1+it}\,dx + \int_{0}^{\infty} \frac{e^{-x(1-it)}}{(1-it)}\,dx\right]$$

$$= \frac{1}{2}\left[\frac{e^{0} - e^{-\infty}}{1+it}\right] + \frac{1}{2}\left[\frac{e^{-\infty} - e^{0}}{(1-it)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1+it} + \frac{1}{1-it}\right]$$

$$= \frac{1}{2}\left[\frac{(1-it)+(1+it)}{(1+it)(1-it)}\right]$$

$$= \frac{2}{2(1+it)(1-it)}$$

$$= \frac{1}{(1+it)(1-it)}$$

$$= \frac{1}{1+t^2}$$

Inversion Formula for finding pdf:

The formula to obtain p.d.f in the case of the continuous r.v. due given characteristic function is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} c(t)\,dt$$

Find the p.d.f whose characteristic function is given by $c(t) = e^{-\frac{1}{2}t^2}$. The inversion formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} c(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\left(e^{-\frac{1}{2}t^2}\right)dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(2itx + t^2)}\,dt$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[t^2 + 2itx + (ix)^2 - (ix)^2\right]}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\left[-\frac{1}{2}(t+ix)^2 - \frac{ix^2}{2}\right]}\,dt$$

$$= \frac{1}{2\pi} e^{-\frac{ix^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(t+ix)^2}\,dt$$

$$= \frac{e^{-\frac{ix^2}{2}}}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(t+ix)^2}\,dt \qquad \text{Put } t+ix = u$$

$$= \frac{e^{-ix^2/2}}{2\pi}\int_{-\infty}^{\infty} e^{-u^2/2}\,du$$

$$= \frac{e^{-\frac{x^2}{2}}}{2\pi}\sqrt{2\pi} = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

The formula to obtain p.m.f. of discrete r.v. for given characteristic function is

$$P(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-itx} c(t)\,dt$$

Obtain the p.m.f this characteristic function is $c(t) = (q + pe^{it})^n$ by inverse formula

$$P(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-itx}(q+pe^{it})^n\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-itx}\left[\sum_{y=0}^{n}\binom{n}{y} q^{n-y}(pe^{it})^y\right]dy$$

$$P(x) = \sum_{y=0}^{n}\binom{n}{y}p^y q^{n-y}\frac{1}{\sqrt{2\pi}}\int e^{it(y-x)}\,dy$$

$$= \binom{n}{x}p^x q^{n-x} \quad \text{if } y = x$$

$$= 0 \quad \text{otherwise.}$$

Which is the p.m.f of Binomial distribution

Obtain p.m.f whose characteristic function is

is $c(t) = e^{m(e^{it}-1)}$

$$P(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-itx}\sum_{r=0}^{\infty}\frac{(me^{it})^r}{r!}\,dt$$

$$= \sum_{r=0}^{\infty}\frac{e^{-m}m^r}{r!}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{it(r-x)}\,dt$$

$$= \frac{e^{-m}m^r}{r!} \quad \text{when } r = x$$

$$= 0 \quad \text{otherwise}$$

which is the p.m.f of poisson distribution

Obtain p.m.f of whose characteristic function is given $c(t) = p(1-qe^{it})^{-1}$

$$P(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-itx}c(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-itx}p(1-qe^{it})^{-1}$$

$$= \frac{p}{2\pi}\int_{-\pi}^{\pi} e^{-itx}\sum_{r=0}^{\infty}(qe^{it})^r\,dt$$

$$= \sum_{r=0}^{\infty} q^r p\,\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{it(r-x)}\,dt$$

$$= q^r p \quad \text{if } r = x$$

Which is the p.m.f of Geometric distribution.

End of Unit-IV