Unit-iv

Vector spaces

Notations:

V : The given vectors space (elements are vectors)

script K : The vectors consists of arbitary field is u, v, w vectors in V.

k : The given number field a, b, c, k : scalars in K.

R : Real field

C : Complex field

$\forall x \subset A$: For every x subset of A.

$\exists x \in A$: There exists an x in A.

$A \subseteq B$: A is subset of B

$A \cap B$: A intersection B

$A \cup B$: ϕ.

Definition of vector space

Let V be an non-empty set with two operations.

1. Vector addition : this assigns to any $u, v \in V$ a sum $u+v$ in V.

2. Scalar multiplication : this assigns to any $u, v \in V$, $k \in K$, $ku \in V$

This V is called a vector space over the arbitary field K] if the following axioms holds for any vectors $u, v, w \in V$

Axioms:

1. $(u+v)+w = u+(v+w)$ (ass)

2. There is a vector in V denoted by 0 and called the zero vector such that any $u \in V$, $u+0 = 0+u = u$

3. For each $u \in V$ there is a vector in V denoted by $(-u)$ negative of u, such that $u+(-u) = (-u)+u = 0$ null vector

4. $u+v = v+u$ com

5. $k(u+v) = ku + kv$ for any scalar $k \in K$

6. $(a+b)u = au + bu$ for any scalar $a, b \in K$

7. $(ab)u = a(bu)$ for any scalar $a, b \in K$

8. $1u = u$ for unit scalar $1 \in K$

The first 4 axioms are only concerned with additive structure of V. V is a commutative under addition. proof

a. Any sum $v_1 + v_2 + v_3 + \dots + v_m$ of vectors requires [redacted] parantheses and does not depend on the order of summands

b. The 0 is unique and the negative of the vector u is unique.

c) cancellation law:

if $u+w=v+w$ then $u=v$ subtraction within V is defined by $u-v=u+(-v)$ where $-v$ is unique negative of v. Remaining 4 axioms are concerned with the action of the field K of scalar on the vector space V.

Theorem:

Let V be an vector space over a field K.

i) For any scalar $k \in K$ and $0 \in V$, $k.0=0$

ii) For $0 \in K$, $u \in V$, $0u=0$

iii) If $ku=0$ where $k \in K$ and $u \in V$, then $k=0$ or $u=0$

iv) For any $k \in K$ and $u \in V$, $(-k)u=k(-u)=-ku$

Examples of vector space:

1. Space K^n

Let K be an arbitrary field, K^n be the set of all n-tuples of elements of K. K^n is a vector space over K using the following operations

i) Vector Addition: $(a_1, a_2, \ldots, a_n)+(b_1, b_2, \ldots b_n)$
$=(a_1+b_1, \ldots$

ii) Scalar multiplication:

$$k(a_1, a_2, \ldots a_n) = (ka_1, ka_2, \ldots ka_n)$$

The zero vector in space K^n is an n-tuple of zeros, $0 = (0, 0, \ldots, 0)$ and the negative of vector is denoted by $-(a_1, a_2, \ldots, a_n) = (-a_1, -a_2, \ldots, -a_n)$

2) polynomial space. $p(t)$

Let $p(t)$ denotes the set of all polynomials $p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_s t^s$ $(s = 1, 2, \ldots)$ where $a_i \in$ a field K. Then $p(t)$ is a vector space over K using the following operations

i) vector Addition:

$p(t) + q(t)$ in $p(t)$ is the usual operation of addition of polynomials.

ii) Scalar multiplication:

~~k p(t)~~ $k\,p(t)$ in $p(t)$ is the usual operation the product of the scalar k and polynomial $p(t)$.

The zero polynomial 0 is the zero vector in $p(t)$

3) polynomial space $p(t)$

Let $P_n(t)$ be a set of all $p(t)$ over a field K, where the degree of $p(t)$ is less than or equal to n. That is $p(t) = a_0 + a_1 t + \dots + a_s t^s$

where $s \leq n$

Then $P_n(t)$ is a vector space over K. with respect to the usual operations of addition of polynomials and of multiplication of the polynimal by a constant. Be include the zero polynomial 0 as an element of $P_n(t)$. Even through its degree is undefined

4) Matrix space $M_{m,n}$:

Matrix space is a set of all $m \times n$ matrices with entries in a field K and is a vector space over K with respect the usual operations of matrix addition and scalar multiplication of matrices

5) Function Space $F(x)$.

Let X be a non-empty set and Let K be an arbitrary field. Let $F(x)$

denote the set of all functions of X into K.

Then $F(X)$ is a vector space over K with respect to the following operations:

Vector addition: The sum of two functions f and g in $F(X)$ is the function $f+g$ in $F(X)$ denoted by $(f+g)(x) = f(x) + g(x)$ $\forall x \in X$

ii) Scalar multiplication:

The product of a scalar $k \in K$ and a function f in $F(X)$ is the function kf in $F(X)$ denoted by $(kf)(x) = k.f(x)$ for all $x \in X$.

The zero vector in $F(X)$ is the zero function 0, which maps every $x \in X$ into the zero element $0 \in K$ ie; $0(x) = 0$ $\forall x \in X$.

For any function f in $F(X)$. The function $-f$ in $F(X)$ denoted by $(-f)(x) = -f(x)$ $\forall x \in X$ is the negative of the function f

6) Fields and Sub-Fields

Suppose the field E is a extension of field K. ie, Suppose E is a field that contains K

as a sub field. Then E may be viewed a vector space over K using the following operation

i) vector Addition

$u+v$ in E is the usual addition in E

ii) Scalar Multiplication

ku in E where $k \in K$ and $u \in E$, is the usual product of k and u as elements of E

Linear Combinations:

Let V be a vector space over a field K. A vector v in V is a linear combination of vectors $u_1, u_2, \ldots, u_m$ in V if there exists scalars $a_1, a_2, \ldots a_m$ in K such that $v = a_1 u_1 + a_2 u_2 + \ldots + a_m u_m$. alternatively linear combination of $u_1, u_2 \ldots u_m$ if there is a solution to the vector equations $v = x_1 u_1 + x_2 u_2 + \ldots + x_m u_m$

where $x_1, x_2, \ldots, x_m$ are unknown scalars

Eg:

For linear

Suppose we want to express $v = (3, 7, -4)$ in R^3 as a linear combination of the vectors

$u_1 = (1, 2, 3)$, $u_2 = (2, 3, 7)$; $u_3 = (3, 5, 6)$

we must find scalars x, y, z, such that

$v = x u_1 + y u_2 + z u_3$ ⟶ ①

ie, $(3, 7, -4) = x(1, 2, 3) + y(2, 3, 7) + z(3, 5, 6)$

$$\Rightarrow \quad x + 2y + 3z = 3$$
$$2x + 3y + 5z = 7$$
$$3x + 7y + 6z = -4$$

By solving this system of equation we can get $x = 2$; $y = -4$, $z = 3$.

By eqn ①

$v = x u_1 + y u_2 + z u_3$

$v = 2u_1 - 4u_2 + 3u_3$

Subspaces

Let V be an vector space over a field K and let W be a subset of V. Then W is a subspace of V if W is itself a vector space over

K with respect to the operations of

vector addition and scalar multiplication on V.

The way in which one shows that any set W is a vector space is to show that W satisfies the 8 axioms of vector space

If W is a subset of the vector space V then the some of the axioms automatically hold in W, since they already hold in V.

The simple criteria for identifying the sub space is given in the following theorem

Theorem:

Suppose W is a subset of a vector space V. Then W is a subspace of V if the following two conditions hold

a) The zero vector 0 belongs to W

b) For every $u, v \in$ ~~to~~ W, $k \in K$

i) vector addition

$u + v \in W$

ii) Scalar multiplication

$ku \in W$

The property i) in b) states that W is

closed under vector addition and propery ii) condition b) states that W is closed under scalar multiplication.

By combining both these properties the condition b) can be re written as. for every $u, v \in W$ & $a, b \in K$, the linear combination $au + bv \in W$.

Example:

Consider the vector space $V = R^3$.

a) Let U consists of all vectors in R^3 whose entries are equal ie, $U = \{(a, b, c) : a = b = c\}$

for example:

$(1, 1, 1), (-3, -3, -3), (7, 7, 7)$ and $(-2, -2, -2)$

are vectors in U. Geometrically U is the line through the origin 0 and the point $(1, 1, 1)$.

clearly $0 = (0, 0, 0) \in U$, since all the entries are equal further suppose u, v are arbitrary vectors in U, say $u = (a, a, a)$ & $v = (b, b, b)$.

Then for any scalar $k \in R$. The following are all vectors in U.

$u+v = \{a+b, a+b, a+b\}$

& $ku = \{ka, ka, ka\}$

Thus U is a subspace of R^3

b) Let W be any plane in R^3 passing through the origin. Then $0 = (0,0,0) \in W$, since we assumed W passes through the origin O.

Suppose u and v are vectors in W. The sum and the multiple are $u+v$ and ku of u also lie in the plane W. Thus W is the subspace of R^3.

Intersection of space:

Let U and W be subspaces of V. The intersection $U \cap W$ is also a subspace of V. Clearly $0 \in U$ & $0 \in W$, since U & W are subspaces where $0 \in U \cap W$. Suppose u & $v \in U \cap W$. Then u and $v \in W$. Since U and W are subspaces for any scalars $a, b \in k$, $au+bv \in U$ and $au+bv \in W$. Thus $au+bv \in U \cap W$. Therefore $U \cap W$ is a subspace of

Theorem

The intersection of any number of subspace of a vector space V. is a also a Subspace of V.

Solution space of a homogeneous system:

Consider a system $AX = B$ of a linear equation in 'n' unknowns. Then every solution u may be viewed as a vector in K^n. Thus the solution set of such a system is a subset of K^n. Suppose the system is homogeneous ie, the system has the form $AX = 0$.

Let W be its solution set. Since $A0 = 0$, The zero vector $0 \in W$.

Suppose u & $v \in W$. then u & v are the solution of $AX = 0$ In other words, $Au = 0$ and $Av = 0$. $\therefore$ for any scalars a and b $A(au+bv)$. we have $A(au+bv) = aAu + bAv$

$$= a0 + b0$$

$$= 0 + 0$$

$$= 0$$

Thus $au + bv \in W$, since it is a solution of $Ax = 0$. Accordingly W is a subspace of K^n

Theorem:

The solution set W of a homogeneous system $Ax=0$ in 'n' unknowns is a subspace of K^n.

The solution set of a non-homogeneous system $Ax=B$ is not a subspace of K^n. In fact, the zero vector $0 \notin$ its solution set.

Basis and Dimension:

Def: 1

A set $S = \{u_1, u_2, \dots u_n\}$ of vectors is a basis of vector V, if it has the following two properties.

i) S is linearly independent

ii) S spans V

Def: 2

A set $S = \{u_1, u_2, \dots u_n\}$ of vectors is a basis of vector V, if every $v \in V$ can be written uniquely as a linear combination the basis vectors.

Theorem:

Let V be a vector space & one basis has m elements and another basis has n elements then $m=n$.

Dimension:

A vector space V is said to be finite

dimension n or n dimensional (written as $\dim V = n$)

if V has a basis with n elements.

The vector space $\{0\}$ is defined to have dimension 0. Suppose a vector space V does not have a finite basis. Then V is said to be of infinite dimension or to be infinite dimensional.

The above theorem is a consequence of the following lemma

Lemma:

Suppose $\{v_1, v_2, \dots, v_n\}$ spans V, and suppose $\{w_1, w_2, \dots, w_m\}$ linearly independent then $m \leq n$ and V is spanned by a set of the form $\{w_1, w_2, \dots w_m\}$ $\{v_{i_1}, v_{i_2}, \dots v_{i_{n-m}}\}$. Thus n+1 are more vectors in V linearly independent.

Example of bases:

1. vector space K^n

Consider the following n vectors in K^n

$e_1 = (1, 0, 0, \dots, 0)$

$e_2 = (0, 1, 0, 0, \dots, 0)$

$\vdots$

$e_n = (0, 0, 0, 0, \dots, 1)$

These vectors are linearly independent.

Any vector $u = (a_1, a_2, \ldots, a_n)$ in K^n can be written as linear combination of the above vectors $e_1, e_2, \ldots e_n$ as $v = a_1e_1 + a_2e_2 + \ldots + a_ne_n$

Accordingly the vectors form a basis of K^n called the usual or standard basis of K^n. Thus K^n has dimension n. In particular, any other basis of K^n has n elements

2. Vector space M

Vector space $M = M_{r,s}$ of all $r \times s$ matrices

The following 6 matrices form a basis of the vector space $M_{2,3}$ of all 2×3 matrices over K:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

More generally in vector space $M = M_{r,s}$ matrices

Let E_{ij} be the matrix with ij entry 1 and 0's elsewhere.

Then all such matrices form a basis of $M_{r,s}$ called the usual or standard basis of $M_{r,s}$.

Accordingly $\dim M_{r,s} = rs$.

3. Vectors space $P_n(t)$ of all polynomials of degree $\leq n$.

The set $S = \{1, t, t^2 \dots t^n\}$ of $n+1$ polynomials is a basis of $P_n(t)$. Specifically any polynomial $f(t)$ of degree $\leq n$ can be expressed as a linear combination of these powers of t and these polynomials are linearly independent

$\therefore \dim P_n(t) = n+1$.

4. Vector space $P(t)$ of all polynomials

Consider any finite set $S = \{f_1(t), f_2(t) \dots f_m(t)\}$ of polynomials in $P(t)$, and let m denotes the largest of the degrees of the polynomials.

Then any polynomial $g(t)$ of degree exceeding m cannot be expressed as a linear combination of the elements of S. Thus S cannot be a basis of $P(t)$.

ie, The dimension of $P(t)$ is infinite.

Theorem 1:

Let V be a vector space of dimension n. Then

i) Any $n+1$ or more vectors in V are linearly dependent.

ii) Any linearly independent set $S = \{u_1, u_2, \dots u_n\}$ with n elements is a basis of V.

iii) Any spanning set $T = \{v_1, v_2, \ldots v_n\}$ of V with n elements is a basis of V.

Theorem 2:

Suppose S spans a vector space V. Then

i) Any maximum number of linearly independent vectors is of form a basis of V.

ii) Suppose one deletes from S every vector that is a linear combination of preceding vectors in S. Then the remaining vectors form a basis of V.

Theorem 3:

Let V be a vectorspace of finite dimension and let $S = \{u_1, u_2, \ldots, u_n\}$ be a set of linearly independent vectors in V. Then S is part of a basis of V. That is S may be extended to a basis of V.

Eg:

a) The following four vectors in R^4 form a matrix in echelon form

$(1,1,1,1)$ $(0,1,1,1)$ $(0,0,1,1)$ $(0,0,0,1)$

Thus the vectors are linearly independent and since $\dim R^4 = 4$, the four vectors form a basis of R^4.

b) The following n+1 polynomials in $P_n(t)$ are of increasing degree:

$1, t-1, (t-1)^2, \ldots (t-1)^n$

Therefore no polynomial is a linear combination of preceding polynomials, Hence the polynomials are linearly independent. Furthermore, they form a basis of $P_n(t)$. Since $\dim P_n(t) = n+1$

c) consider any four vectors in R^3, say

$(257, -132, 58), (43, 0, -17), (521, -317, 94), (328, -512, -731)$

By theorem ①, the four vectors must be linearly dependent, since they come the 3-dimensional vector space R^3.

Inner product spaces:

Def:

Let V be a real vector space. Suppose to each pair of vectors $u, v \in V$ there assigned a real number, denoted by $\langle u, v \rangle$. This function is called a real inner product on V, if it satisfies the following axioms

1. Linear property

$\langle a u_1 + b u_2, v \rangle = a\langle u_1, v \rangle + b\langle u_2, v \rangle$

2. Symmetric property:

$\langle u, v \rangle = \langle v, u \rangle$

3. Positive definite property

$\langle u, u \rangle \geq 0$; $\langle u, u \rangle = 0$ iff $u = 0$

vector space V with a inner product is called a real inner product space

Norm of a vector

By the 3rd axiom of an inner product is non-negative $\langle u, u \rangle \geq 0$ for any vector thus, its positive square root exists we use the notation.

$$\|u\| = \sqrt{\langle u, u \rangle}$$

This non negative number is called a norm or length of u.

1. If $\|u\| = 1$. $\langle u, u \rangle = 1$, then u is called a unit vector and is said to be normalized

2. Every non-zero vector v in V can be multiplied by the reciprocal of its length to obtain the unit vector

$$\hat{v} = \frac{1}{\|v\|} \cdot v$$

which is positive multiple of V.

This process is called normalizing V.

Eg:

Euclidean n-space R^n

consider the vector space R^n. The dot product or scalar product in R^n is defined by

$$u \cdot v = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

where $u = (a_i)$, $v = (b_i)$.

This function defines a inner product on R^n.

The norm $\|u\|$ of the vector $u = a_i$ in this space is as follows.

$$\|u\| = \sqrt{u \cdot u} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.$$

On the other hand by pythagorean theorem, the distance from the origin O in R^3 to a point $P(a, b, c)$ is given by $\sqrt{a^2+b^2+c^2}$. Since the pythagorean theorem is the consequences of the axioms of Euclidean geometry, the vector space or thogon with the above inner product and norm is called Euclidean n-space.

Eg:

Let $u = (1, 3, -4, 2)$

$v = (4, -2, 2, 1)$

$w = (5, -1, -2, 6)$ in R^4

a) Show $\langle 3u - 2v, w\rangle = 3\langle u, w\rangle - 2\langle v, w\rangle$

By Definition,

$\langle u, w\rangle = 5 - 3 + 8 + 12 = 22$ and

$\langle v, w\rangle = 20 + 2 - 4 + 6 = 24$

Note that, $3u - 2v = (-5, 13, -16, 4)$ Thus

$\langle 3u-2v, w\rangle = -25-13+32+24 = 18$

As expected, $\langle 3u, w\rangle - 2\langle u, w\rangle = 3(22) - 2(22)$

$= 18 = \langle 3u - 2v, w\rangle$

b) Normalize u and v

By definition,

$$\|u\| = \sqrt{1+9+16+4} = \sqrt{30} \text{ and}$$

$$\|v\| = \sqrt{16+4+4+1} = 5$$

We normalize u and v to obtain the following unit vectors in the directions of u and v respectively

$$\hat{u} = \frac{1}{\|u\|} u = \left(\frac{1}{\sqrt{30}}, \frac{3}{\sqrt{30}}, \frac{-4}{\sqrt{30}}, \frac{2}{\sqrt{30}}\right) \text{ and}$$

$$\hat{v} = \frac{1}{\|v\|} \cdot v = \left(\frac{4}{5}, \frac{-2}{5}, \frac{2}{5}, \frac{1}{5}\right)$$

Function space $C[a,b]$ and polynomial space $P(t)$:

The notation $C[a,b]$ is used to denote the vector space of all continuous functions on the closed interval $[a,b]$, that is, where $a \le t \le b$. The following defines an inner product on $C[a,b]$ where $f(t)$ and $g(t)$ are functions in $C[a,b]$.

$$\langle f, g\rangle = \int_a^b f(t)\, g(t)\, dt.$$

It is called the usual inner product on $C[a,b]$.

The vector space $P(t)$ of all polynomials is a subspace of $C[a,b]$ and hence the above is also an inner product on $P(t)$.

Eg:

Consider $f(t) = 3t - 5$ and $g(t) = t^2$ in the polynomial space $p(t)$ with inner product $\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt$.

a) Find $\langle f, g \rangle$

b) Find $\|f\|$ & $\|g\|$

a) $\langle f, g \rangle = \int_0^1 (3t-5)(t^2)\, dt$

$= \int_1^0 (3t^3 - 5t^2)\, dt$

$= \left[\frac{3t^4}{4} - \frac{5t^3}{3} \right]_0^1$

$= \frac{3}{4} - \frac{5}{3}$

$= \frac{9-20}{12}$

$\langle f, g \rangle = \frac{-11}{12}.$

$(3t-5)^2$

$9t^2 - 2(3t)(5)$

$30t$

b) $\|f\|^2 = \langle f, f \rangle = \int_0^1 f(t)^2\, dt$

$= \int_0^1 (9t^2 - 30t + 25)\, dt$

$= \left[9\frac{t^3}{3} - 30\frac{t^2}{2} + 25t \right]_0^1$

$\|f\|^2 = 13$

$\frac{9}{3} - 15 + 25$

$\|f\| = \sqrt{13}$

$3 - 15 + 25$

$3 + 10$

$\|g\|^2 = \langle g, g \rangle = \int_0^1 g(t)^2\, dt$

$= \int_0^1 (t^2)^2\, dt$

$$= \int_0^1 (t^4)\, dt$$

$$= \frac{t^5}{5}$$

$$\|g\|^2 = \frac{1}{5}$$

$$\|g\| = \sqrt{\frac{1}{5}}$$

Matrix space

$M = M_{m,n}$

Let $M = M_{m,n}$, the vector space of all $m \times n$ matrices. An inner product is defined on M by $\langle A, B \rangle = tr(B^T . A)$ where $tr()$ is the trace ie, the sum of the diagonal elements. if $A = [a_{ij}]$ & $B = [b_{ij}]$, then $\langle A, B \rangle = tr(B^T . A)$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\, b_{ij}$$

The norm of A^2 $\|A\|^2 = \langle A . A \rangle$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2$$

ie, $\langle A, B \rangle$ is the sum of the products of the corresponding entries in A and B. $\langle A, A \rangle$ is the sum of the squares of the entries of A.

Hillbert space.

Let V be the vector space of all infinite sequences of real numbers satisfying $\sum_{i=1}^{\infty} a_i^2 = a_1^2 + a_2^2 + \dots < \infty$. ie, the sum converges.

Addition and scalar multiplication are defined in V component-wise, ie, if $u = (a_1, a_2, \dots)$ and $v (b_1, b_2, \dots)$ then $u+v = (a_1+b_1, a_2+b_2, \dots$

and $ku = k(a_1, a_2, \dots)$

$= ka_1, ka_2, \dots$

An inner product is defined in V by $\langle u, v \rangle = a_1 b_1 + a_2 b_2 + \dots$ The above sum converges absolutely for any pair of points V.

Hence the inner product is well defined. This inner product space is called hillbert space or l_2- space.

Linear Transformation:

Cauchy - Schwarz inequality

Statement:

For any vectors u & v in an inner product V,

$\langle u, v \rangle^2 \leq \langle u, u \rangle \langle v, v \rangle$

or

$|\langle u, v \rangle| \leq \|u\| . \|v\|$

Eg:

Consider real vectors $u = (a_1, a_2, \ldots, a_n)$ & $v = (b_1, b_2, \ldots, b_n)$

$$\langle u, v\rangle^2 = (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

$$\|u\|^2 = a_1^2 + a_2^2 + \ldots + a_n^2$$

$$\|v\|^2 = b_1^2 + b_2^2 + \ldots + b_n^2$$

$$\langle u, v\rangle^2 \leq (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2)$$

$$\leq \|u\|^2 \cdot \|v\|^2$$

$$\Rightarrow |\langle u, v\rangle| \leq \|u\| \cdot \|v\|$$

Eg:

Let f and g be continuous functions on the unit interval $[0, 1]$. Then by Cauchy Schwarz inequality $\left[\int_0^1 f(t)\, g(t)\, dt\right]^2 \leq \int_0^1 f^2(t)\, dt \cdot \int_0^1 g^2(t)\, dt$

i.e., $[\{\langle f, g\rangle\}]^2 \leq \|f\|^2 \cdot \|V\|^2$

where V is the inner product space $C[0, 1]$

Theorem

Let V be a inner product space. Then the norm in V satisfies the following properties

1. $\|v\| \geq 0$; $\|v\| = 0$ iff $v = 0$
2. $\|kv\| = |k| \cdot \|v\|$
3. $\|u + v\| \leq \|u\| + \|v\|$

The 3rd property is called triangular property because if $v, u, u+v$ as the side of the triangle with sides u and v, then the property 3 states that the length of one side of a triangle cannot be greater than the sum of lengths of the other 2 sides.

Angle between vectors.

For any non-zero vectors u and v in an inner product space V, the angle between u and v is defined to be the angle θ ∋ $0 \le \theta \le \pi$ and $\cos\theta = \frac{\langle u, v\rangle}{\|u\| \cdot \|v\|}$ by cauchy schwarz inequality $-1 \le \cos\theta \le +1$ and so the angle exists and is unique.

Eg:

1. Consider vectors $u = (2, 3, 5)$ and $v = (1, -4, 3)$ in R^3 then find the angle between u and v

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\| \|v\|}$$

where,

$$\langle u, v\rangle = 2 - 12 + 15$$

$$= 5$$

$$\|u\| = \sqrt{2^2 + 3^2 + 5^2}$$

$$= \sqrt{38}$$

$$\|v\| = \sqrt{1 + 16 + 9}$$

$$= \sqrt{26}$$

$$\cos\theta = \frac{5}{\sqrt{38}\cdot\sqrt{26}}$$

Note

* θ is an acute angle since $\cos\theta$ is positive.

2. Let $f(t) = 3t-5$, $g(t) = t^2$ in the polynomial space $P(t)$ with inner product $\langle f, g\rangle = \int_0^1 f(t)\cdot g(t)\, dt$

$\langle f, g\rangle = -11/12$, $\|f\| = \sqrt{13}$ $\|g\| = \sqrt{1/5}$

$$\cos\theta = \frac{-11/12}{\sqrt{13}\sqrt{1/5}}$$

$$= \frac{-11\sqrt{5}}{12\sqrt{13}}$$

Since $\cos\theta$ is negative. θ is an obtuse angle

Linear transformation or Linear mapping:

Let U and V be vector spaces over the same field K. A mapping $F: V \to U$ is called a linear mapping or linear transformation. if it satisfies the following 2 conditions

i) For any vectors u & $w \in V$, $F(u+w) = F(u) + F(w)$

ii) For any scalar k, the vector $v \in V$.

$F(kv) = kF(v)$

In words $F: V \to U$ is linear if it 'preserves' the two basic operations of vector space that of vector addition and scalar multiplication.

A linear mapping $F: V \to U$ is completely characterized by the condition $F(au + bv) = aF(u) + bF(v)$.

The term linear transformation rather than linear mapping is frequently used for linear mapping of the form $F: R^n \to R^m$

Eg1:

a) Let $F: R^3 \to R^3$ be the "projection" mapping into xy-plane that is, F is the mapping defined by $F(x, y, z) = F(x, y, 0)$. We show that F is linear. Let $v = (a, b, c)$ and $w = (a', b', c')$ then

$$F(v+w) = F(a+a', b+b', c+c') = (a+a', b+b', 0)$$
$$= (a, b, 0) + (a', b', 0) = F(v) + F(w)$$

and for any scalar k

$$F(kv) = (ka, kb, kc) = (ka, kb, 0) = k(a, b, 0) = kF(v)$$

Thus F is linear.

b) Let $G: R^2 \to R^2$ be the "translation" mapping defined by $G(x, y) = (x+1, y+2)$. [That is, G adds the vector $(1, 2)$ to any vector $v = (x, y)$ in R^2]. Note that

$G(0) = G(0,0) = (1,2) \neq 0$

Thus, the zero vector is not mapped in to the zero vector. Hence G is not linear.

Eg : 2

(Derivative and Integral mappings) consider the vector space $V = P(t)$ of polynomials over the real field R. Let $u(t)$ and $v(t)$ be any polynomials in V and let k be any scalar.

a) Let $D : V \to V$ be the derivative mapping. One proves in calculus that.

$$\frac{d(u+v)}{dt} = \frac{du}{dt} + \frac{dv}{dt} \quad \text{and} \quad \frac{d(ku)}{dt} = k\frac{du}{dt}$$

That is, $D(u+v) = D(u) + D(v)$ and $D(ku) = kD(u)$.

Thus the derivative mapping is linear.

b) Let $J : V \to R$ be an integral mapping, say

$$J(f(t)) = \int_0^1 f(t)\,dt$$

One also proves in calculus that,

$$\int_0^1 [u(t) + v(t)]\,dt = \int_0^1 u(t)\,dt + \int_0^1 v(t)\,dt$$

and

$$\int_0^1 ku(t)\,dt = k\int_0^1 u(t)\,dt$$

That is, $J(u+v) = J(u) + J(v)$ and $J(ku) = kJ(u)$

Thus the integral mapping is linear.

Eg 3: (zero and Identity Mappings)

a) Let $F: V \to U$ be the mapping that assigns the zero vector $0 \in U$ to every vector $v \in V$. Then, for any vectors $v, w \in V$ and any scalar $k \in K$, we have

$$F(v+w) = 0 = 0+0 = F(v) + F(w) \text{ and}$$

$$F(kv) = 0 = k0 = kF(v)$$

Thus F is linear. We call F the zero mapping, and we shall usually denote it by 0.

b) Consider the identity mapping $I: V \to V$, which maps each $v \in V$ into itself. Then, for any vectors $v, w \in V$ and any scalars $a, b \in K$, we have

$$I(av+bw) = av+bw = aI(v) + bI(w)$$

Thus I is linear.

Theorem:

Let V and U be vector spaces over a field K. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of V and let $u_1, u_2, \ldots u_n$ be any vectors in U. Then there exists a unique linear mapping $F: V \to U$ ∋

$$F(v_1) = u_1, \; F(v_2) = u_2, \ldots \; F(v_n) = u_n$$

Matrices as linear mapping

Let A be any real $m \times n$ matrix

It is none that A determines a mapping

$F_A : K^n \to K^m$ by $F_A(u) = Au$ where the vectors

in K^n and K^m are written as columns.

By matrix multiplication $F_A(v+w) = A(v+w)$

$= Av + Aw$

$= F_A(v) + F_A(w)$

It is a vector addition.

$F_A(kv) = A(kv) = k(Av) = k \cdot F_A(v)$

It is a scalar multiplication.

In other words, using A to represents the

mapping

$A(v+w) = Av + Aw$

& $A(kv) = k(Av)$

Thus matrix mapping A is linear

vector space isomorphism:

Definition:

Two vector spaces V and U over

field K are isomorphic written as $V \cong U$.

If there exists a bijective (one-one and onto) linear mapping

$F : V \to U$

The mapping F is called an isomorphism between V and U

Eg:

Consider any vector space V of dimension n and let S be any basis of V. Then the mapping

$v \mapsto [v]_S$ which maps each vector $v \in V$ into its co-ordinate vector is an isomorphism between V and K^n