UNIT - III

Riemann Integrability

Partitions and Riemann (or Darboux) Sums

(i) Partition (or dessection or net) of a closed interval.

Let $I = [a, b]$ be a finite closed interval. If $a = x_0 < x_1 < x_2 < \dots x_n = b$, then the finite ordered set $P = \{x_0, x_1, x_2 \dots x_n\}$ is known as partition of I. The n closed intervals $I_1 = [x_0, x_1]$, $I_2 = [x_1, x_2]$ $I_r = [x_{r-1}, x_r] \dots I_n = [x_{n-1}, x_n]$ determined by P are known as the segments of the partition P. The length of the r^{th} sub-interval $I_r = [x_{r-1}, x_r]$ is denoted by δ_r. Thus

$\delta_r = x_r, x_{r-1}$

By changing the partition points, the partition can be changed and therefore, we can obtain an infinite number of partitions of the interval $[a, b]$. The set (or family) of all partitions of $[a, b]$ is denoted by $P[a, b]$.

Note:

$$\sum_{r=1}^{n} \delta_r = \delta_1 + \delta_2 + \dots \delta_n$$

$$= (x_1 - x_0) + (x_2 - x_1) + \dots (x_n - x_0)$$

$$= x_n - x_0$$

$$= b - a$$

(ii) Norm of a partition

The length of the greatest of all the intervals $[x_{r-1}, x_r]$ of the partition P will be called its norm and is denoted by $\|P\|$ or $\mu(P)$. Thus

$$\|P\| = \mu(P) = \max\{\delta_r : 1 \leq r \leq n\}$$

(iii) Refinement of a partition:

If P_1, P_2 be two partitions of $[a,b]$ such that $P_2 \overset{\text{contains (is a superset of)}}{\supseteq} P_1$, i.e., every point of the partition P_1 is as well as point of P_2, i.e., P_2 is finer than P_1 (or P_2 is refinement of P_1)

If P_1, P_2 are two partition of $[a,b]$ then $P_1 \overset{\text{union}}{\cup} P_2$ is called a common refinement of P_1 and P_2

(iv) Upper and lower (Riemann or Darboux) sums

The function f which is defined in $[a,b]$ is also necessarily bounded in In each sub-interval I_r. Let M_r, m_r be the bounds (i.e., supremum and the infimum of f in I_r. i.e., in $[x_{r-1}, x_r]$

Then the two sums ~~with be~~ of f corresponding to the partition P will be, Upper Darboux sum or Upper Riemann Sum

$$U(P,f) = M_1\delta_1 + M_2\delta_2 + \cdots M_n\delta_n = \sum_{r=1}^{n} M_r\delta_r$$

Lower Darboux sum or Lower Riemann Sum

$$L(P,f) = m_1\delta_1 + m_2\delta_2 + \cdots m_n\delta_n = \sum_{r=1}^{n} m_r\delta_r$$

Note: For every portion P of $[a,b]$,

we have $\quad L(P,f) \leq U(P,f)$

since for $\forall r \quad m_r \leq M_r \Rightarrow m_r\delta_r \leq M_r\delta_r$

$$\Rightarrow \sum_{r=1}^{n} m_r\delta_r \leq \sum_{r=1}^{n} M_r\delta_r$$

$$\Rightarrow L(P,f) \leq U(P,f)$$

(v) oscillatory sum:

$$U(P,f) - L(P,f) = \sum M_r\delta_r - \sum m_r\delta_r$$

$$= \sum (M_r - m_r)\delta_r$$

$$= \sum O_r\delta_r$$

Where O_r denotes the oscillation of the function in subinterval $[x_{r-1}, x_r]$. The sum $\sum O_r\delta_r$ is called the oscillatory sum and is denoted by $\omega(P,f)$. As $O_r \geq 0 \ \forall r$, each osillatory sum consists of the sum of a finite no. of non-negative terms.

Theorem 2.

Let f be a bounded function defined on $[a, b]$ and let m, M be the infimum and supremum of f on $[a, b]$. Then for any partition P of $[a, b]$, we have

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a)$$

Proof:

Let $P = \{a = x_0, x_1, x_2 \ldots x_{r-1}, x_r \ldots x_n = b\}$ be any portion of $[a, b]$. Since f is bounded on $[a, b]$, so f is bounded on each sub-interval $[x_{r-1}, x_r]$, $r = 1, 2, \ldots n$.

Let m_r and M_r be infimum and supremum of f on $[x_{r-1}, x_r]$. Then for every value of r, we have, $m \leq m_r \leq M_r \leq M$

$$\Rightarrow m\delta_r \leq m_r\delta_r \leq M_r\delta_r \leq M\delta_r$$

$$\Rightarrow \sum_{r=1}^{n} m\delta_r \leq \sum_{r=1}^{n} m_r\delta_r \leq \sum_{r=1}^{n} M_r\delta_r \leq \sum_{r=1}^{n} M\delta_r$$

$$\Rightarrow m\sum_{r=1}^{n}\delta_r \leq L(P, f) \leq U(P, f) \leq M\sum_{r=1}^{n}\delta_r$$

$$\Rightarrow m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a)$$

It follows that the numbers $L(P, f)$ and $U(P, f)$ form a bounded set.

Example: compute $L(P,f)$ and $U(P,f)$ if 5

(i) $f(x) = x^2$ on $[0,1]$ and

$P = \{0, 1/4, 2/4, 3/4, 1\}$ be a partition of $[0,1]$

HW (ii) $f(x) = x$ on $[0,1]$ and

$P = \{0, 1/3, 2/3, 1\}$ be a partition of $[0,1]$

HW (iii) $f(x) = x$ on $[0,1]$ and

$P = \{0, 1/4, 2/4, 3/4, 1\}$ on $[0,1]$

Solution: (i) $f(x) = x^2$; $P\{0, 1/4, 2/4, 3/4, 1\}$

(i) Here partition P divides $[0,1]$ into sub. intervals

$I_1 = [0, 1/4]$ $I_2 = [1/4, 2/4]$ $I_3 = [2/4, 3/4]$ $I_4 = [3/4, 1]$

The length of the intervals are given by

$\delta_1 = \delta_2 = \delta_3 = \delta_4 = 1/4$

Let M_r and m_r be the supremum and infimum of f in interval I_r, $r = 1, 2, 3, 4 \ldots$

Since $f(x) = x^2$ is increasing on $[0,1]$, we have

$m_1 = 0$, $M_1 = 1/16$

$m_2 = 1/16$, $M_2 = 4/16$

$m_3 = 4/16$, $M_3 = 9/16$

$m_4 = 9/16$ $M_4 = 1$

$\therefore L(P,f) = \sum m_r \delta_r = m_1\delta_1 + m_2\delta_2 + m_3\delta_3 + m_4\delta_4$

$= [0 \times 1/4] + [1/16 \times 1/4] + [4/16 \times 1/4] + [9/16 \times 1/4]$

$= 0 + \frac{1}{64} + \frac{1}{16} + \frac{9}{64}$

$= \frac{1+4+9}{64} = \frac{14}{64} = \frac{7}{32}$

$$U(P, f) = \sum_{r=1}^{n} M_r \delta_r = M_1 \delta_1 + M_2 \delta_2 + M_3 \delta_3 + M_4 \delta_4$$

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$$= \left[\frac{1}{16} \times \frac{1}{4}\right] + \left[\frac{4}{16} \times \frac{1}{4}\right] + \left[\frac{9}{16} \times \frac{1}{4}\right] + \left[1 \times \frac{1}{4}\right]$$

$$= \frac{15}{32}$$

Upper and lower Riemann integrals.

Let f be defined on $[a,b]$. Then for every partition P of $[a,b]$, we have

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a) \quad \text{(1)}$$

where m and M are the infimum and Supremum of f on $[a,b]$

From (1) it follows that the set

$$\{L(P,f) : P \text{ in a partition of } [a,b]\}$$

of lower sum is bounded above by $M(b-a)$ and, consequentey has the least upper bound.

Ilrly from (1) it also follows that the set

$$\{U(P,f) : P \text{ in a partition of } [a,b]\}$$

of upper sum is bounded below by $m(b-a)$ and consequentey has the ~~least~~ greatest lower bound.

Definition:

Upper integral: The infimum of the set of the upper sums $U(P,f)$ is called the upper integral of f over $[a,b]$ and is denoted by $\overline{\int_a^b} f(x)\,dx = \operatorname{Inf}\{U(P,f) : P \text{ is a partition of } [a,b]\}$

Lower integral: The supremum of the set of the lower sums $L(P,f)$ is called the lower integral of f over $[a,b]$ and is denoted by $\underline{\int_a^b} f(x)\,dx = \sup\{L(P,f) : P \text{ is a partition of } [a,b]\}$

Riemann integrable: A bounded function f is said to be Riemann integrable or simply integrable over $[a,b]$, if its upper and lower integral are equal and is denoted by

$$\int_a^b f(x)\,dx$$

The family (or class) of all bounded functions which are Riemann integrable on $[a,b]$ is denoted by $R[a,b]$. If f is R-integrable on $[a,b]$ we express it by writing $f \in R[a,b]$ or R simply.

Note:

$\rightarrow$ a : lower ~~limit~~ limit, b : upper limit

$\rightarrow$ $\underline{\int_a^b} f(x)\,dx$ (or) $\underline{\int_a^b} f$ $\qquad \rightarrow \int_a^b f(x)\,dx = \int_a^b f$

$\rightarrow$ $\overline{\int_a^b} f(x)\,dx$ or $\overline{\int_a^b} f$

Theorem I: The lower R-integral cannot exceed the upper R-integral i.e., (8)

$$\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$$

Proof:

Let $P[a,b]$ denote the set of all partitions of $[a,b]$

Let $P_1, P_2 \in P[a,b]$. Since no lower sum can exceed any upper sum, we have

$$L(P_1, f) \le U(P_2, f) \longrightarrow (1)$$

(1) is true for each $P_1 \in P[a,b]$ keeping P_2 fixed, we see that the set

$\{L(P_1, f) : P_1 \in [a,b]\}$ has an upper bound $U(P_2, f)$. Again we know that

$$\underline{\int_a^b} f(x)\,dx = \sup\{L(P_1, f) : P_1 \in [a,b]\}$$

But ~~st~~ supremum is $\le$ any upper bound, Hence we get

$$\underline{\int_a^b} f(x)\,dx \le U(P_2, f) \longrightarrow (2)$$

which is true for each $P_2 \in P[a,b]$. From (2) it follows that the set $\{U(P_2, f) : P_2 \in P[a,b]\}$ has a lower bound $\underline{\int_a^b} f(x)\,dx$. Also W.K.T

$$\overline{\int_a^b} f(x)\,dx = \inf\{U(P_2, f) : P_2 \in [a,b]\}$$

But any lower bound $\le$ Infimum, Hence we get

$$\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx.$$

*: Necessary and sufficient condition for Integrability:

Theorem: A necessary and sufficient condition for the integrability of a bounded function f is, that to every $\epsilon > 0$, there corresponds $\delta > 0$ such that for every partition P, whose norm is $\leq \delta$, the oscillatory sum $w(P,f)$ is $< \epsilon$

i.e., $U(P,f) - L(P,f) < \epsilon$

Proof: The condition is necessary.

The bounded function f being integrable.

$$\int_a^{-b} f(x)\,dx = \int_{-a}^{b} f(x)\,dx = \int_a^b f(x)\,dx \longrightarrow (1)$$

Let ϵ be any positive integer, Therefore there exist $\delta > 0$ such that for every partition P whose norm is $\leq \delta$

$$U(P,f) < \int_a^{-b} f(x)\,dx + \frac{\epsilon}{2} = \int_a^b f(x)\,dx + \frac{\epsilon}{2} \longrightarrow (2)$$

and $$L(P,f) > \int_{-a}^{b} f(x)\,dx - \frac{\epsilon}{2} = \int_a^b f(x)\,dx - \frac{\epsilon}{2} \longrightarrow (3)$$

Multiply (−) sign in both side of equation (3)

$$-L(P,f) < \int_{-a}^{b} f(x)\,dx - \frac{\epsilon}{2} \longrightarrow (4)$$

Now by adding (2) and (4) we get,

$$U(P,f) - L(P,f) < \int_a^{-b} f(x)\,dx + \frac{\epsilon}{2} - \int_{-a}^{b} f(x)\,dx - \frac{\epsilon}{2}$$

$$\Rightarrow \quad w(P,f) = U(P,f) - L(P,f) < \epsilon$$

For every partition P with $\|P\| \leq \delta$

Hence the condition is necessary.

Sufficient Condition.

Let ϵ be any positive number. There exist a partition P such that

$$U(P,f) - L(P,f) = \left[U(P,f) - \int_a^{-b} f(x)\,dx\right] + \left[\int_a^{-b} f(x)\,dx - \int_{-a}^{b} f(x)\,dx\right] + \left[\int_{-a}^{b} f(x)\,dx - L(P,f)\right] < \epsilon$$

Each one of the three numbers

$$U(P,f) - \int_a^{-b} f(x)\,dx, \quad \int_a^{-b} f(x)\,dx - \int_{-a}^{b} f(x)\,dx,$$

$$\int_{-a}^{b} f(x)\,dx - L(P,f)$$

being non-negative, it follows that

$$0 < \int_a^{-b} f(x)\,dx - \int_{-a}^{b} f(x)\,dx < \epsilon$$

As ϵ is an arbitrary positive number, we see that the non-negative number

$$\int_a^{-b} f(x)\,dx - \int_{-a}^{b} f(x)\,dx$$

is less than every positive number and hence

$$\int_a^{-b} f(x)\,dx - \int_{-a}^{b} f(x)\,dx = 0$$

$$\Rightarrow \int_a^{-b} f(x)\,dx = \int_{-a}^{b} f(x)\,dx$$

So that f is integrable

Hence the theorem.

Algebra of Integrable functions.

1) Integral of sum of functions.

Statement:

If f and $g \in R[a,b]$, then $(f+g) \in R[a,b]$

and

$$\int_a^b (f+g)\,dx = \int_a^b f\,dx + \int_a^b g\,dx$$

Proof:

Since $f \in R[a,b]$ and $g \in R[a,b]$ for a given $\varepsilon > 0$, there exist partitions P_1 and P_2 of $[a,b]$ such that

$$\left.\begin{array}{l} U(P_1, f) - L(P_1, f) < \varepsilon/2 \\ U(P_2, f) - L(P_2, f) < \varepsilon/2 \end{array}\right\} \rightarrow (1)$$

Let $P = P_1 \cup P_2$ the P is a common refinement of P_1 and P_2 on $[a,b]$

Now $U(P, f+g) - L(P, f+g)$

$$\leq [U(P,f) + U(P,g)] - [L(P,f) + L(P,g)]$$

$$= [U(P,f) - L(P,f)] + [U(P,g) - L(P,g)]$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \longrightarrow (2)$$

Hence $f+g \in R[a,b]$

Further $U(P, f+g) \leq U(P,f) + U(P,g)$

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$\Rightarrow \inf U(P, f+g) \le \inf U(P,f) + \inf U(P,g)$

$$\Rightarrow \overline{\int_a^b} (f+g)\,dx \le \overline{\int_a^b} f\,dx + \overline{\int_a^b} g\,dx$$

$$\Rightarrow \int_a^b (f+g)\,dx \le \int_a^b f\,dx + \int_a^b g\,dx \longrightarrow (3)$$

($\because f, g$ + $f+g \in R[a,b]$)

Also

$$L(P, f+g) \ge L(P,f) + L(P,g)$$

$\Rightarrow \sup L(P, f+g) \ge \sup L(P,f) + \sup L(P,g)$

$$\Rightarrow \underline{\int_a^b} (f+g)\,dx \ge \underline{\int_a^b} f\,dx + \underline{\int_a^b} g\,dx$$

$$\Rightarrow \underline{\int_a^b} (f+g)\,dx \ge \int_a^b f\,dx + \int_a^b g\,dx \longrightarrow (4)$$

($\because f, g$ and $f+g \in R[a,b]$)

From (3) + (4) we have

$$\int_a^b (f+g)\,dx = \int_a^b f\,dx + \int_a^b g\,dx.$$

Hence the Theorem.

Integral of Scalar Multiplication

Theorem: If $f \in R[a,b]$ and $k \in R$ then $kf \in R[a,b]$ and $\int_a^b kf\,dx = k\int_a^b f\,dx$

Proof:

Since $f \in R[a,b]$, we have

$$\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f$$

and f is bounded on $[a,b]$

Now $|kf| = |k||f| \Rightarrow kf$ is bounded on $[a,b]$

Let $P = (a = x_0, x_1, x_2 \dots x_n = b)$ be any ~~positive~~ partition of $[a,b]$ and let m_r and M_r be the infimum and supremum of f in the sub-interval $[x_{r-1}, x_r]$ and $k > 0$. Then km_r and kM_r are infimum and supremum of kf.

$$\therefore \underline{\int_a^b} kf = \left[\sum_{r=1}^{n} km_r \Delta x_r\right] = k\sup\left[\sum_{r=1}^{n} m_r \Delta x_r\right]$$

$$= k\underline{\int_a^b} f = k\int_a^b f = k\overline{\int_a^b} f$$

$$= k\inf\left[\sum_{r=1}^{n} M_r \Delta x_r\right]$$

$$= \inf\left[\sum_{r=1}^{n} kf\, \Delta x_r\right] = \overline{\int_a^b} kf$$

Thus $\underline{\int_a^b} kf = \overline{\int_a^b} kf$

Hence, $kf \in R[a,b]$ and.

$$\int_a^b kf = k\int_a^b f$$

If $k<0$, then kM_r and km_r are the infimum and supremum of kf in the sub interval $[x_{r-1}, x_r]$

So,

$$\overline{\int_a^b} kf = \sup\left[\sum_{r=1}^{n} kM_r \Delta x_r\right]$$

$$= k\inf\left[\sum_{r=1}^{n} M_r \Delta x_r\right] \quad (\because k>0)$$

$$= k\overline{\int_a^b} f = k\int_{\bar{a}}^{b} f$$

$$= k\sup\left[\sum_{r=1}^{n} m_r \Delta x_r\right] = \inf\left[\sum_{r=1}^{n} km_r \Delta x_r\right]$$

$$= \underline{\int_a^b} kf \quad (\because k<0)$$

Thus $\overline{\int_a^b} kf = \underline{\int_a^b} kf$

Hence $kf \in R[a,b]$ and

$$\int_a^b kf = k\int_a^b f$$

Hence the theorem.

Integrability of a product functions.

Statement:

If f and g are Integrable over $[a,b]$ then fg is also integrable over $[a,b]$

Proof:

Since $f \in R[a,b]$ and $g \in R[a,b]$

$\therefore$ they are bounded on $[a,b]$

Let $M > 0$ be a real number such that

$|f(x)| \leq M$ and $|g(x)| \leq M \quad \forall x \in [a,b]$

$$\therefore |(fg)(x)| = |f(x)g(x)| \quad \forall x \in [a,b]$$

$$\leq M^2 \quad \forall x \in [a,b]$$

This shows that fg is bounded on $[a,b]$

Let $\epsilon > 0$ be given

Since $f \in [a,b]$ there exists a partition P_1 of $[a,b]$ such that

$$U(P_1, f) - L(P_1, f) < \frac{\epsilon}{2M}$$

llrly $g \in [a,b]$ there exists a partition P_2 of $[a,b]$ such that

$$U(P_2, g) - L(P_2, g) < \frac{\epsilon}{2M}$$

Let $P = P_1 \cup P_2$ be a refinement of P_1 and P_2

Then we have,

$$\left.\begin{aligned} U(P,f) - L(P,f) &< \frac{\epsilon}{2M} \\ U(P,g) - L(P,g) &< \frac{\epsilon}{2M} \end{aligned}\right\} \longrightarrow (1)$$

Let m_r, M_r, m_r', M_r', m_r'', M_r'' be the infimum and supremum of f, g and fg respectively over the Sub-Interval

$I_r = [x_{r-1}, x_r]$. Then for all $x, y \in I_r$,

we have,

$$|(fg)(x) - (fg)(y)| = |f(x)g(x) - f(y)g(y)|$$

$$= |f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y)|$$

$$= |g(x)[f(x) - f(y)] + f(y)[g(x) - g(y)]|$$

$$\leq |g(x)||f(x) - f(y)| + f(y)|g(x) - g(y)|$$

$$\leq M|f(x) - f(y)| + M|g(x) - g(y)| \longrightarrow (2)$$

$$\left.\begin{aligned} \text{Now, } |f(x) - f(y)| &\leq M_r - m_r \\ \text{and } |g(x) - g(y)| &\leq M_r' - m_r' \end{aligned}\right] \longrightarrow (3)$$

$$\therefore M_r'' - m_r'' < M(M_r - m_r) + M(M_r' - m_r')$$

using (2) & (3)

Multiplying both sides by Δx_r and adding on we have

$$\sum_1 [M_r'' - m_r''] \leq M \sum_1 (M_r - m_r)\Delta x_r + M \sum_1 (M_r' - m_r')\Delta x_r$$

$$U(P,fg) - L(P,fg) \leq M[U(P,f) - L(P,f)] + M[U(P,g) - L(P,g)]$$

from (1)

$$< M \cdot \frac{\epsilon}{2M} + M\frac{\epsilon}{2M} = \epsilon$$

Hence $fg \in R[a,b]$

Integrability of the quotient function:

Staliment:

If f and g are two functions of R-integral on $[a,b]$ and $|g(x)| \geq \lambda$ $\forall x \in [a,b]$ where λ is a positive number, then the quotient function f/g is also R-integrable on $[a,b]$

Proof:

Since $f, g \in R[a,b]$ they are bounded on $[a,b]$. Let M be a positive number such that $|f(x)| \leq M$ $\forall x \in [a,b]$

Also $|g(x)| \geq \lambda$ $\forall x \in [a,b]$

$$\therefore \left|\left(\frac{f}{g}\right)(x)\right| = \left|\frac{f(x)}{g(x)}\right| = \frac{|f(x)|}{|g(x)|} \leq \frac{M}{\lambda} \longrightarrow (1)$$

$\therefore \frac{f}{g}$ is also bounded on $[a,b]$

Let $\epsilon > 0$ be given. Since $f \in R[a,b]$ there exists a partition P_1 of $[a,b]$ such that

$$U(P_1, f) - L(P_1, f) < \frac{\epsilon}{2M}\lambda^2 \longrightarrow (2)$$

llrly $g \in R[a,b]$ therefore there exists a partition P_2 of $[a,b]$ such that

$$U(P_2, g) - L(P_2, g) < \frac{\epsilon}{2M}\lambda^2 \longrightarrow (3)$$

Let $P = P_1 \cup P_2$ be a refinement of P_1 and P_2 then from (2) & (3) we have,

$$\left.\begin{array}{l} U(P, f) - L(P, f) < \frac{\epsilon}{2M}\lambda^2 \\ U(P, g) - L(P, g) < \frac{\epsilon}{2M}\lambda^2 \end{array}\right] \rightarrow (4)$$

Let m_r, M_r, m_r' M_r' and m_r'', M_r'' be the infimum and supremum of f, g and f/g respectively over the sub interval $I_r = [x_{r-1}, x_r]$

then $\forall x, y \in I$ we have,

$$\left|\frac{f}{g}(x) - \frac{f}{g}(y)\right| = \left|\frac{f(x)}{g(x)} - \frac{f(y)}{g(y)}\right|$$

$$= \frac{\cancel{|f(x)g(y) - f(y)g(y)|}}{\cancel{|g(x) - g(y)|}}$$

$$= \left|\frac{f(x)g(y) - f(y)g(x)}{g(x)g(y)}\right|$$

$$= \frac{|f(x)g(y) - f(y)g(y) + f(y)g(y) - f(y)g(x)|}{|g(x)g(y)|}$$

$$= \frac{|[f(x) - f(y)]g(y) + f(y)[g(y) - g(x)]|}{|g(x)g(y)|}$$

$$\leq \frac{|g(y)||f(x) - f(y)|}{|g(x)||g(y)|} + \frac{|f(y)||g(x) - g(y)|}{|g(x)||g(y)|}$$

$$\leq \frac{M}{\lambda^2}|f(x) - f(y)| + \frac{M}{\lambda^2}|g(x) - g(y)| \quad \text{(5)}$$

Now m_r and M_r are the infimum and supremum of f respectively over I_r

$$\therefore |f(x) - f(y)| \leq M_r - m_r \quad \forall x, y \in [a, b] \rightarrow \text{(6)}$$

Illy we have

$$|g(x) - g(y)| \leq M_r' - m_r', \quad \forall x, y \in [a, b] \rightarrow \text{(7)}$$

$$\therefore \left|\frac{f}{g}(x) - \frac{f}{g}(y)\right| \leq \frac{M}{\lambda^2}(M_r - m_r) + \frac{M}{\lambda^2}(M_r' - m_r') \quad \text{(8)}$$

Hence $M_r'' - m_r'' \leq \frac{M}{\lambda^2}(M_r - m_r) + \frac{M}{\lambda^2}(M_r' - m_r')$ → (9)

Multiplying (9) by Δx_r and adding on both both sides, we get

$$U(P, f/g) - L(P, f/g) \leq \frac{M}{\lambda^2}[U(P,f) - L(P,f)] + \frac{M}{\lambda^2}[U(P,g) - L(P,g)]$$

$$\leq \frac{M}{\lambda^2} \cdot \frac{\epsilon\lambda^2}{2M} + \frac{M}{\lambda^2} \cdot \frac{\epsilon\lambda^2}{2M}$$

$$\leq \epsilon/2 + \epsilon/2$$

$$= \epsilon$$

Hence $f/g \in R[a,b]$

Fundamental theorem of Integral calculus.

Statement : Let f be a continuous function on $[a, b]$ and let φ be a differentiable function on $[a, b]$ such that $\varphi'(x) = f(x)$, $a \le x \le b$,

then $\int_a^b f(t)\,dt = \varphi(b) - \varphi(a)$

Proof: Since f is continuous on $[a, b]$ then the integral function F of f defined by

$F(x) = \int_a^x f(t)\,dt$ $\quad x \in [a, b]$ is differentiable.

and $F'(x) = f(x)$, $\quad x \in [a, b] \longrightarrow (1)$

But we have $\varphi'(x) = f(x)$ $\quad x \in [a, b] \longrightarrow (2)$

From (1) and (2) we have

$$F'(x) = \varphi'(x), \quad \forall\, x \in [a, b]$$

$$\Rightarrow F'(x) - \varphi'(x) = 0 \quad \forall\, x \in [a, b]$$

or $\frac{d}{dF}[F(x) - \varphi(x)] = 0 \quad \forall\, x \in [a, b]$

$\therefore$ $F(x) - \varphi(x) = c$, some constant

$$\Rightarrow F(x) = \varphi(x) + c$$

$\therefore$ $F(a) = \varphi(a) + c$

and $F(b) = \varphi(b) + c$

Hence $F(a) - F(b) = \varphi(a) - \varphi(b) \longrightarrow (3)$

But $F(b) = \int_a^b f(t)\,dt = 0$ $\quad F(a) = \int_a^b f(t)\,dt = 0$

$\therefore$ $F(b) - F(a) = \int_a^b f(t)\,dt - \int_a^b f(t)\,dt = 0 \longrightarrow (4)$

from (3) + (4) we have

$$\int_a^b f(t)\,dt = \varphi(a) - \varphi(b)$$

First Mean Value Theorem:

Statement: If $f(x)$ is continuous on $[a,b]$ then there exist a point c in $[a,b]$ such that

$$\int_a^b f(x)\,dx = (b-a)f(c)$$

Proof: Since the function f is continuous on $[a,b]$ it is bounded on $[a,b]$

Let m and M be the infimum and supremum of $f(x)$ respectively then

$$m \le f(x) \le M \quad \forall\, x \in [a,b] \quad \text{(or)}$$

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx \quad \text{(or)}$$

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

Hence there exist a number μ between m and M such that $\int_a^b f(x)\,dx = \mu(b-a)$

Since f is continuous on $[a,b]$ by intermediate value theorem there exist a point

$c \in [a,b]$ such that $\mu = f(c)$

Hence $\int_a^b f(x)\,dx = f(c)(b-a)$.

Second mean value Theorem:

Statement: If $f(x)$ and $g(x)$ be two continuous functions on $[a,b]$ and $g(x) > 0$ $\forall x \in [a,b]$, then there exist a point $c \in [a,b]$ such that $\int_a^b f(x)\, g(x)\,dx = f(c)\int_a^b g(x)\,dx$

Proof: since $f(x)$ is continuous on $[a,b]$ it is bounded on $[a,b]$, Thus where

$m = \inf f(x)$ and $M = \sup f(x)$

Also $g(x) > 0$ $\forall x \in [a,b]$

$\therefore m\,g(x) \le f(x)\,g(x) \le M g(x)$ $\forall x \in [a,b]$

Now both f and g are continuous on $[a,b]$

So R-integrable over $[a,b]$

$\therefore \int_a^b m\,g(x) \le \int_a^b f(x)\cdot g(x)\,dx \le \int_a^b M g(x)\,dx$

$\Rightarrow m\int_a^b m g(x) \le \int_a^b f(x) g(x)\,dx \le M\int_a^b g(x)\,dx$

This shows that there exist a number μ between m and M such that

$\int_a^b f(x)g(x)\,dx = \mu\int_a^b f(x)\,dx$, $m \le \mu \le M$

since f is continuous on $[a,b]$ by intermediate theorem there exist a point $c \in [a,b]$

Such that $\mu = f(c)$ and

Hence $\int_a^b f(x)g(x)\,dx = f(x)\int_a^b g(x)\,dx$; $a \le c \le b$

Riemann Stieltjes Integral :-

Riemann integral $\int_a^b f$ of a bounded function f on a closed & finite interval $[a, b]$ is the limit of certain sums associated with the function f and the partition $P = [x_r]_0^n$ of $[a, b]$. These sums referred to here are the upper sum $\Sigma M_r \delta_r$ and lower sum $\Sigma m_r \delta_r$ where M_r & m_r are least u.b & g.l.b of $f(x)$ over the component interval $[x_{r-1}, x_r]$ &

$\Delta_r = x_r - x_{r-1} = \delta_r$

Thus apart from the bounded function f, the identity x can also plays a role in the funct of these sums.

For each component interval x_r, x_{r-1}, the identity x, measure the length it by $\delta_r = x_r - x_{r-1}$, in this role we ~~reg~~ recognized it has a being a monotonically non-decreasing, if $\alpha(x)$ be same there monotically non-decreasing of $[a,b]$ and we using g in the place of identity funct. in the formulation of upper & lower sums. In the Riemann theory if we arrive at unique limit then this limit also called an integral, it is more general than the Riemann integral. It is called the R-Stieltjes integral.

<u>Upper & lower Sums:-</u>

Let $P = [x_r]_0^n$ be the partition and the funct. f is bounded on $[a,b]$ and let α be the monotonically non-decreasing funct. defined in the same closed interval $[a,b]$. M_r & m_r be the l.u.b, g.l.b and $\Delta\alpha_r = \alpha(x_r) - \alpha(x_{r-1})$

we have

$$U[f,\alpha,P] = \sum_{r=1}^{n} M_r \Delta\alpha_r$$

is the upper sum w.r. to α

of f in $[a,b]$.

III^{ly} $L[f,\alpha,P] = \sum_{r=1}^{n} m_r \Delta\alpha_r$ is the lower sum w.r.to α of f $[a,b]$

Properties of Upper & lower Sums:-

1) For every partition $P[a,b]$ of f we have $L[f,\alpha,P] \leq U[f,\alpha,P]$

$\because$ $\Delta\alpha_r = \alpha(x_r) - \alpha(x_{r-1}) \Rightarrow$

$m_r \leq M_r \;\forall\; r = 1, 2 \ldots n$

2) Let P be the partition of f in $[a,b]$ then we have

$m[\alpha(b)-\alpha(a)] \leq L[f,\alpha,P] \leq U[f,\alpha,P] \leq M[\alpha(b)-\alpha(a)] \Rightarrow$

$m \leq m_r \leq M_r \leq M.$

3) Let P^* be the refinement of P in $[a,b]$ then

$U[f,\alpha,P^*] \leq U[f,\alpha,P]$

$L[f,\alpha,P^*] \geq L[f,\alpha,P]$

4) If P_1 & P_2 be the partition of f in $P[a,b]$, we have

$$L[f,\alpha,P_1] \le U[f,\alpha,P_2]$$

5) For every lower sum always less than or equal to upper sum

i.e) $L[f,\alpha,P_1] \le U[f,\alpha,P]$

$$L[f,\alpha,P] \le U[f,\alpha,P_2]$$

Definition :-

We have $\inf_P U[f,\alpha,P] = \sum_{r=1}^{n} M_r \Delta\alpha_r$

$$= \int_a^b f(x)\,d\alpha(x)$$

$$\sup_P L[f,\alpha,P] = \sum_{r=1}^{n} m_r \Delta\alpha_r$$

$$= \int_a^b f(x)\,d\alpha(x)$$

when the above lower & upper exists and are equal

i.e) $\int_a^b f\,d\alpha = \int_a^b f\,d\alpha = \int_a^b f\,d\alpha$

is said to R-Stieltjes integral.

the common $\int f d\alpha$ is called R-S integral value with respect to α of the function f defined in $[a,b]$.

Theorem:-

If the function 'f' is bounded on $[a,b]$ & let 'α' is monotonically non-decreasing function on $[a,b]$ then we have $\underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha$.

Proof:-

If 'f' is bounded on $[a,b]$, then $\exists$ partition $P = [x_r]_0^n$ then we have $\inf_P U[f,\alpha,P] = \overline{\int_a^b} f d\alpha$

is RS upper sum for monotonically α of f in $[a,b]$.

and $\sup_P L[f,\alpha,P] = \underline{\int_a^b} f d\alpha$

Let P_1 & P_2 be the partition of f in $[a,b]$ then we have

$$\left.\begin{array}{l} \sup\limits_{P_1} L[f,\alpha,P_1] = \underline{\int_a^b} f\,dx \\ \inf\limits_{P_2} U[f,\alpha,P_2] = \overline{\int_a^b} f\,dx \end{array}\right] \text{ — } (1)$$

From (1) we have $(m_r \le M_r)$

$$L[f,\alpha,P_1] \le U[f,\alpha,P_2] \text{ — } (2)$$

In the above inequality P_1 is fixed and P_2 is varying we have

$$L[f,\alpha,P_1] \le \overline{\int_a^b} f\,dx \text{ — } (3)$$

From (2) & (3) we have

$$L[f,\alpha,P_1] \le U[f,\alpha,P_2] \le \overline{\int_a^b} f\,dx \text{ — } (4)$$

Let P_2 is fixed and P_1 is varying then we have

$$L[f,\alpha,P_1] \le \underline{\int_a^b} f\,dx \le \overline{\int_a^b} f\,dx$$

(5)

From (4) & (5) comparing we get $L[f, \alpha, P_1] \le U[f, \alpha, P_2] \le$

$$\int_a^b f\,d\alpha \le \bar{\int_a^b} f\,d\alpha$$

$$\Rightarrow \underline{\int_a^b} f\,d\alpha \le \int_a^b f\,d\alpha$$

Hence Proved.

(1) If the function f is bounded on $[a,b]$ and α is monotonically on $[a,b]$ then function $f(x) = k$ in R_α on $[a,b]$

<u>Soln</u>:- f is bounded on $[a,b]$ then the partition $P = [x_r]_0^n$ and let α is a monotonically non-decreasing function on $[a,b]$.

$I_r = [x_{r-1}, x_r]$

$\delta_r = [x_r - x_{r-1}]$

$\Delta \alpha_r = [x_{r-1}, x_r]$

$\Delta\alpha_r = \alpha(x_r) - \alpha(x_{r-1})$ $\exists$ $l.u.b$ & $g.l.b$, m_r & M_r respectively.

$\therefore M_r = k = m_r$

$\therefore \Delta\alpha_r \geq 0 \; \forall \; r = 1, 2 \ldots$

RS upper sum is denoted by

$$\inf_P U[f, \alpha, P] = \sum_{r=1}^{n} M_r \Delta\alpha_r$$

$$= \sum_{r=1}^{n} k \Delta\alpha_r = k \sum_{r=1}^{n} \Delta\alpha_r$$

$$= k[\alpha(x_r) - \alpha(x_{r-1})]$$

$$= k[\alpha(b) - \alpha(a)]$$

III^{ly} RS lower sum

$$\inf_P L[f, \alpha, P] = \sum_{r=1}^{n} m_r \Delta\alpha_r$$

$$= \sum_{r=1}^{n} k \Delta\alpha_r = k \sum_{r=1}^{n} \Delta\alpha_r$$

$$= k(\alpha(x_r) - \alpha(x_{r-1}))$$

$$= k(\alpha(b) - \alpha(a))$$

$\therefore \inf_P U[f, \alpha, P] = k(\alpha(b) - \alpha(a)) = \bar{\int} f d\alpha$

& $\sup_P L[f, \alpha, P] = k(\alpha(b) - \alpha(a)) = \underline{\int_a^b} f d\alpha$

$\because \underline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha = \overline{\int_a^b} f\,d\alpha = k[\alpha(b) - \alpha(a)]$

$\Rightarrow f \in R[a,b]$.

Note:-

Since $\underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha$

(ie) $\underline{\int_a^b} f(x)\,d\alpha(x) \leq \overline{\int_a^b} f(x)\,d\alpha(x)$

put $\alpha(x) = x$ in above

inequality

$\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx$

$\Rightarrow \alpha(x) = x$ the RS integral converted into R-integral

~~Theorem~~:- Let $f(x)$ be a bounded function and α a monotonic non-decreasing function defined over $[a,b]$. Then the function $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$ of f in $[a,b]$

then $f \notin R[a,b]$

with respect to α

Soln:

f is bounded on $[a,b]$

$\exists$ a $P = [x_r]_0^n$

$\Delta x_r = [x_{r-1}, x_r]$

$\Delta \alpha_r = \alpha(x_r) - \alpha(x_{r-1})$

M_r & m_r are l.u.b &

g.l.b of f in $I = [x_{r-1}, x_r]$

$U[f, \alpha, P] = \sum M_r \Delta \alpha_r$

$= \sum_{r=1}^{n} 1 . \Delta \alpha_r$

$= \sum \Delta \alpha_r$

$= [\alpha(b) - \alpha(a)]$

IIIly

$L[f, \alpha, P] = \sum m_r \Delta \alpha_r$

$= \sum_{r=1}^{n} 0 . \Delta \alpha_r$

$= 0$

$\therefore \inf_P [f, \alpha, P] = \bar{\int_a^b} f \, d\alpha = \alpha(b) - \alpha(a)$

& $\sup_P [f, \alpha, P] = \underline{\int_a^b} f \, d\alpha = 0$

$\therefore \bar{\int_a^b} f \, d\alpha \neq \underline{\int_a^b} f \, d\alpha$

$\therefore f \notin RS[a,b]$

Necessary & Sufficient condition for RS integral :-

Let the function f is bounded and α is monotonically non-decreasing function on $[a,b]$ then the necessary & sufficient condition that f in RS integral if for every $\epsilon > 0$ $\exists$ a partition P of $[a,b]$ $\ni$ $U[f,\alpha,P] - L[f,\alpha,P] < \epsilon$

Proof :- Necessary

f is RS $[a,b]$

$$\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha$$

$\because$ f is bounded on $[a,b]$ $\exists$ a

Partition $P = [x_r]_0^n$

$\therefore$ We have $\inf_P U[f,\alpha,P] = \overline{\int_a^b} f\,d\alpha$

$$\sup_P L[f,\alpha,P] = \underline{\int_a^b} f\,d\alpha$$

$$L[f,\alpha,P] \geq \underline{\int_a^b} f\,d\alpha - \epsilon/2 \quad \text{--- (1)}$$

& $U[f,\alpha,P] \leq \int_a^b f\,d\alpha + \epsilon/2$ — ②

f is bounded over $[a,b]$ $\exists$ P_1 & P_2
of f over $[a,b]$.

$$L[f,\alpha,P_1] \geq \int_a^b f\,d\alpha - \epsilon/2 \quad \text{— ③}$$

&
$$U[f,\alpha,P_2] \leq \int_a^b f\,d\alpha + \epsilon/2 \quad \text{— ④}$$

$$P^* = P_1 \cup P_2, \quad P^* \supset P_1 \text{ \& } P^* \supset P_2$$

From ③ & ④ We have

$$L[f,\alpha,P^*] \geq L[f,\alpha,P_1] \geq \int_a^b f\,d\alpha - \epsilon/2$$

$$\Rightarrow L[f,\alpha,P^*] \geq \int_a^b f\,d\alpha - \epsilon/2$$

$$\Rightarrow -L[f,\alpha,P^*] \leq -\int_a^b f\,d\alpha + \epsilon/2 \quad \text{— ⑤}$$

$$U[f,\alpha,P^*] \leq U[f,\alpha,P_2] \leq \int_a^b f\,d\alpha + \epsilon/2$$

$$U[f,\alpha,P^*] \leq \int_a^b f\,d\alpha + \epsilon/2 \quad \text{— ⑥}$$

From ⑤ & ⑥ We have

$U[f,\alpha,P^*] - L[f,\alpha,P^*] < \int_a^b f\,d\alpha - \int_a^b f\,d\alpha + \epsilon$

$\Rightarrow U[f,\alpha,P^*] - L[f,\alpha,P^*] < \epsilon$

$\Rightarrow f \in RS[a,b]$

(ii) <u>Sufficient</u> :-

f is bounded on $[a,b]$ & $\epsilon > 0$ $\exists$ a partition of $P^* = \{x_i\}_0^n$ of $[a,b]$ $\ni U[f,\alpha,P^*] - L[f,\alpha,P^*] < \epsilon$

We know that

$$\inf_{P^*} U[f,\alpha,P^*] = \int_a^b f\,d\alpha$$

& $$\sup_{P^*} L[f,\alpha,P^*] = \int_a^b f\,d\alpha$$

$$U[f,\alpha,P^*] \le \int_a^b f\,d\alpha \quad \text{--- (1)}$$

& $$L[f,\alpha,P^*] \ge \int_a^b f\,d\alpha \quad \text{--- (2)}$$

$$\Rightarrow -L[f,\alpha,P^*] \le -\int_a^b f\,d\alpha \quad \text{--- (3)}$$

from (1) & (3)

$0 \leq U[f,\alpha,P^*] - L[f,\alpha,P^*] \leq \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha < \epsilon$

$\Rightarrow \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha < \epsilon$

$\Rightarrow \overline{\int_a^b} f\,d\alpha < \underline{\int_a^b} f\,d\alpha$ ——— (*)

But we know that

$\overline{\int_a^b} f\,d\alpha \geq \underline{\int_a^b} f\,d\alpha$ ——— (**)

from (*) & (**) we have

$\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$

$\therefore f \in RS[a,b]$.

(1). If the functⁿ 'f' is bounded & 'α' is monotonically non-decreasing over $[a,b]$ then f is RS integral w.r.to α. over $[a,b]$

Solⁿ:

f is bounded over $[a,b]$ for every $\epsilon > 0$ $\exists$ a partition $P = [x_r]_0^n$

$\Delta x_r = [x_{r-1}, x_r]$ $\alpha(x_r) = k$

$\alpha(x_{r-1}) = k$

$\Delta \alpha_r = \alpha(x_r) - \alpha(x_{r-1})$ $[x_{r-1}, x_r]$

$U[f, \alpha, P] = \sum M_r \Delta \alpha_r$

$L[f, \alpha, P] = \sum m_r \Delta \alpha_r$

$U[f, \alpha, P] - L[f, \alpha, P] = \sum (M_r - m_r) \Delta \alpha_r$

$= 0$

$\therefore f \in RS$ integral w.r.to k.

① Examine if the integral $\int_0^1 x \, d(x^2)$ is RS integral.

Soln :-

$f(x) = x$

f is bounded over $[0,1]$ $\exists$

$P = [x_r]_0^n$,

$\alpha(x) = x^2$

where $\alpha(x)$ is monotonically non-decreasing function

$$\Delta x_r = \alpha(x_r) - \alpha(x_r - 1)$$

M_r & m_r are exists

$$\Delta x_r = \alpha\left(\frac{r}{n}\right) - \alpha\left(\frac{r-1}{n}\right)$$

$$\Delta x_r = \left(\frac{r}{n}\right)^2 - \left(\frac{r-1}{n}\right)^2$$

$$= \frac{\cancel{r^2}}{n^2} - \frac{\cancel{r^2}}{n^2} - \frac{1}{n^2} + \frac{2r}{n^2}$$

$$\Delta x_r \Rightarrow \frac{1}{n^2}(2r-1)$$

For every $\epsilon > 0 \ \exists$

$$U[f, P, \alpha] - L[f, \alpha, P]$$

$$= \sum (M_r - m_r) \Delta x_r$$

$$= \sum \left(\frac{r}{n} - \frac{r-1}{n}\right) \frac{2r-1}{n^2}$$

$$= \frac{1}{n^3} \sum_{r=1}^{n} (\cancel{r} - r + 1)(2r-1)$$

$$= \frac{1}{n^3} \sum_{r=1}^{n} (2r) - \frac{1}{n^{\cancel{3}}} \cdot \cancel{n}$$

$$U - L = \frac{1}{n^3} 2[1+2+\dots+n] - \frac{1}{n^2}$$

$= \frac{1}{n^3} \cdot n(n+1) - \frac{1}{n^2}$

Real Analysis

Asora.

$= \cancel{\frac{1}{n^2}} + \frac{1}{n} - \cancel{\frac{1}{n^2}}$

$= \frac{1}{n} < \epsilon$

$f \in RS[0,1]$

Algebra of Rs Integral :-

Theorem :-

If the function f_1 & f_2 be Rs integrable functions with respect to 'α' defined over $[a,b]$, then the sum of f_1 & f_2 is also Rs integrable function over $[a,b]$

(ie $\int (f_1+f_2)\, d\alpha = \int f_1\, d\alpha + \int f_2\, d\alpha$.

Proof :- part - I

$f_1 \in RS[a,b]$ & $f_2 \in RS[a,b]$

$\exists$ a partition P of $[a,b]$, for each

$\epsilon > 0$ $\exists$

$U[f_1, \alpha, P] - L[f_1, \alpha, P] < \epsilon$ — (1)

& $U[f_2, \alpha, P] - L[f_2, \alpha, P] < \epsilon$ — (2)

Adding ④ & ②

$U[f_1,\alpha,P] + U[f_2,\alpha,P] - [L[f_1,\alpha,P] + L[f_2,\alpha,P]] < 2\epsilon$

$\because f_1 \; \& \; f_2 \in R[a,b]$

$\exists \quad M_r, m_r \rightarrow f$

$M_r', m_r' \rightarrow f_1 \; \& \; M_r'', m_r'' \rightarrow f_2$

We know that

$f_1 + f_2 \leq M_r' + M_r''$

(or)

$M_r \leq M_r' + M_r''$

$$\sum_{r=1}^{n} M_r \Delta\alpha_r \leq \sum_{r=1}^{n} M_r' \Delta\alpha_r + \sum_{r=1}^{n} M_r'' \Delta\alpha_r$$

(ie) $U[f, \alpha, P] = U[f_1, \alpha, P] + U[f_2, \alpha, P]$ — ④

We know that

$U[f_1, \alpha, P] < \int^{b} f_1 \, d\alpha + \epsilon$ — ⑤

$U[f_2, \alpha, P] < \int_{a}^{b} f_2 \, d\alpha + \epsilon$ — ⑥

Adding ⑤ & ⑥

$U[f_1, \alpha, P] + U[f_2, \alpha, P] \leq \int_{a}^{b} f_1 \, d\alpha +$

$\int_{a}^{b} f_2 \, d\alpha + 2\epsilon$ ⑦

$$U[f, \alpha, P] \leq \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha + 2\epsilon \quad (8)$$

|||ly

$$L[f, \alpha, P] \geq \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha - 2\epsilon \quad (9)$$

Subtg (8) - (9) we get

$$U[f, \alpha, P] - L[f, \alpha, P] \leq 4\epsilon$$

$$\therefore f_1 + f_2 \in RS[a, b].$$

Part - II :-

$$U[f_1, \alpha, P] - \int_a^b f_1 \, d\alpha < \epsilon \quad (1)$$

$$L[f_2, \alpha, P] - \int_a^b f_2 \, d\alpha < \epsilon \quad (2)$$

Adding (1) & (2)

$$U[f_1, \alpha, P] + U[f_2, \alpha, P] < \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha + 2\epsilon$$

$$U[f, \alpha, P] < \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha + 2\epsilon$$

$$0 \leq U[f, \alpha, P] < \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha + 2\epsilon \quad (3)$$

We know that

$$U[f, \alpha, P] - \int_a^b f \, d\alpha < \epsilon \quad \text{—— (4)}$$

Compare (3) & (4)

$$\int_a^b f \, d\alpha = \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha$$

$$\Rightarrow f_1 + f_2 \in Rs[a, b].$$

Properties of Rs integral:

1. If f and g are 2 funct: both are Rs Integral w.r.to α. Then $f+g$ is also Rs integral.

   (ie) $\int_a^b (f+g) d\alpha = \int_a^b f d\alpha + \int_a^b g d\alpha$

2. If f is a function which is Rs integral w.r.to α $[a, b]$ & k be a Scalar then kf is also Rs integral w.r.to α $[a, b]$

   (ii) $\int_a^b (kf) d\alpha = k \int_a^b f d\alpha$.

3. If $f \in$ Rs integral $[\alpha, a, b]$ then $-f \in Rs(\alpha, a, b)$