UNIT - II

Limit of a function of two variables

Let f be a function of two variables, x and y. The limit of $f(x,y)$ as (x,y) approaches (a,b) as L,

$$\boxed{\text{i.e., } \lim_{(x,y)\to(a,b)} f(x,y) = L}$$

if for each $\epsilon > 0$ there exists a small enough $\delta > 0$ such that for all points (x,y) in a δ disk around (a,b), except possibly for (a,b) itself, the value of $f(x,y)$ is no more than ϵ away from L

using symbols, we write the following:

For any $\epsilon > 0$, there exists a number $\delta > 0$ such that

$$\boxed{|f(x,y) - L| < \epsilon}$$

whenever,

$$0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$$

Limit laws for functions of two variables:

Let $f(x, y)$ and $g(x,y)$ be defined for all $(x,y) \neq (a,b)$ in a neighbourhood around (a,b) and assume the neighbourhood is contained completely inside the domain of f. Assume that L and M are real numbers such that

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

$$\lim_{(x,y)\to(a,b)} g(x,y) = M$$

and let c be a constant. Then each of the following statements holds:

Constant law : $\lim_{(x,y)\to(a,b)} c = c$

Identity laws : $\lim_{(x,y)\to(a,b)} x = a$

$\lim_{(x,y)\to(a,b)} y = b$

Sum law:

$$\lim_{(x,y)\to(a,b)} [f(x,y) + g(x,y)] = L + M$$

Difference law :

$$\lim_{(x,y)\to(a,b)} [f(x,y) - g(x,y)] = L - M$$

Constant Multiple law :

$$\lim_{(x,y)\to(a,b)} [c f(x,y)] = cL$$

Product law :

$$\lim_{(x,y)\to(a,b)} [f(x,y)\, g(x,y)] = LM$$

Quotient law :

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M} \quad \text{for } M \neq 0$$

Power law :

$$\lim_{(x,y)\to(a,b)} [f(x,y)^n] = L^n$$

for any positive integer n

Root law :

$$\lim_{(x,y)\to(a,b)} \sqrt[n]{f(x,y)} = \sqrt[n]{L}$$

for all L if n is odd and positive, and for $L \geq 0$ if n is even and positive.

1) **Example:** Find each of the following limits

a. $\lim_{(x,y)\to(2,-1)} (x^2-2xy+3y^2-4x+3y-6)$

solution:

using sum and difference laws to separate the terms

i.e., $\lim_{(x,y)\to(2,-1)} (x^2-2xy+3y^2-4x+3y-6)$

$$=\left(\lim_{(x,y)\to(2,-1)} x^2\right)-\left(\lim_{(x,y)\to(2,-1)} 2xy\right)+\left(\lim_{(x,y)\to(2,-1)} 3y^2\right)-\left(\lim_{x,y\to(2,-1)} 4x\right)$$
$$+\left(\lim_{(x,y)\to(2,-1)} 3y\right)-\left(\lim_{(x,y)\to(2,-1)} 6\right)$$

Next, use the constant multiple law on the second, third, fourth and fifth limits

$$=\left(\lim_{(x,y)\to(2,-1)} x^2\right)-2\left(\lim_{(x,y)\to(2,-1)} xy\right)+3\left(\lim_{(x,y)\to(2,-1)} y^2\right)-4\left(\lim_{x,y\to(2,-1)} x\right)$$
$$+3\left(\lim_{(x,y)\to(2,-1)} y\right)-\left(\lim_{(x,y)\to(2,-1)} 6\right)$$

Now use the power law on the first and third limits and the product law on the second limit

$$=\left(\lim_{(x,y)\to(2,-1)} x\right)^2-2\left(\lim_{(x,y)\to(2,-1)} x\right)\left(\lim_{(x,y)\to(2,-1)} y\right)+3\left(\lim_{(x,y)\to(2,-1)} y\right)^2$$
$$-4\left(\lim_{(x,y)\to(2,-1)} x\right)+3\left(\lim_{x,y\to(2,-1)} y\right)-\left(\lim_{(x,y)\to(2,-1)} 6\right)$$

Last, use the identity laws on the first six limits and the constant law on the last limit

$$\lim_{(x,y)\to(2,-1)} (x^2-2xy+3y^2-4x+3y-6)$$

$$= (2)^2 - 2(2)(-1) + 3(-1)^2 - 4(2) + 3(-1) - 6$$

$$= -6$$

Continuity of a function of two variables:

A function $f(x,y)$ is continuous at a point (a,b) in its domain if the following conditions are satisfied.

1. $f(a,b)$ exists
2. $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists
3. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Example 1: Show that the function

$$f(x,y) = \frac{3x+2y}{x+y+1}$$

is continuous at point $(5,-3)$

Solution: There are 3 conditions to be satisfied, per the definition of continuity.

1. $f(a,b)$ exists

i.e., $f(a,b) = f(5,-3) = \dfrac{3(5)+2(-3)}{5-3+1} = 3$

2. $\lim\limits_{(x,y)\to(5,-3)} f(x,y) = \lim\limits_{(x,y)\to(5,-3)} \dfrac{3x+2y}{x+y+1} = \dfrac{15-6}{5-3+1} = 3$

3. $\lim\limits_{(x,y)\to(5,-3)} f(x,y) = f(a,b)$. Hence continuity exists

Example 2:

Examine whether the function

$$f(x,y) = \begin{cases} x^2+4y & \text{when } (x,y) \neq (1,2) \\ 0 & \text{when } (x,y) = (1,2) \end{cases}$$

(i) has a limit as $x \to 1$ and $y \to 2$

(ii) is continuous at $(1,2)$

Solution:

(i) $$\lim_{\substack{x\to 1\\ y\to 2}} f(x,y) = \lim_{\substack{x\to 1\\ y\to 2}} x^2+4y$$

$$= 1^2+4(2) = 1+8 = 9$$

So that the limit exist and is equal to 9

(ii) since $f(1,2) = 0$ and $\lim\limits_{\substack{x\to 1\\ y\to 2}} f(x,y) = 9$

we have $\lim\limits_{\substack{x\to 1\\ y\to 2}} f(x,y) \neq f(1,2)$. Hence the function is not continuous.

Example 3:

Show that the function

$$f(x,y) = \begin{cases} e^{-|x-y|/x^2-2xy+y^2}, & f(x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

is continuous.

Given that

$$f(x,y) = e^{-|x-y|/x^2-2xy+y^2}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} e^{-|x-y|/x^2-2xy+y^2}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} e^{-|x-y|/(x-y)^2}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} e^{-1/(x-y)}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} e^{-1/0}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} e^{-\infty} = 0$$

The function $f(x,y)$ is continuous at a point $(0,0)$

Example 3:

Show that $f(x,y) = \begin{cases} \dfrac{2xy}{\sqrt{x^2+y^2}} & ; (x,y) \neq (0,0) \\ & (x,y) = (0,0) \end{cases}$ is continuous.

solution

Given that

$$f(x,y) = \frac{2xy}{\sqrt{x^2+y^2}} \quad \text{put } y = x$$

$$= \lim_{(x,y)\to(0,0)} \frac{2x \cdot x}{\sqrt{x^2+x^2}} = \lim_{(x,y)\to(0,0)} \frac{2x^2}{x\sqrt{1+1}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{2x}{\sqrt{2}} = \lim_{(x,y)\to(0,0)} \frac{\sqrt{2}\,\sqrt{2}\,x}{\sqrt{2}}$$

$$= \lim_{(x,y)\to(0,0)} \sqrt{2}\,x = 0$$

$\therefore$ The function $f(x,y)$ is continuous at the point $(0,0)$.

Derivatives of a function of Two variables:

In derivatives functions of one variable, the derivative is an instantaneous rate of change of y as a function of x. Leibniz notation for the derivative is dy/dx, which implies that y is the dependent variable and x is the independent variable. For a function $z = f(x, y)$ of two variables, x and y are the independent variables and z is the dependent variable. Hence derivatives of a function of two variables can be found using partial derivatives.

Partial derivatives:

Let $z = f(x, y)$ be a function of two independent variables x and y. Then the partial derivative of z w.r.t x is the ordinary derivative of z when y is regarded as a constant and is denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or f_x.

For the point (x_0, y_0) such a partial derivative is defined by the limit

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

Similarly, holding x constant, we define the partial derivative of $f(x, y)$ w.r.to y at (x_0, y_0) by the limit

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

The derivative functions f_x, f_y are in general functions of x and y and hence may themselves have partial derivatives w.r.t x and y. we are thus led to higher order partial derivatives.

$$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial x \partial y} \text{ etc..}$$

Differentiability of two variables

Suppose a single valued function $z = f(x,y)$ is defined in the domain D given by

$a < x < b$, $c < y < d$

TO find the condition under which the total differential of f may be expressed by the relation

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \longrightarrow (1)$$

suppose now x and y are connected by a functional relation,

Say $y = \varphi(x)$ $\longrightarrow$ (2)

or x and y are expressed in terms of a parameter t by the functional relations

$$x = \chi_1(t), \quad y = \chi_2(t) \longrightarrow (3)$$

We may find the total derivatives of z w.r.t x and w.r.t t by the formulae

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} \longrightarrow (4)$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \longrightarrow (5)$$

We may also find these derivatives by differentiating directly the transformed function

$$z = f(x, \phi(x)) \longrightarrow (6)$$

and
$$z = f(\chi_1(t), \chi_2(t)) \longrightarrow (7)$$

In order that the relation (1) shall constitute a satisfactory definition of total differential, the numerical values of $\frac{dz}{dx}$ and $\frac{dz}{dt}$ as given by (4) and (5) must be the same as obtained by differentiating directly the relation (6) and (7) respectively.

Total differentiability

A function $z = f(x, y)$ is said to be totally differentiable at (x_0, y_0), if we have the simultaneous double limit.

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) - x f_x(x_0, y_0) - y f_y(x_0, y_0)}{\Delta(x, y)} = 0$$

where $\Delta(x, y) = \sqrt{[(\Delta x)^2 + (\Delta y)^2]}$

continuity in the two variables in necessary condition for total differentiability but not a sufficient condition.

Theorem: If the function $f(x,y)$ is totally differentiable at the point (x_0, y_0) then it is continuous in (x,y) together at (x_0, y_0)

Proof: The proof follows at once from total differentiability. In order that the limit on the right hand side of total differentiability is zero, it is necessary that the numerator shall have the limit zero, that is

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left[f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) - \Delta x f_x(x_0,y_0) - \Delta y f_y(x_0,y_0) \right] = 0$$

which shows that $f(x,y)$ is continuous in (x,y) together at (x_0,y_0).

Remark: A necessary and sufficient condition for total differentiability can be expressed analytically by means of polar coordinates. If we put $\underline{\Delta x = r\cos\theta}$, $\underline{\Delta y = r\sin\theta}$ in the L.H.S of total differentiability, it becomes

~~It~~ i.e., Total differentiability is given as.

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) - x f_x(x_0,y_0) - y f_y(x_0,y_0)}{\Delta(x,y)} = 0$$

Where $\boxed{\Delta(x,y) = \sqrt{(\Delta x)^2 + (\Delta y)^2}}$

~~[illegible]~~

~~Δx =~~

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x_0 + r\cos\theta, y_0 + r\sin\theta) - f(x_0,y_0) - \cos\theta f_x(x_0,y_0) - \sin\theta f_y(x_0,y_0)}{\sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}}$$

A necessary and sufficient condition that the limit zero as $\Delta x, \Delta y$ tend to zero simultaneously, that is $f(x,y)$ is totally differentiable at (x_0,y_0) is that

$$\frac{1}{r}\left\{f(x_0+r\cos\theta, y_0+r\sin\theta) - f(x_0,y_0) - \cos\theta f_x(x_0,y_0) - \sin\theta f_y(x_0,y_0)\right\}$$

$$\frac{1}{r}\left\{f(x_0+r\cos\theta, y_0+r\sin\theta) - f(x_0,y_0)\right\}$$

converges uniformly in the closed interval $[0,\pi]$ to the limit $\cos\theta f_x(x_0+y_0) + \sin\theta f_y(x_0,y_0)$ as r tends to zero.

Maxima and Minima of functions of two variables.

If c be any interior point of the domain $[a, b]$ of a function f, then

(i) $f(c)$ is said to be a maximum value of the function f, if there exist some neighbourhood $]c-\delta, c+\delta[$ of c, such that

$$f(c) > f(x) \quad \forall \; x \in \;]c-\delta, c+\delta[$$

other than c

(ii) $f(c)$ is said to be a minimum value of the function f, if there exist some neighbourhood $]c-\delta, c+\delta[$ of c, such that

$$f(c) < f(x) \quad \forall \; x \in \;]c-\delta, c+\delta[\text{ other than } c$$

(iii) $f(c)$ is said to be an extreme value of f, if it is either maximum or a minimum value

A necessary condition for the Existence of Extreme values.

If $f(c)$ be an extreme value of a function f, then $f'(c)$, in case it exist is zero

Proof: since $f(c)$ is an extreme value of $f(x)$, it follows that $f(x)$ has either a maximum value or a minimum value at $x = c$

Since $f'(c)$ exist, we have

$$Lf'(c) = Rf'(c) = f'(c) \longrightarrow (1)$$

Let $f(x)$ have a maximum value at $x = c$. So, there exist a $\delta > 0$, such that

$$c-\delta < x < c \Rightarrow f(x) < f(c) \longrightarrow (2)$$
$$c < x < c+\delta \Rightarrow f(x) < f(c) \longrightarrow (3)$$

From (2)

$$f(x) - f(c) < 0 \quad \text{and} \quad x - c < 0$$

whenever $c-\delta < x < c$

$$\Rightarrow \frac{f(x)-f(c)}{x-c} > 0 \longrightarrow (4)$$

Taking limits as $x \to c$

$$\Rightarrow Lf'(c) \geq 0 \longrightarrow (5)$$

Again from (3), $f(x) - f(c) < 0$ and $x - c > 0$

whenever $c < x < c+\delta$

$$\Rightarrow \frac{f(x)-f(c)}{x-c} < 0 \longrightarrow (6)$$

Taking limits as $x \to c$

$$\Rightarrow Rf'(c) \leq 0 \longrightarrow (7)$$

From (1), (5) and (7) we have

$$f'(c) = 0$$

Again, if $f(x)$ has a minimum value at $x = c$, by similar argument, we have $f'(c) = 0$

Problem: Find the points of Maxima and minima of $f(x) = 3\cos^2 x + \sin^6 x$, $-\pi/2 < x < \pi/2$

Solution: Let $y = 3\cos^2 x + \sin^6 x$, $-\pi/2 < x < \pi/2 \rightarrow (1)$

then $\frac{dy}{dx} = -6\cos x \sin x + 6\sin^5 x \cos x \rightarrow (2)$

For maximum and minimum value of y, we have, $\frac{dy}{dx} = 0$ or $6\sin x \cos x(\sin^4 x - 1) = 0 \rightarrow (3)$

But for $-\pi/2 < x < \pi/2$, $\cos x \neq 0$ and $(\sin^4 x - 1) \neq 0$

Hence from $(3) \Rightarrow \sin x = 0$ (or) $x = 0$ for $-\pi/2 < x < \frac{\pi}{2}$

Now from (2) $d^2y/dx^2 = -6(\cos^2 x - \sin^2 x) + 6(5\sin^4 x \cos^2 x - \sin^6 x)$

$\therefore$ At $x = 0$, $\frac{d^2y}{dx^2} = -6 < 0$

Hence y is maximum when $x = 0$

Example: Prove that the function f defined by $f(x) = |x-2|\,|x-3|$ $\forall x \in R$ has a minimum value 0 at 2, 3 and a maximum value $1/4$ at $5/2$

Solution: Let $y = f(x)$, Then we have

$$y = |x-2||x-3| = \begin{cases} (2-x)(3-x) & \text{if } x \leq 2 \\ (x-2)(3-x) & \text{if } 2 \leq x \leq 3 \\ (x-2)(x-3) & \text{if } x \geq 3 \end{cases}$$

$$y = f(x) = \begin{cases} 6-5x+x^2, & \text{if } x \leq 2 \\ -6+5x-x^2, & \text{if } 2 \leq x \leq 3 \\ x^2-5x+6, & \text{if } x \geq 3 \end{cases}$$

Clearly $f(x)$ is continuous everywhere.

Also $f(2) = f(3) = 0$

Here $Lf'(2) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$

$= \lim_{h \to 0} \frac{6 - 5(2-h) + (2-h)^2 - 0}{-h}$

$= -1$

$$R f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0} \frac{-6+5(2+h) - (2+h)^2 - 0}{h}$$

$$= 1$$

Since $L f'(2) \neq R f'(2)$ so $f'(x)$ does not exist at $x = 2$ $\Rightarrow$ $x = 2$ is a critical pt of y.

Again, $L f'(2) < 0$ and $R f'(2) > 0$

$\Rightarrow$ $f(x)$ is minimum at $x = 2$

Further, minimum value at $x = 2$ is given by $f(2) = 0$

Next $$L f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{-h}$$

$$= \lim_{h\to 0} \frac{-6+5(3-h) - (3-h)^2 - 0}{-h} = -1$$

and $$R f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h\to 0} \frac{(3+h)^2 - 5(3+h) + 6 - 0}{h} = 1$$

Since $L f'(3) \neq R f'(3)$ so $f'(x)$ does not exist at $x = 3$ $\Rightarrow$ $x = 3$ is a critical point of y.

Also, $L f'(3) < 0$ and $R f'(3) > 0$

$\Rightarrow$ $f(x)$ is minimum at $x = 3$

Further minimum value at $x = 3$ is given by $f(3) = 0$

Now, for $2 \le x \le 3$, $y = -6 + 5x - x^2$

$dy/dx = 5 - 2x$ and $d^2y/dx^2 = -2$

Since d^2y/dx^2 is negative, the value of x for which y is maximum is given by

$dy/dx = 0$ or $5 - 2x = 0$ or $x = 5/2$

Maximum value at $x = 5/2 \Rightarrow f(5/2) = -6 + 25/2 - 25/4$

$= 1/4$ //