12/2 Unit : 3

Interpolation with unequal Intervals :

The values of the interpolation problem are unequal then we have the problem of interpolation with unequal intervals. Newton's Forward, Backward formulae and central difference formula will not hold good. Hence we introduce the idea of divided differences. These divided differences take into the consideration of the changes of the values of the function $f(x)$ and also the changes in the values of the arguments (x)

Divided Differences :

* Let the function $y=f(x)$ assume the values $f(x_0)$ $f(x_1), \dots f(x_n)$ corresponding to the arguments $x_0, x_1, x_2, \dots x_n$ the 1st divided difference of $f(x)$ for the argument x_0, x_1 is denoted as $f(x_0, x_1)$

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

The second order difference is defined as,

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

Form the divided difference Table for the following data.

	x	$f(x)$ (y)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	-2	-792					
			450	-102			
1	0	108					
			-60		18		
2	3	-72		24		-2.24	
			60		-2.14		
3	5	48		-39			0.1782
			-96			-0.4585	
4	7	-144			-5.8		
				-68			
			-108				
5	8	-252					

(3) 12/21 Form the divided difference table

If $f(x) = x^3 + x + 2$ for the argument

x	$f(x)$ (y)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	2					
		2				
1	4		2			
		8		2.667		
2	12		10		-1.084	
		28		-1.667		0.417
3	32		5		1.000	
		38		2.333		
4	70		12			
		62				
5	132					

Properties of divided difference:

* The value of any divided difference is independent of the order of arguments. that is the divided differences are symmetrical in all their arguments

* The operator Δ is linear.

* The 'n' th divided difference of polynomial of degree 'n' are constant.

Newton's Divided difference formula:

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + \dots + (x-x_0)(x-x_1)\dots(x-x_{n-1}) f(x_0,x_1,x_2\dots x_n)$$

Using Newton's Divided difference formula find $f(2)$, $f(8)$ and $f(15)$.

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
4	48				
		52			
5	100		15		
		97		1	
7	294		21		0
		202		1	
10	900		27		0
		310		1	
11	1210		33		
		409			
13	2028				

$x_0 = 4$, $x_1 = 5$, $x_2 = 7$, $x_3 = 10$

$f(x_0) = 48$, $f(x_0,x_1) = 52$, $f(x_0,x_1,x_2) = 15$

$f(x_0,x_1,x_2,x_3) = 1$

$x=2$

$$f(x_2) = 48 + (2-4)(52) + (2-4)(2-5)(15) + (2-4)(2-5)(2-7)(1)$$

$$= 48 + (-104) + 90 - 30$$

$$= 48 - 104 + 90 - 30$$

$$= 4$$

$$f(x=8) = 48 + (8-4)(52) + (8-4)(8-5)(15) + (8-4)(8-5)(8-7)(1)$$

$$= 48 + 208 + 180 + 12$$

$$= 448$$

$f(15) = 48(15-4)(52) + (15-4)(15-5)(15) +$
$(15-4)(15-5)(15-7)(1)$

$= 27456 + 1650 + 880$

$= 3410$

From the following table find $f(6)$ using Newton's Interpolation formula.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	1			
		4		
2	5		-0.667	
		0		0.071
7	5		-0.167	
		-1		
8	4			

$x_0 = 1,\ x_1 = 2,\ x_2 = 7,\ x_3 = 8$

$f(x_0) = 1 \quad \cancel{f(x_1) = 5} \quad \cancel{f(x_2) = 5} \quad f(x_3) = 4$

$f(x_0, x_1) = 4,\ f(x_0, x_1, x_2) = -0.667,$

$f(x_0, x_1, x_2, x_3) = 0.071$

$f(x) = f(x_0) + (x - x_0)(f(x_0, x_1) + (x - x_0)(x - x_1)$
$f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2)$
$f(x_0, x_1, x_2, x_3)$

$f(x=6) = 1 + (6-1)4 + (6-1)(6-2)(-0.667) +$
$(6-1)(6-2)(6-7)(0.071)$

$= 1 + 20 - 13.34 - 1.42$

$= 6.24$

From the following table find the pressure ~~at~~ temp (372.1)

temp	Pressure y	Δy	$\Delta^2 y$	$\Delta^3 y$
361°	154.9			
		2.17		
367°	167.9		-0.004	
				0.00073
		2.10		
378°	191.0		+0.015	
		2.39		0.000094
387°	212.5		0.012	
		2.64		
399°	244.2			

$$f(372.1) = (154.9) + (372.1-361)(2.17) + (372.1-361)(372.1-367)(-0.004) + (372.1-361)(372.1-367)(372.1-378)(0.00073)$$

$$= (154.9) + (11.1)(2.17) + (11.1)(5.1)(-0.004) + (11.1)(5.1)(-5.9)(0.00073)$$

$$= 154.9 + 24.084 - 0.226444 - 0.24382$$

$$= 178.5$$

Find the equation $y = f(x)$ at least degree and passing through points $(-1,-21)(1,15)(2,12)(3,3)$ also find y at $x=0$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-1	-21			
		18		
1	15		-7	
		-3		1
2	12		-3	
		-9		
3	3			

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3) + \cdots$$

$x_0 = -1, \quad x_1 = 1, \quad x_2 = 2, \quad x_3 = 3$

$f(x_0) = -21, \quad f(x_0,x_1) = 18, \quad f(x_0,x_1,x_2) = -7$

$f(x_0,x_1,x_2,x_3) = 1$

$$f(x=0) = -21 + (0+1)(18) + (0+1)(0-1)(-7) + (0+1)(0-1)(0-2)(1)$$

$$= -21 + 18 + 7 + 2$$

$$= 6$$

Lagrange's Interpolation

The forward and Backward interpolation formula of newton can be used only when the values of independent variable x are equally spaced. further the differences must become ultimately small.

Incases where the values of independent variable are not equally spaced and where the differences of the dependent variable are not small. In such cases we us lagrange's Interpolation formula.

Let $y = f(x)$ be a function, such that the values of $y_0, y_1, \ldots\ldots y_n$ corresponding to $x = x_0, x_1, \ldots\ldots x_n$

ie, $y_i = f(x_i)$; $i = 0, 1, 2 \ldots\ldots n$!

Now there are $(n+1)$ paired values (x_i, y_i) and hence can be represented by polynomial function of degree n in x

Let us select $f(x)$ as follows

$$f(x) = a_0 (x - x_1)(x - x_2) \ldots\ldots (x - x_n)$$
$$+ a_1 (x - x_0)(x - x_2) \ldots\ldots (x - x_n)$$
$$+ a_2 (x - x_0)(x - x_1)(x - x_3) \ldots\ldots (x - x_n)$$
$$(x - x_0)(x - x_1) \ldots\ldots (x - x_{i-1})(x - x_{i+1}) \ldots (x - x_n) \quad (1)$$
$$(x - x_{n-1})$$

the term in which a_i, occurs now the factor $(x - x_i)$ lacking that is true for all values of x

Sub in ①, $x = x_0$, $y = y_0$ we get

$$y_0 = a_0(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)$$

$$\therefore a_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)}$$

III^{ly}

Let us set $x = x_i$ and $y = y_i$

we have

$$a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)}$$

In the same way we get

$$a_2 = \frac{y_2}{(x_2 - x_0)(x_2 - x_1) \cdots (x_2 - x_n)}$$

$$a_n = \frac{y_n}{(x_n - x_1)(x_n - x_2) \cdots (x_n - x_{n-1})}$$

Sub the values of a's in ①

we get

$$y = f(x) = \frac{(x - x_1)(x - x_2) \cdots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)} y$$

$$y = f(x) = \frac{(x-x_1)(x-x_2)\cdots\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_n)} y_0$$

$$+ \frac{(x-x_0)(x-x_2)\cdots\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots\cdots(x_1-x_n)} y_1$$

$$+ \cdots\cdots$$

$$+ \frac{(x-x_0)(x-x_1)\cdots\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)} y_i$$

$$+ \cdots\cdots$$

$$+ \frac{(x-x_0)(x-x_1)\cdots\cdots(x_i-x_{n-1})}{(x_n-x_0)(x_0-x_1)\cdots\cdots(x_n-x_{n-1})} y_n \quad (2)$$

equation (2) is called lagrange's Interpolation formula for Unequal Intervals.

Problem:

Using lagrange's interpolation formula for find y (10) from the following table.

x	y
x_0 5	y_0 12
x_1 6	y_1 13
x_2 9	y_2 14
x_3 11	y_3 16

$y = f(10)$

$$= \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13) + \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16).$$

$$= \cancel{\left(\frac{+148}{+24}\right)} + \cancel{\left(\frac{65}{15}\right)} + \cancel{\left(\frac{286}{+24}\right)} + \cancel{\frac{280}{60}}$$

$$= \cancel{[illegible] + [illegible] + 11.667 + [illegible]}$$

$$= \cancel{[illegible]}$$

$$= \frac{(4)(1)(-1)}{(-1)(-4)(-6)}(12) + \frac{(5)(4)(-1)}{(1)(-3)(-5)}(13) + \frac{(5)(4)(-1)}{(4)(3)(-2)}(14) + \frac{(5)(6)(1)}{(6)(5)(2)}(16)$$

$$= +2 - 4.33 + 11.67 + 5.33$$

$$= 14.67$$

2) Using lagrange's formula find y(9.5) given

x :	7	8	9	10
y :	3	1	1	9

3)

x :	-1	0	2	3
y :	-8	3	1	12

find $x = 2.5$

2) $$f(9.5) = \frac{(9.5-8)(9.5-9)(9.5-10)}{(7-8)(7-9)(7-10)}(3) + \frac{(9.5-7)(9.5-9)(9.5-10)}{(8-7)(8-9)(8-10)}(1) + \frac{(9.5-7)(9.5-8)(9.5-10)}{(9-7)(9-8)(9-10)}(1) + \frac{(9.5-7)(9.5-8)(9.5-9)}{(10-7)(10-8)(10-9)}(9)$$

$$= \frac{(1.5)(0.5)(-0.5)}{(-1)(-2)(-3)}(3) + \frac{(2.5)(0.5)(-0.5)}{(1)(-1)(-2)}(1) + \frac{(2.5)(1.5)(-0.5)}{(2)(1)(-1)} + \frac{(2.5)(1)(0.5)}{(3)(2)(1)}(9)$$

$$= 0.1875 - 0.3125 + 0.9375 + 2.8125$$

$$= 3.625$$

0.9375
3024
9.5625
5.3125
143.4375

8) $y(2.5) = \frac{(2.5-0)(2.5-2)(2.5-3)}{(-1-0)(-1-2)(-1-3)}(-8) +$

$\frac{(2.5+1)(2.5-2)(2.5-3)}{(0+1)(0-2)(0-3)}(3) +$

$\frac{(2.5+1)(2.5-0)(2.5-3)}{(2+1)(2-0)(2-3)}(1) +$

$\frac{(2.5+1)(2.5-0)(2.5-2)}{(3+1)(3-0)(3-2)}(12) -$

$\Rightarrow -2.5 - 0.4375 + 0.7292 + 4.375$

$\Rightarrow 2.$

17/2) Inverse interpolation.

For the given set of values x and y we are finding the values of y corresponding to some $x = x_K$ (which is not given in the Table)

Here we treat y as function of x

The problem is given some y is equate to y_R $(y = y_R)$ we should find corresponding x, the process of finding x given y is called inverse interpolation.

In such a case we will take y as independent variable and x as dependent variable and use lagrange's interpolation formula.

Taking y as independent variable and x as dependent variable

Lagrange's Formula:

$$= \frac{(y-y_1)(y-y_2)(y-y_3)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)\cdots(y_0-y_n)}(x_0)$$

$$+ \frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)}(x_1)$$

$$+ \cdots + \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-1})}(x_n$$

1) Find the value of x.

When $y = 13.5$ (ans 97.6557503)

x	93	96.2	100	104.2	108.7
y	11.38	12.8	14.7	17.07 ~~12.~~	19.91

2) Find the age corresponding to the annuity value $y = 13.6$ (43)

age (x)	30	35	40	45	50
Annuity value (y)	15.9	14.9	14.1	13.3	12.5

17) $f(y) = 13.5$

$$= \frac{(13.5-12.8)(13.5-14.7)(13.5-17.07)(13.5-19.91)}{(11.38-12.8)(11.38-14.7)(11.38-17.07)(11.38-19.91)}(93)$$

$$+ \frac{(13.5-11.38)(13.5-14.7)(13.5-17.07)(13.5-19.91)}{(12.8-11.38)(12.8-14.7)(12.8-17.07)(12.8-19.91)}(96.2)$$

$$+ \frac{(13.5-11.38)(13.5-12.8)(13.5-17.07)(13.5-19.91)}{(14.7-11.38)(14.7-12.8)(14.7-17.07)(14.7-19.91)}(100)$$

$$+ \frac{(13.5-11.38)(13.5-12.8)(13.5-14.7)(13.5-19.91)}{(17.07-11.38)(17.07-12.8)(17.07-14.7)(17.07-19.91)}(104)$$

$$+ \frac{(13.5-11.38)(13.5-12.8)(13.5-14.7)(13.5-17.07)}{(19.91-11.38)(19.91-12.8)(19.91-14.7)(19.91-17.07)}(108.7)$$

$$= \frac{(0.7)(-1.2)(-3.57)(-6.41)(93)}{(-1.42)(-3.32)(-5.69)(-8.53)} + \frac{(2.12)(-1.2)(-3.57)(-6.41)(96.2)}{(1.42)(-1.9)(-4.27)(-7.11)}$$

$$+ \frac{(2.12)(0.7)(-3.57)(-6.41)(100)}{(3.32)(1.9)(-2.37)(-5.21)} + \frac{(2.12)(0.7)(-1.2)(-6.41)}{(5.69)(4.27)(2.37)(-2.84)}(104.02)$$

$$+ \frac{(2.12)(0.7)(-1.2)(-3.57)}{(8.53)(7.11)(5.21)(2.84)}(108.7)$$

$= -7.813 + 68.372 + 43.599 - 7.261 + 0.765$

$= 97.662$

2)

Age	30	35	40	45	50
Annuity	15.9	14.9	14.1	13.3	12.5

$y = 13.6$

$$
\begin{aligned}
&= \frac{(13.6-14.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)}{(15.9-14.9)(15.9-14.1)(15.9-13.3)(15.9-12.5)}(30) \\
&+ \frac{(13.6-15.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)(35)}{(14.9-15.9)(14.9-14.1)(14.9-13.3)(14.9-12.5)} \\
&+ \frac{(13.6-15.9)(13.6-14.9)(13.6-13.3)(13.6-12.5)}{(14.1-15.9)(14.1-14.9)(14.1-13.3)(14.1-12.5)}(40) \\
&+ \frac{(13.6-15.9)(13.6-14.9)(13.6-14.1)(13.6-12.5)(45)}{(13.3-15.9)(13.3-14.9)(13.3-14.1)(13.3-12.5)} \\
&+ \frac{(13.6-15.9)(13.6-14.9)(13.6-14.1)(13.6-13.3)(50)}{(12.5-15.9)(12.5-14.9)(12.5-14.1)(12.5-13.3)}
\end{aligned}
$$

$= 43$