UNIT V

Concept of bivariat Distribution

There is a possibility of assigning defining more than one random variable on the same sample space. eg. considering height and weight of every person. To describe such experiment mathematically we study two dimensional random variable

Definition

Let X and Y be two random variables defined on the same sample space S then the function (X,Y) that assign a point in $R^2 (= R \times R)$ is called two-dimensional random variable.

Let (X,Y) be a two-dimensional random variable defined on the sample space S and $w \in S$. The value of (X,Y) at w is given by the pair of real numbers $[X(w), Y(w)]$

$$X(w) = (X_1, X_2, \ldots X_n)$$

$$Y(w) = (Y_1, Y_2, \ldots Y_n)$$

and the product set

$$X(w)\, Y(w) = (X_1, X_2 \ldots X_n)\,(Y_1, Y_2 \ldots Y_n)$$

into a probability space by defining the probability of the ordered pair (x_i, y_j) to the

$P(X = x_i, Y = y_j)$ which we write $p(x_i, y_j)$

The function p on $X(S) \times Y(S)$ defined by

$$p_{ij} = P(X = x_i \cap Y = y_j) = p(x_i, y_j)$$

is called the joint probability function of X and Y is represented as

X \ Y	y_1	y_2	...	y_j	y_m	Total
x_1	p_{11}	p_{12}		p_{1j}	p_{1m}	$P_{1.}$
x_2	p_{21}	p_{22}		p_{2j}	p_{2m}	$P_{2.}$
⋮						
x_i	p_{i1}	p_{i2}		p_{i3}	p_{im}	$P_{i.}$
⋮						
x_n	p_{n1}	p_{n2}		p_{n3}	p_{nm}	p_{ni}
Total	$p_{.1}$	$p_{2.}$		$p_{.3}$	$p_{.m}$	1

Joint probability mass function

If (X, Y) is a two dimensional discrete random variable, then joint discrete function of X, Y are also called the joint probability mass function of X, Y is denoted by p_{XY} and defined as

$P_{X,Y}(x_i, y_i) = P(X = x_i, Y = y_i)$ of (X, Y)

$P_{XY}(x_i, y_i) = 0$

Marginal probability Function:

Let (X, Y) be a discrete two-dimensional r.v which takes up countable no of values (x_i, y_i) then the prob distribution of X is determined as

$p_X(x_i) = P(X = x_i)$

$= P(X = x_i \cap Y = y_1) + P(X = x_i \cap Y = y_2) + \dots + P(X = x_i \cap Y = y_m)$

$= P_{i1} + P_{i2} + \dots + p_{im}$

$= \sum_{j=1}^{m} p_{ij}$

$= \sum_{j=1}^{m} (x_i, y_i)$

$= p_{i\cdot}$

is known as marginal probability mass function or discrete marginal density function X.

also. $\sum_{i=1}^{n} p_{i\cdot} = p_{1\cdot} + p_{2\cdot} + \dots + p_{n\cdot}$

$= \sum_{i=1}^{n} \sum_{j=1}^{m} p_{ij} = 1$

Similarly

$$P_y(y_j) = \sum_{i=1}^{n} p_{ij}$$

$$= \sum_{i=1}^{n} p(x_i, y_j)$$

$$= p_{\cdot j}$$

which is the probability mass function of y.

Conditional probability function.

Let (X, Y) be a discrete two-dimensional random variable. Then the conditional discrete density function or conditional probability mass function of X given $Y = y$ denoted by

$$p_{X/Y}(x, y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

lly the conditional probability mass function $p_{Y/X})$ is similarly defined

as $$p_{Y/X}(y/x) = \frac{P(X = x, Y = y)}{P(X = x)}$$

Two dimensional distribution function

The distribution function of the two dimensional random variable (X, Y) is a real valued function F defined for all real x and y be the relation :

$$F_{xy}(x,y) = p(X \leq x,\ Y \leq y)$$

Marginal distribution function

From the joint distribution function $F_{xy}(x,y)$ it is possible to obtain the individual distribution functions $F_x(x)$ and $F_y(y)$ which are termed as marginal distribution function of x and y respectively with respect to the joint distribution function $F_{xy}(x,y)$

$$F_x(x) = p(X \leq x) = p(X \leq x,\ Y < \infty)$$

$$= \lim_{y \to \infty} F_{xy}(x,y)$$

$$= F_{xy}(x,\infty)$$

$|||^{ly}$ $F_y(x) = p(Y \leq y) = p(X < \infty,\ Y \leq \infty)$

$$= \lim_{x \to \infty} F_{xy}(x,y)$$

$$= F_{xy}(\infty,y)$$

$F_x(x)$ is termed as marginal distribution of X corresponding to the joint distribution function $F_{xy}(x,y)$

$|||^{ly}$

$F_y(y)$ is termed as marginal distribution of the random variable y corresponding to joint distribution function $F_{xy}(x,y)$

Discrete r.v's :

$$F_X(x) = \sum_{y} P(X \leq x, Y = y)$$

$$F_Y(y) = \sum_{x} P(X = x, Y \leq y)$$

continous r.v's

$$F_X(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{xy}(x,y)\, dx .$$

$$F_Y(y) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} f_{xy}(x,y)\, dy .$$

conditioo probability density functioo

Let (X,Y) be two jointly distributed continou random variable with joint density function $f(x,y)$

The conditional density function of y is defined as

$$f(x/y) = \frac{f(x,y)}{f(y)}$$

The conditional denity functioo of x is given by

$$f(y/x) = \frac{f(x,y)}{f(x)}$$

Independent random Variable.

Two random variable x and y are said to be independent if

i) $P(X = x_i, Y = y_j) = P(X = x_i) \cdot P(Y = y_j)$
if x and y are discret

ii) $f(x, y) = f(x) \cdot f(y)$
if x and y are continious

Discret random Variables

1. The joint probability distn of two random variables X and y is given by $P(X=0, Y=1) = 1/3$, $P(X=1, Y=-1) = 1/3$ and $P(X=1, Y=1) = 1/3$.
Find i) Marginal distributions of X and y
ii) conditional probability distribution of X given $Y=1$

Solution:

Y \ X	-1	0	1	Marginal Y
-1	0	0	1/3	1/3
0	0	0	0	0
1	0	1/3	1/3	2/3
Marginal X	0	1/3	2/3	1

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Marginal distribution of x is.

Values of x	-1	0	1
$P(X=x)$	0	1/3	2/3

Marginal distribution of y is.

Values of y	-1	0	1
$P(Y=y)$	1/3	0	2/3

Conditional probability distribution of x given y is.

$$P(X=x/Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}.$$

$$P(X=-1/Y=1) = \frac{P[(X=-1) \cap P(Y=1)]}{P(Y=1)}$$

$$= 0$$

$$P(X=0/Y=1) = \frac{P[X=0] \cap P(Y=1)}{P(Y=1)}$$

$$= \frac{1/3}{2/3} = 1/2$$

$$P(X=1/Y=1) = \frac{P(X=1] \cap P(Y=1)}{P(Y=1)}$$

$$= \frac{1/3}{2/3} = 1/2$$

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2. The joint probability distribution of 2 r.v's are given below

$y \backslash x$	1	2	3	Total
1	0.1	0.1	0.2	0.4
2	0.2	0.3	0.1	0.6
Total	0.3	0.4	0.3	

i) Find the marginal prob of x, y.

ii) Find the conditional probabilities of x given y = 1

iii) Find the conditional probabilities of y given x = 2.

i.) Marginal prob of x

$$P[X = x_i] = p_{i.} = \sum_j p_{ij}$$

$$P(X=1] = 0.1 + 0.2 = 0.3$$

$$P[X=2] = 0.1 + 0.3 = 0.4$$

$$P[X=3] = 0.2 + 0.1 = 0.3$$

ii) Marginal prob of y

$$P[Y = y_j] = p_{.j} = \sum_i p_{ij}$$

$P[y=1] = 0.1 + 0.1 + 0.2 = 0.4$

$P[y=2] = 0.2 + 0.3 + 0.1 = 0.6$

2. conditional prob of y given x = 2

$$p(y_i / x=1) = \frac{p[y = y_i \cap x=2]}{p(x=2)}$$

$$p(y=1/x=1) = \frac{P(y=1 \cap x=2)}{P(x=2)}$$

$$= 0.1/0.4$$

$$= 0.25$$

$$p(y=2/x=1) = \frac{P(y=2 \cap x=1)}{P(x=1)}$$

$$= 0.3/0.4$$

$$= 0.75$$

3. The joint prob of two random variable are given below

	X	-1	0	1
y	0	1/15	2/15	1/15
	1	3/15	2/15	1/15
	2	2/15	1/15	2/15

1. Find the marginal prob of x and y

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2. Find the conditional prob of x given $y = 1$

3. Find the conditional prob of y given $x = 0$

i) Marginal prob of x

$P[X = x_i] = p_{i.} = \sum_j p_{ij}$

$P[X = -1] = \frac{1}{15} + \frac{3}{15} + \frac{2}{15}$

$= \frac{1+3+2}{15}$

$= 6/15$

$P[X = 0] = 2/15 + \frac{2}{15} + \frac{1}{15}$

$= \frac{2+2+1}{15}$

$= 5/15$

$P[X = 1] = 1/15 + 1/15 + 2/15$

$= 4/15$

Marginal prob of y

$P[Y = y_j] = p_{.j} = \sum_i p_{ij}$

$P[Y = 0] = \frac{1}{15} + \frac{2}{15} + \frac{1}{15}$

$= 4/15$

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$p[y=1] = 3/15 + 2/15 + 1/15$

$= 6/15$

$p[y=2] = 2/15 + 1/15 + 2/15$

$= 5/15$

conditional prob of x given y = 1

$$p(x/y=1) = \frac{p(x=x_i \cap y=1)}{p(y=1)}$$

$$p\left(\frac{x=-1}{y=1}\right) = \frac{p(x=-1 \cap y=1)}{p(y=1)}$$

$$= \frac{3/15}{6/15}$$

$$= 3/6 = 1/2$$

$$p(x=0/y=1) = \frac{p(x=0 \cap y=1)}{p(y=1)}$$

$$= \frac{2/15}{6/15} = 2/6 = 1/3$$

$$p(x=1/y=1) = \frac{p(x=1 \cap y=1)}{p(y=1)}$$

$$= \frac{2/15}{6/15} = 1/6$$

conditional prob of y given $x=0$

$$p(y/x=0) = \frac{p(y=y_j \cap x=0)}{p(x=0)}$$

$$p(y=0/x=0) = \frac{p(y=0 \cap x=0)}{p(x=0)}$$

$$= \frac{2/15}{5/15}$$

$$= 2/5$$

$$p(y=1/x=0) = \frac{p(y=1 \cap x=0)}{p(x=0)}$$

$$= \frac{2/15}{5/15}$$

$$= 2/15$$

$$p(y=2/x=0) = \frac{p(y=1 \cap x=0)}{p(x=0)}$$

$$= \frac{1/15}{5/15}$$

$$= 1/5$$

A two dimensional r.v (X, Y) have bivariate distribution given by

$$P(X=x, Y=y) = \frac{x^2+y}{32}$$

for $x = 0, 1, 2, 3$

$y = 0, 1$

Find the marginal distributions of x and y.

Solution:

y \ x	0	1	2	3	Marginal distn of y
0	0	1/32	4/32	9/32	14/32
1	1/32	2/32	5/32	10/32	18/32
Marginal distn of x	1/32	3/32	9/32	19/32	

continous Random variable.

1. If x and y are two random variables having joint density function

$$f(x,y) = \begin{cases} \frac{1}{8}(6-x-y) & 0 \le x < 2, \; 2 \le y \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Find

i) $P(X<1 \cap Y<3)$

ii) $P(X+Y<3)$

iii) $P(X<1 / Y<3)$

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i) $p(X<1 \cap Y<3) = \int_{-\infty}^{1}\int_{-\infty}^{3} f(x,y)\,dx\,dy$

$= \int_{0}^{1}\int_{2}^{3} \frac{1}{8}(6-x-y)\,dx\,dy$

$= 3/8$

ii) $p(X+Y<3) = \int_{0}^{1}\int_{2}^{3-x} \frac{1}{8}(6-x-y)\,dx\,dy$

$= 5/24$

iii) $P(X<1 / Y<3) = \dfrac{p(X<1 \cap Y<3)}{p(Y<3)}$

$= \dfrac{3/8}{5/8}$

$= 3/5$

2. The joint probability density function of two random variables (X,Y) is given by

$$f(x,y) = \frac{9(1+x+y)}{2(1+x)^4(1+y)^4} \qquad \begin{matrix} 0<x<\infty \\ 0<y<\infty \end{matrix}$$

1) Find the marginal density function of X and Y.

2. Find the conditional density function of X and Y

Solution:

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \frac{9}{2}\int_{0}^{\infty} \frac{(1+x+y)}{(1+x)^4 (1+y)^4}\, dy$$

$$= \frac{9}{2}\int_{0}^{\infty} \frac{(1+y) + x}{(1+x)^4 (1+y)^4}\, dy$$

$$= \frac{9}{2(1+x)^4} \int_{0}^{\infty} \frac{(1+y)}{(1+y)^4}\, dy + x \int_{0}^{\infty} \frac{1}{(1+y)^4}\, dy$$

$$= \frac{9}{2(1+x)^4} \left\{ \int_{0}^{\infty} \left[\frac{1+y}{(1+y)^4}\right] dy + x \int_{0}^{\infty} \frac{1}{(1+y)^4}\, dy \right.$$

$$= \frac{9}{2(1+x)^4} \left[\frac{1}{-2(1+y)^2}\right]_0^{\infty} + \left[\frac{x}{-3(1+y}\right]_0^{\infty}$$

$$= \frac{9}{2(1+x)^4} \left[\frac{1}{2} + \frac{x}{3}\right]$$

$$= \frac{9}{2(1+x)^4} \cdot \frac{3+2x}{6}$$

$$f(x) = \frac{\cancel{9}\,3}{2(1+x)^4} \cdot \frac{3+2x}{\cancel{6}\,2}$$

$$= \frac{3}{4} \cdot \frac{3+2x}{(1+x)^4}$$

$$f(y) = \int f(x,y)\, dx$$

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$$= \frac{9}{2}\int_0^\infty \frac{(1+x+y)}{(1+x)^4\,(1+y)^4}\,dx$$

$$= \frac{9}{2(1+x)^4}\int_0^\infty \frac{(1+x)+y}{(1+x)^4\,(1+y)^4}\,dx$$

$$= \frac{9}{2(1+y)^4}\int_0^\infty \frac{1}{(1+x)^3}\,dx + y\int \frac{1}{(1+x)^4}\,dx$$

$$= \frac{9}{2(1+y)^4}\left[\frac{1}{-2(1+x)^3}\right]_0^\infty + \left[\frac{y}{-3(1+x)}\right]_0^\infty$$

$$= \frac{9}{2(1+y)^2}\left[\frac{1}{2}+\frac{y}{3}\right]$$

$$= \frac{\cancel{9}\,3}{2(1+y)^4}\;\frac{3+2y}{\cancel{6}\,2}$$

$$f(x) = \frac{3}{4}\;\frac{3+2y}{(1+y)^4}$$

$$f(x/y) = \frac{f(x,y)}{f(y)}$$

$$= \frac{9(1+x+y)\,/\,2(1+y)^4)}{3/4 \times \frac{3+2y}{(1+y)^4}}$$

$$= \frac{6\times(1+x+y)}{(1+x)^4\,(3+2y)}$$

$$f(y/x) = \frac{f(x,y)}{f(x)}$$

$$= \frac{9(1+x+y)}{2(1+x)^4 (1+y)^4}$$

$$= \frac{4(1+x)^4}{3(3+2x)}$$

$$= \frac{6(1+x+y)}{(1+x)^4 (3+2x)}$$