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UNIT IV

Moment Generating function

Let X be a random variable its moment generating function is denoted by $M_x(t)$ and where $f(x)$ is its probability function hence

$$M_x(t) = E(e^{tx})$$

$$= \sum_x e^{tx} p(x) \quad \text{for discrete r.v}$$

$$= \int e^{tx} f(x)\, dx \quad \text{for continous r.v}$$

here the integeration or summation being extended to entire range of x
t - the real parameter and $-h < t < h$.

Relationship between moments and M.G.F

Consider MGF of r.v X

$$M_x(t) = E(e^{tx})$$

We know that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

$$M_x(t) = E(e^{tx})$$

$$= E\left[1 + \frac{t(x)}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots\right]$$

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$$= E\left[1 + \frac{t}{1!}x + \frac{t^2}{2!}x^2 + \frac{t^3}{3!}x^3 + \cdots\right]$$

$$= 1 + \frac{t}{1!}E(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \cdots$$

$$= 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots \quad \text{--- (1)}$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

$$\mu_r' = E(x^r) = \begin{cases} \sum x^r p(x) \rightarrow x \text{ is discrete r.v} \\ \int_{-\infty}^{\infty} x^r f(x)\,dx \rightarrow x \text{ is continuous r.v} \end{cases}$$

is the rth moment of x about origin.

Thus co-eff of $\frac{t^r}{r!}$ in $M_x(t)$ gives μ_r'

Since $M_x(t)$ generate moments it is known as moment generating function.

Diff (1) w.r.t r times and sub $t=0$

$$\frac{d^r}{dt^r} M_x(t)\Big|_{t=0} = \left[\frac{\mu_r'}{r!} r! + \mu_{r+1}' t + \mu_{r+2}' \frac{t^2}{2!} + \cdots\right]$$

$$\Rightarrow \mu_r' = \frac{d^r}{dt^r} M_x(t)\Big|_{t=0}$$

$\mu_r' = E[(x-a)^r]$ is the rth moment

about the point $x = a$

Limitations of M.G.F

1. A random variable x may have no moments although its m.g.f exists

2. A random variable X can have MGF or all moments yet the MGF does not generate moments

3. A random variable x can have all or some moments but MGF does not exists except perhaps some points

Properties of moment generating function

1. $M_{cx}(t) = M_x(ct)$ where c is a constant

Proof:

LHS $M_{cx}(t) = E(e^{tcx})$

RHS $M_x(ct) = E(e^{ctx})$

$= LHS$

$\therefore M_{cx}(t) = M_x(ct)$

2. Additive property of M.G.F

The MGF of the sum of a number of independent random variables is equal to the product of their respective M.G.F's symbolically

If $X_1, X_2 \ldots X_n$ are independent r.v's then M.G.F of their sum $X_1+X_2+\cdots+X_n$ is given by

$$M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t)\, M_{X_2}(t) \cdots M_{X_n}(t)$$

Proof:

By def

$$M_{X_1+X_2+\cdots+X_n}(t) = E\left[e^{t(X_1+X_2+\cdots+X_n)}\right]$$

$$= E\left[e^{tX_1} \cdot e^{tX_2} \cdots e^{tX_n}\right]$$

$$= E(e^{tX_1}) \cdot E(e^{tX_2}) \cdots E(e^{tX_n}) \text{ (}$$

$$= M_{X_1}(t)\, M_{X_2}(t) \cdots M_{X_n}(t)$$

since $X_1, X_2 \ldots X_n$ are independent

3. Effect of change of origin and scale on M.G.F.

Let us transform X to a new r.v U by changing both the origin and scale in X

$$U = \frac{X-a}{h} \quad \text{where } a \text{ and } h \text{ are constants}$$

M.G.F of U about origin is

$$M_U(t) = E(e^{tU}) = E\left[e^{t[(X-a)/h]}\right]$$

$$= E\left[e^{tx/h} \cdot e^{-at/h}\right]$$

$$-at/h \quad \left[\ e^{tx/h}\ \right]$$

$= e^{-at/h} M_x(t/h)$

where $M_x(t)$ is M.G.F of x about origin

Uniqueness Theorem

The MGF of a distribution if it exists, uniquely determines the distribution

Importance of uniqueness theorem

This theorem implies that corresponding to a given probability distribution, there is only one MGF (provided it exists) and corresponding to a given m.g.f there is only one probability distribution

Hence $M_x(t) = M_y(t)$
$\Rightarrow$ x and y are identically distributed

Cumulants

cumulants generating function $k(t)$ is defined as $K_x(t) = \log_e M_x(t)$

$$K_x(t) = \log_e M_x(t)$$

where

$$K_x(t) = K_1(t) + K_2 \frac{t^2}{2!} + \cdots + K_r \frac{t^r}{r!} = \log M_x(t)$$

$$\log(x) = \log\left(1 + \mu_1'(t) + \mu_2'\frac{t^2}{2!} + \mu_3'\frac{t^3}{3!} + \cdots + \mu_r'\frac{t^r}{r!} + \cdots\right)$$

where K_r = co-eff of $\frac{t^r}{r!}$ in $K_x(t)$ is called rth cumulant.

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$K_1(t) + K_2\frac{t^2}{2!} + K_3\frac{t^3}{3!} + K_r\frac{t^r}{r!} = \left(\mu_1'(t) + \mu_2'\frac{t^2}{2!} + \mu_3'\frac{t^3}{3!} + \cdots\right)$$

$$- \frac{1}{2}\left(\mu_1'(t) + \mu_2'\frac{t^2}{2!} + \mu_3'\frac{t^3}{3!} + \cdots\right)^2$$

$$+ \frac{1}{3}\left(\mu_1'(t) + \mu_2'\frac{t^2}{2!} + \mu_3'\frac{t^3}{3!} + \cdots\right)^3 - \frac{1}{4}\left(\mu_1'(t) + \mu_2'\frac{t^2}{2!} + \cdots\right)^4 + \cdots$$

comparing the co-eff of like powers of t on both sides. We get the relationship between moments and cumulants

$$K_1(t) = \mu_1' = \text{mean.}$$

$$\frac{K_2}{2!} = \frac{\mu_2'}{2!} - \frac{(\mu_1')^2}{2!} \Rightarrow K_2 = \mu_2' - (\mu_1')^2$$

$$\frac{K_3}{3!} = \frac{\mu_3'}{3!} - \frac{1}{2}\left(\frac{2\mu_1'\mu_2'}{2!}\right) + \frac{\mu_1'^3}{3}$$

$$= \frac{\mu_3'}{3!} - \frac{3\mu_1'\mu_2'}{3!} + \frac{2(\mu_1')^3}{3!}$$

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$$k_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = \mu_3$$

$$\frac{k_4}{4!} = \frac{\mu_4'}{4!} - \frac{1}{2}\left(\frac{\mu_2'^2}{4} + \frac{2\mu_1'\mu_3'}{3!} + \dots\right) + \frac{1}{3}\left(\frac{3\mu_1'^2\mu_2'}{2}\right) - \frac{\mu_1'^4}{4}$$

$$\frac{k_4}{4!} = \frac{\mu_4'}{4!} - \frac{1}{2}\left(\frac{3\mu_2'^2 + 4\mu_1'\mu_3'}{12} + \dots\right) + \frac{1}{3}\left(\frac{12\mu_1'^2\mu_2'}{2\times 4}\right) - \frac{6\mu_1'^4}{4\times 3\times 2}$$

$$k_4 = \mu_4' - 3\mu_2'^2 - 4\mu_1'\mu_3' + 12\mu_1'^2\mu_2' - 6\mu_1'^4$$

$$= \mu_4' - 3\mu_2'^2 - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 + 6\mu_2'\mu_1'^2 - 3\mu_1'^4 - 3\mu_1'^4$$

$$= (\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4) - 3(\mu_2'^2 - 2\mu_2'\mu_1'^2 + \mu_1'^4)$$

$$= \mu_4 - 3(\mu_2' - \mu_1'^2)^2$$

$$k_4 = \mu_4 - 3\mu_2^2 \qquad \mu_2 = k_2$$

$$k_4 = \mu_4 - 3\mu_2^2$$

$$\mu_4 = k_4 + 3k_2^2$$

$\therefore$ Mean $= k_1$ $\qquad$ $\mu_2 = k_2$ (Variance)

$\mu_3 = k_3$ $\qquad$ $\mu_4 = k_4 + 3k_2^2$

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Properties of cummulants

1. Additive property of cumulants.

The r^{th} cumulant of the sum of the independent random variables is equal to the sum of the r^{th} cumulants of the individual variables

symbolically

$$K_r(X_1+X_2+\cdots+X_n) = K_r(X_1)+K_r(X_2)+\cdots+K_r(X_n)$$

where $i=1,2,\ldots n$ are independent random variables.

2. Effect of change of origin and scale on cumulants

If $u = \frac{X-a}{h}$

$$M_u(t) = e^{(-at/h)} M_x(t/h)$$

Except first cumulant, all cumulants are independent of change of origin.

But the cumulants are not invariant of change of scale

Characteristic function

Let X be r.v that characteristic fn is denoted by $C_x(t)$

$$C_x(t) = E[e^{itx}]_2$$

$$= e^{itx} p(x) \quad \text{if } x \text{ is discrete}$$

$$= \int e^{itx} f(x)dx \quad \text{if } x \text{ is continuous}$$

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Properties of C.F

1) $c(0) = \int_{-\infty}^{\infty} d f(x) = 1$

2. $|c(t)| \leq 1$

3. $C_x(t)$ is continuous everywhere

ie $C_x(t)$ is a continuous function of t in $(-\infty, \infty)$

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Chebyshev's Inequality

Statement:

If x is a random variable with mean μ and variance σ^2 then for any +ve no k we have.

$$P\{|x-\mu| \geq k\sigma\} \leq 1/k^2 \quad \text{--- (1)}$$

$$P\{|x-\mu| < k\sigma\} \geq 1-1/k^2 \quad \text{--- (2)}$$

Proof:

If x is a continous r.v by def

$$\sigma^2 = \sigma_x^2 = E(x-E(x))^2$$

$$= E(x-\mu)^2$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx.$$

where $f(x)$ is p.d.f of x.

$$= \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu-k\sigma}^{\mu+k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$\geq \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)\,dx \quad \text{--- (1)}$$

We know that

$x \leq \mu - k\sigma$ and $x \geq \mu + k\sigma$

$\Rightarrow |x-\mu| \geq k\sigma$ --- (2)

substituting ② in ①

$$\Rightarrow \sigma^2 \geq k^2\sigma^2 \left[\int_{-\infty}^{\mu-k\sigma} f(x)\,dx + \int_{\mu+k\sigma}^{\infty} f(x)\,dx \right]$$

$$= k^2\sigma^2 \left[P(X \leq \mu - k\sigma) + P(X \geq \mu + k\sigma) \right]$$

$$= k^2\sigma^2 \, P(|X-\mu| \geq k\sigma)$$

$$\Rightarrow P(|X-\mu| \geq k\sigma) \leq \frac{\sigma^2}{k^2\sigma^2} = 1/k^2$$

$$\Rightarrow P\{|X-\mu| \geq k\sigma\} \leq \frac{1}{k^2} \quad \text{—— ③}$$

Also.

$$P\{|X-\mu| \geq k\sigma\} + P\{|X-\mu| < k\sigma\} = 1$$

$$\Rightarrow P[|X-\mu| < k\sigma] = 1 - P[|X-\mu| \geq k\sigma]$$

$$\geq 1 - 1/k^2$$

$$\therefore P\{|X-\mu| < k\sigma] \geq 1 - 1/k^2 \quad \text{—— ④}$$

$\therefore$ from ③ and ④ we have ~~proved~~ established the theorem

Weak law of large numbers

Statement:

Let $X_1, X_2, \ldots X_n$ be a sequence of random variable and $\mu_1, \mu_2 \ldots \mu_n$ be their respective expectation and let

$$B_n = Var(X_1 + X_2 + \cdots + X_n) < \infty$$

then

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$$P\left\{\left|\frac{X_1+X_2+\dots X_n}{n}-\frac{M_1+M_2+\dots M_n}{n}\right|<\epsilon\right\}\geq 1-\eta$$

$\forall$ $n > n_0$ where ϵ and η are arbitrary small positive numbers provided.

$$\lim_{n\to\infty}\frac{B_n}{n^2}\to 0$$

Proof:

Using chebychev's inequality

$P[|X-M| < K\sigma] \geq 1-1/K^2$ to the random variables $(X_1+X_2+\dots+X_n)/n$ we get $\epsilon > 0$

$$P\left\{\left|\left(\frac{X_1+X_2+\dots+X_n}{n}\right)-E\left(\frac{X_1+X_2+\dots+X_n}{n}\right)\right|<\epsilon\right\}$$

$$\geq 1-\frac{B_n^2}{n^2\epsilon^2}$$

as $Var(X_1+X_2+\dots X_n) = B_n$

$$Var\left(\frac{X_1+X_2+\dots X_n}{n}\right) = \frac{1}{n^2}B_n = \frac{B_n}{n^2}$$

$$\Rightarrow P\left\{\left|\frac{X_1+X_2+\dots X_n}{n}-\frac{M_1+M_2+\dots+M_n}{n}\right|<\epsilon\right\}\geq$$

$$1-\frac{B_n^2}{n^2\epsilon^2}$$

Assumption about B_n is not done and also n is an increasing value then

$\frac{B_n}{n^2 \epsilon^2} \to 0$ as n is indefinitely large.

Thus having chosen 2 arbitrary small positive numbers ϵ and η a number n_0 can be found so that $\frac{B_n}{n^2 \epsilon^2} < \eta$ will hold for $n > n_0$

consequently

$$P\left\{\left|\frac{X_1+X_2+\dots+X_n}{n} - \frac{\mu_1+\mu_2+\dots+\mu_n}{n}\right| < \epsilon\right\} \geq 1-\eta$$

$\forall\ n > n_0$.

This leads to important result known as Weak law of large number.

With the probability approaching unity or certainity as near as we please we may expect that the arithmetic mean of values actually assumed by n random variables will differ from the arithmetic mean of their expectations by less than any given number, however small provided the no of variables taken is sufficiently large

$\frac{B_n}{n^2} \to 0$ as $n \to \infty$ is fulfilled.

ie

$$\bar{X}_n \xrightarrow{P} \bar{\mu}_n$$

provided $B_n/n^2 \to 0$

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Central limit theorem.

The central limit theorem can be expressed as

If x_i $(i = 1, 2 \dots n)$ be independent random variables such that $E(x_i) = \mu$ and $V(x_i) = \sigma_i^2$ then

under certain very general conditions, the random variable $S_n = X_1 + X_2 + \dots + X_n$ is asymptotically normal with mean μ and S.D σ

where

$$\mu = \sum_{i=1}^{n} M_i \quad \text{and} \quad \sigma^2 = \sum_{i=1}^{n} \sigma_i^2$$

De-Moivre's Laplace theorem

De Moivre's theorem states as

$$x_i = \begin{cases} 1 & \text{with prob } p \\ 0 & \text{" " } q \end{cases}$$

then the ~~distn~~ distribution of random varibles $S_n = X_1 + X_2 + \dots + X_n$

where x_i's are independent, is asymptotically normal as $n \to \infty$