4

LIMITS AND METRIC SPACES

4.1 LIMIT OF A FUNCTION ON THE REAL LINE

In Chapter 2 we define limit of a sequence. We now recall from calculus the definition of limit of a "function of a real variable" on which are based the definitions of continuous function and derivative. Later in the chapter we generalize to a wide class of spaces (called metric spaces) which includes the real line R as a very special case.

Let $a \in R$, and let f be a real-valued function whose domain includes all points in some open interval $(a-h, a+h)$ except possibly the point a itself.

4.1A. DEFINITION. We say that $f(x)$ approaches L (where $L \in R$) as x approaches a if given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon \qquad (0 < |x-a| < \delta).$$

In this case we write $\lim_{x \to a} f(x) = L$ or $f(x) \to L$ as $x \to a$. We sometimes say "f has the limit L at a" instead of "$f(x)$ approaches L as x approaches a."

We emphasize that the point a need not be in the domain of f. You remember from calculus (we hope) that $\lim_{x \to 0}(\sin x / x) = 1$, even though $\sin x / x$ is not defined for $x = 0$. We verify this limit in a later chapter, after we give a rigorous definition of $\sin x$.

Consider Figure 12. In order for $f(x)$ to approach L as x approaches a the following must be true: Given any ϵ parentheses about L there must exist δ parentheses about a such that every arrow which begins inside the δ parentheses (except possibly the arrow, if there is one, that starts at a) must end inside the ϵ parentheses.

Roughly speaking, the following can be seen on the x-y graph of a function f such that $\lim_{x \to a} f(x) = L$: As the x coordinate of a moving point on the graph gets close to a (from either the right or the left), the height $f(x)$ of the point heads toward L. Be sure to think out why this is a geometric interpretation of definition 4.1A. For example, the functions in Figures 13, 14, and 15 all satisfy $\lim_{x \to a} f(x) = L$. (An empty dot, as in Figure 15, indicates a point not on the graph.) On the other hand, the function in Figure 16 has no limit at a. This is because $f(x)$ gets close to 3 when x gets close to a on the left, while $f(x)$ gets close to 1 when x gets close to a on the right. Hence there is no single number L such that $f(x)$ gets close to L when x gets close to a.

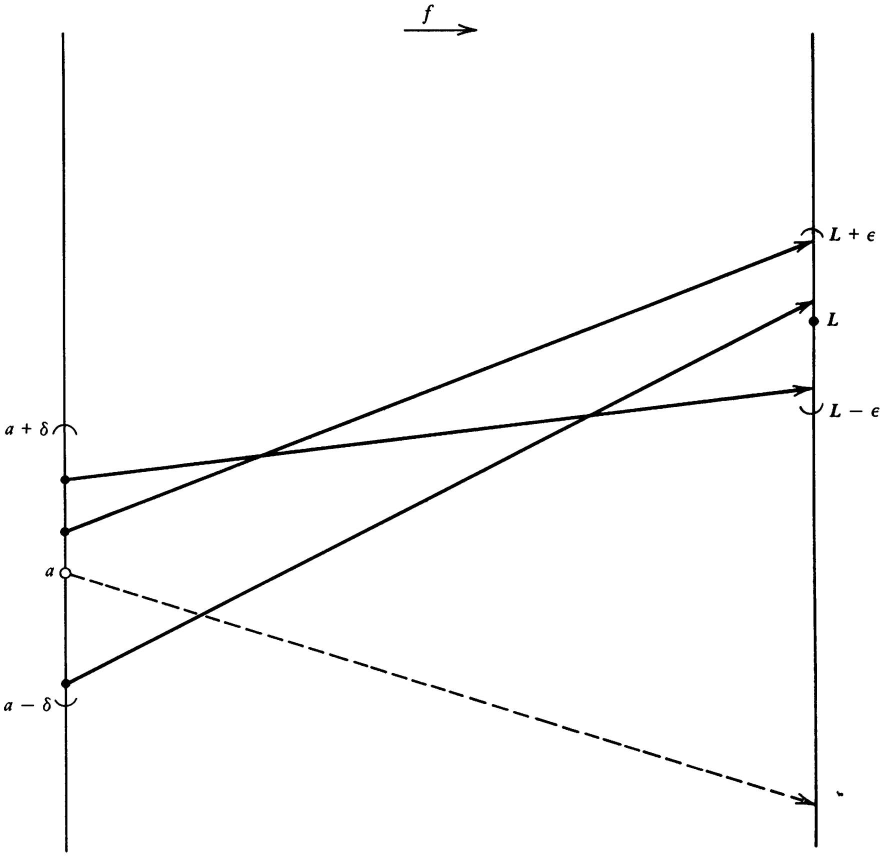

FIGURE 12.

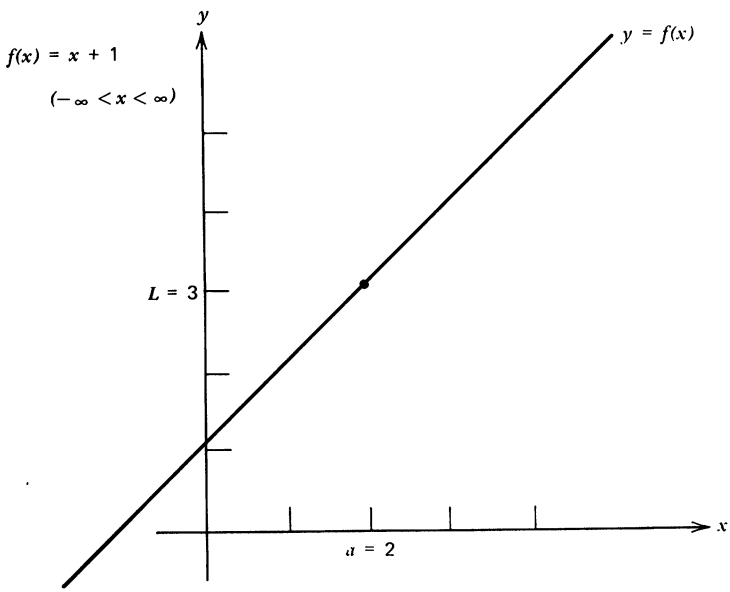

FIGURE 13. Example 1 of $\lim_{x \to a} f(x) = L$

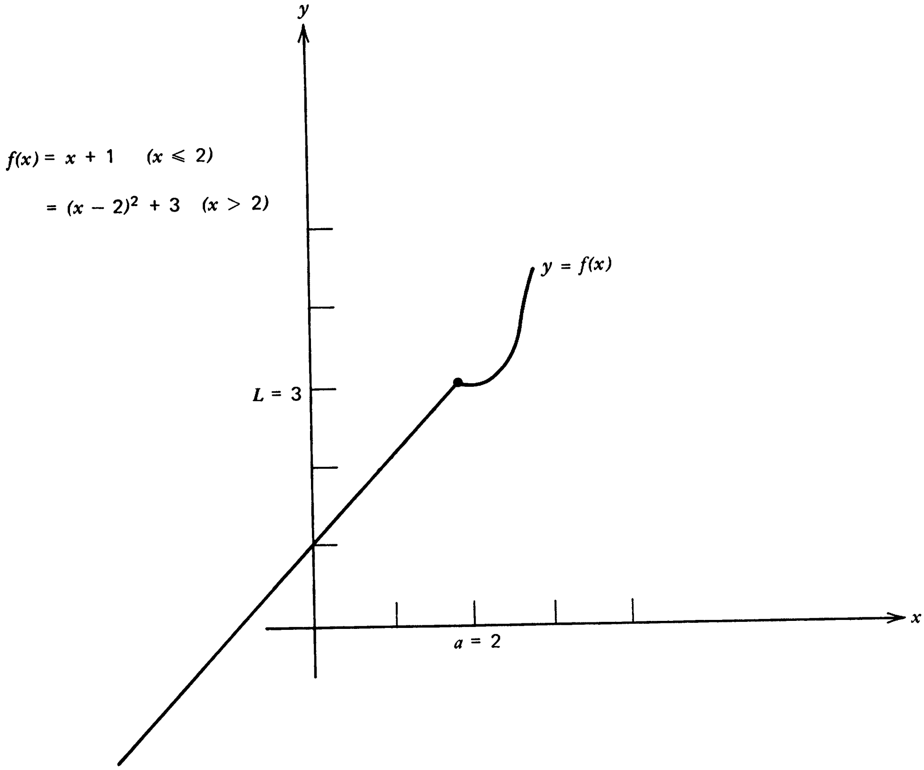

FIGURE 14. Example 2 of $\lim_{x \to a} f(x) = L$

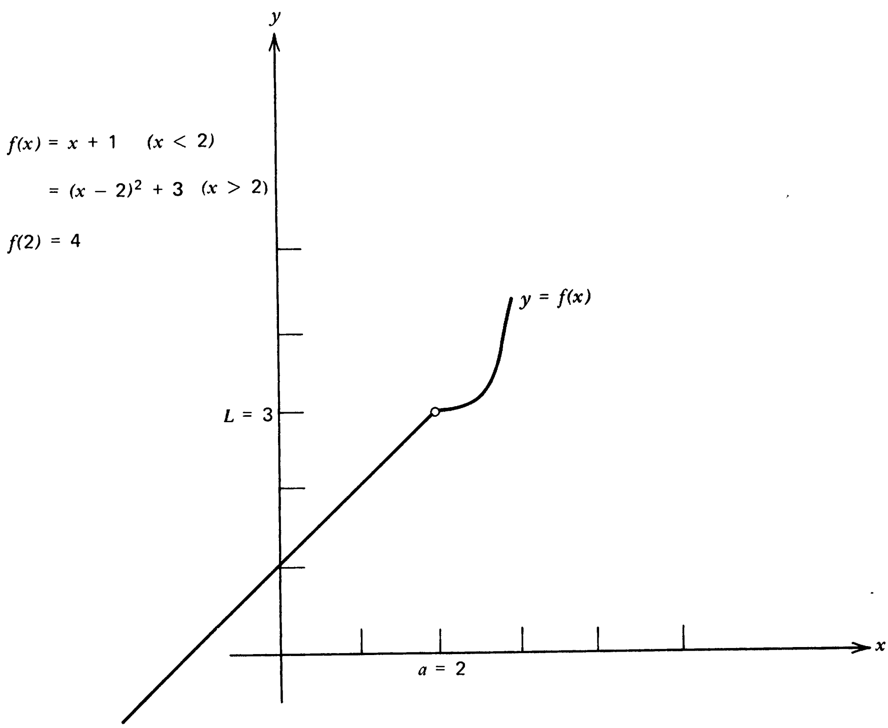

FIGURE 15. Example 3 of $\lim_{x \to a} f(x) = L$

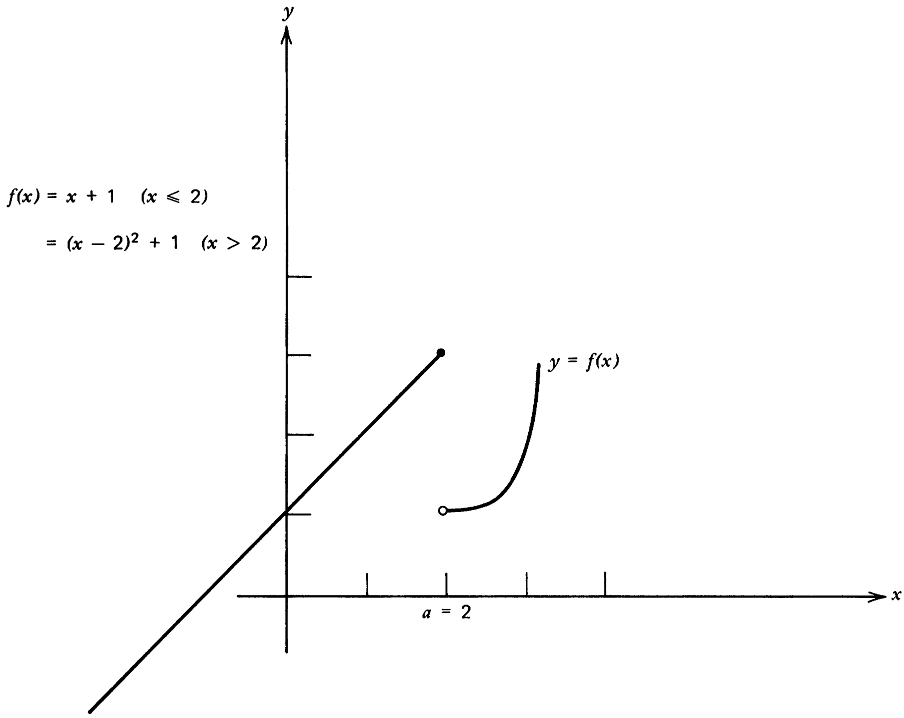

FIGURE 16. Example of a function f such that $f(x)$ does not approach a limit as x approaches a

For a final pictorial example, consider Figure 17. This shows the graph of f where

$$f(x) = \sin \frac{1}{x} \qquad (x \neq 0).$$

Here, as x gets close to $a = 0$, the value $f(x)$ oscillates rapidly. Even if we look on only one side of a, it is clear that there is no number L toward which the value $f(x)$ tends. Hence f has no limit at 0.

Now, for some examples with *proofs*.

First, let us prove $\lim_{x \to 3}(x^2 + 2x) = 15$. Here $f(x) = x^2 + 2x$, $L = 15$, $a = 3$. Given $\epsilon > 0$ we must find $\delta > 0$ such that

$$|(x^2 + 2x) - 15| < \epsilon \qquad (0 < |x - 3| < \delta). \tag{1}$$

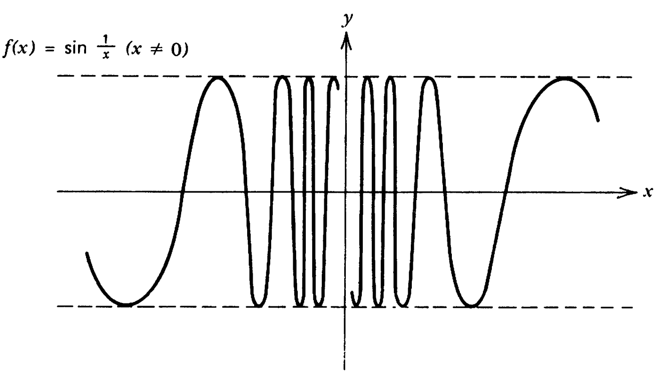

FIGURE 17. The function has no limit at $a = 0$

We first note that $|(x^2+2x)-15|=|x-3|\cdot|x+5|$. We are going to have $|x-3|<\delta$. The question is, how big can $|x+5|$ be? Without making our final choice of δ, let us agree that when we do choose it we will take $\delta \leqslant 1$. Then, if $|x-3|<\delta$, we will have $|x-3|<1$. Thus $x\in(2,4)$ and so $x+5\in(7,9)$. Hence $|x+5|<9$ if $|x-3|<\delta<1$, and so $|x-3|\cdot|x+5|<\delta\cdot 9$ if $|x-3|<\delta$ and $\delta\leqslant 1$. Let $\delta=\min(1,\epsilon/9)$. Then

$$|x-3|\cdot|x+5|<9\delta\leqslant\epsilon \qquad (|x-3|<\delta),$$

which implies (1). Given $\epsilon>0$ we have found a δ [namely $\delta=\min(1,\epsilon/9)$] for which (1) holds, and this proves $\lim_{x\to 3}(x^2+2x)=15$. Note that in this example $a=3$ and $|f(x)-L|<\epsilon$ even for $x=a$.

For a second example we will show that $\lim_{x\to 1}\sqrt{x+3}=2$. Here $f(x)=\sqrt{x+3}$, $L=2, a=1$. Given $\epsilon>0$ we must find $\delta>0$ such that

$$|\sqrt{x+3}-2|<\epsilon \qquad (0<|x-1|<\delta). \tag{2}$$

Multiplying the left side by $|(\sqrt{x+3}+2)/(\sqrt{x+3}+2)|$, we see that (2) is equivalent to

$$\frac{|(\sqrt{x+3})^2-2^2|}{|\sqrt{x+3}+2|}<\epsilon \qquad (0<|x-1|<\delta)$$

or

$$\frac{|x-1|}{|\sqrt{x+3}+2|}<\epsilon \qquad (0<|x-1<\delta). \tag{3}$$

If we agree to take $\delta\leqslant 1$, then $|x-1|<\delta$ implies $x\in(0,2)$ and hence $\sqrt{x+3}+2>\sqrt{3}+2$. Thus if $|x-1|<\delta\leqslant 1$, then

$$\frac{|x-1|}{|\sqrt{x+3}+2|}<\frac{\delta}{\sqrt{3}+2}.$$

If we pick $\delta=\min(1,\epsilon(\sqrt{3}+2))$, then $\delta/(\sqrt{3}+2)\leqslant\epsilon$, hence (3) holds, hence (2) holds, and we are done.

For an example in the other direction we shall *prove* what we have already inferred from Figure 17—namely, that $\sin(1/x)$ does not approach a limit as $x\to 0$. For assume the contrary—that is, assume there exists $L\in R$ such that $\lim_{x\to 0}\sin(1/x)=L$. Then for $\epsilon=1$ there would exist $\delta>0$ such that

$$\left|\sin\frac{1}{x}-L\right|<1 \qquad (0<|x|<\delta). \tag{4}$$

Now

$$\sin\left(2n\pi+\frac{\pi}{2}\right)=\sin\frac{(4n+1)\pi}{2}=1$$

for any $n\in I$. Thus $\sin(1/x)=1$ for $x=2/\pi(4n+1)$ and hence for some $x\in(0,\delta)$, since $\lim_{n\to\infty}2/\pi(4n+1)=0$. For this x (4) implies

$$|1-L|<1. \tag{5}$$

Similarly, $\sin(2n\pi+3\pi/2)=-1$ for $n\in I$. There will thus be an $x\in(0,\delta)$ for which $\sin(1/x)=-1$. By (4) again,

$$|-1-L|<1. \tag{6}$$

The reader should be able to deduce a contradiction from (5) and (6). Hence $\lim_{x\to 0}\sin(1/x)$ does not exist.

4.1B. We wish to emphasize the strong analogy between definition 4.1A and definition 2.2A. Indeed, consider the "table of analogues."

TABLE OF ANALOGUES

2.2A	4.1A
$S=\{s_n\}_{n=1}^{\infty}$	f
n	x
s_n	$f(x)$
L	L
∞	a
ϵ	ϵ
N	δ
$n \geqslant N$	$0<\|x-a\|<\delta$

If we substitute each entry in the right-hand column for the corresponding entry on the left, we change definition 2.2A into definition 4.1A.

However, more than a mechanical process is involved here. Corresponding entries in the table actually "have the same meaning." For example, $S=\{s_n\}_{n=1}^{\infty}$ is the *function* (sequence) involved in definition 2.2A, while f is the function involved in 4.1A. Also, s_n is the value of S at n, while $f(x)$ is the value of f at x. Finally, $n \geqslant N$ means "n is sufficiently close to infinity" (but not, of course, equal to infinity), while $0<|x-a|<\delta$ means "x is sufficiently close to a but not equal to a."

We will now prove a theorem corresponding to 2.7A. The reader should first study the proof and then see how it could be obtained from the proof of 2.7A by mechanical substitution from our table.

4.1C. THEOREM. If* $\lim_{x\to a} f(x)=L$ and $\lim_{x\to a} g(x)=M$, then $f(x)+g(x)$ has a limit as $x\to a$ and, in fact, $\lim_{x\to a}[f(x)+g(x)]=L+M$.

PROOF: Given $\epsilon>0$ we must find $\delta>0$ such that

$$|[f(x)+g(x)]-(L+M)|<\epsilon \qquad (0<|x-a|<\delta). \tag{1}$$

Since $\lim_{x\to a} f(x)=L$, there exists $\delta_1>0$ such that

$$|f(x)-L|<\frac{\epsilon}{2} \qquad (0<|x-a|<\delta_1).$$

Similarly, there exists $\delta_2>0$ such that

$$|g(x)-M|<\frac{\epsilon}{2} \qquad (0<|x-a|<\delta_2).$$

Thus if $\delta=\min(\delta_1,\delta_2)$ and if $0<|x-a|<\delta$, then

$$|f(x)-L|<\frac{\epsilon}{2}, \qquad |g(x)-M|<\frac{\epsilon}{2},$$

and so

$$\begin{aligned}|[f(x)+g(x)]-(L+M)| &= |[f(x)-L]+[g(x)-M]| \\ &\leqslant |f(x)-L|+|g(x)-M|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.\end{aligned}$$

Thus (1) holds for $\delta=\min(\delta_1,\delta_2)$ and the proof is completed.

* In this chapter, whenever we write a hypothesis such as $\lim_{x\to a} f(x)=L$ it is understood that f is a function whose domain contains all points whose distance from a is less than some $h\,(h>0)$ except perhaps the point a itself.

Using the proofs of 2.7D, 2.7G, and 2.7I as models, the reader should now be able to prove the following theorem.

4.1D. THEOREM. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

(a) $\lim_{x\to a}[f(x) - g(x)] = L - M$,
(b) $\lim_{x\to a} f(x)g(x) = L\cdot M$,

and if $M \neq 0$,

(c) $\lim_{x\to a}[f(x)/g(x)] = L/M$.

We occasionally need to handle limits of the form $\lim_{x\to\infty} f(x)$.

4.1E. DEFINITION. We say that $f(x)$ approaches L as x approaches infinity, if given $\epsilon > 0$, there exists $M \in R$ such that

$$|f(x) - L| < \epsilon \qquad (x > M).$$

In this case we write $\lim_{x\to\infty} f(x) = L$, or $f(x) \to L$ as $x \to \infty$.

Definition 4.1E requires, of course, that the domain of the (real-valued) function f contain some interval of the form (c, ∞). Note the very strong resemblance of 4.1E to 2.2A.

For example, let us prove that $\lim_{x\to\infty}(1/x^2) = 0$. Given $\epsilon > 0$ we must find $M \in R$ such that

$$\left|\frac{1}{x^2} - 0\right| < \epsilon \qquad (x > M). \tag{1}$$

Since (1) is equivalent to

$$\frac{1}{x} < \sqrt{\epsilon} \qquad (x > M),$$

it is clear that (1) will hold if we take $M = 1/\sqrt{\epsilon}$.

It is also useful to consider "one-sided" limits.

4.1F. DEFINITION. We say that $f(x)$ approaches L as x approaches a from the right, if given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon \qquad (a < x < a + \delta).$$

In this case we write $\lim_{x\to a+} f(x) = L$. (The number L is called the right-hand limit of f at a.)

We say that $f(x)$ approaches M as x approaches a from the left, if given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - M| < \epsilon \qquad (a - \delta < x < a).$$

In this case we write $\lim_{x\to a-} f(x) = M$. (The number L is called the left-hand limit of f at a.)

Thus the statement $\lim_{x\to a+} f(x) = L$ involves only values of $f(x)$ for x to the right of a, while $\lim_{x\to a-} f(x) = M$ involves only values of $f(x)$ for x to the left of a. It should be obvious to the reader that $\lim_{x\to a} f(x) = L$ if and only if

$$\lim_{x\to a+} f(x) = \lim_{x\to a-} f(x) = L.$$

On the other hand, both $\lim_{x\to a+} f(x)$ and $\lim_{x\to a-} f(x)$ may exist without being equal to each other. For example, if

$$f(x)=x \qquad (0 \leqslant x<1),$$
$$f(x)=3-x \qquad (1 \leqslant x \leqslant 2),$$

then $\lim_{x\to 1-} f(x)=1$ while $\lim_{x\to 1+} f(x)=2$.

It is not difficult to show that theorems analogous to 4.1C and 4.1D also hold for limits of the form $\lim_{x\to\infty} f(x), \lim_{x\to a+} f(x)$, or $\lim_{x\to a-} f(x)$. (See problem 15.)

"One-sided" limits always exist for an important class of functions—namely, monotone functions.

4.1G. DEFINITION. If f is a real-valued function on an interval $J \subset R$, we say that f is nondecreasing of J if

$$f(x) \leqslant f(y) \qquad (x<y;\quad x,y \in J).$$

We say that f is nonincreasing on J if

$$f(x) \geqslant f(y) \qquad (x<y;\quad x,y \in J).$$

We say that f is monotone if f is either nondecreasing or nonincreasing.

Thus definition 4.1G is analogous to definition 2.6A for sequences. As for sequences, we say that a function f on an interval $J \subset R$ is bounded above or bounded below if the range of f is respectively bounded above or bounded below. We then have the following important result analogous to 2.6B.

4.1H. THEOREM. Let f be a nondecreasing function on the bounded open interval (a,b). If f is bounded above on (a,b), then $\lim_{x\to b-} f(x)$ exists. Also, if f is bounded below on (a,b), then $\lim_{x\to a+} f(x)$ exists.

PROOF: If f is bounded above and nondecreasing on (a,b), let

$$M = \underset{x\in(a,b)}{\text{l.u.b.}}\ f(x).$$

Given $\epsilon>0$ the number $M-\epsilon$ is thus not an upper bound for the range of f. Hence there exists $y\in(a,b)$ such that $f(y)>M-\epsilon$. Let $\delta=b-y$. Then

$$f(b-\delta)=f(y)>M-\epsilon.$$

Since f is nondecreasing, this implies

$$f(x)>M-\epsilon \qquad (b-\delta<x<b).$$

Hence since $f(x)\leqslant M$ for all $x\in(a,b)$ we have

$$|f(x)-M|<\epsilon \qquad (b-\delta<x<b).$$

This proves that $\lim_{x\to b-} f(x)=M$.

If f is bounded below, a similar argument will show that $\lim_{x\to a+} f(x)=m$ where $m=\text{g.l.b.}_{x\in(a,b)} f(x)$.

If f is nonincreasing on (a,b), the following result may be proved by applying 4.1H to $-f$ (which will be nondecreasing).

4.1I. THEOREM. Let f be a nonincreasing function on the bounded open interval (a,b). If f is bounded below on (a,b), then $\lim_{x\to b-} f(x)$ exists, while if f is bounded above on (a,b) then $\lim_{x\to a+} f(x)$ exists.

We then have the important corollary 4.1J.

4.1J. COROLLARY. If f is a monotone function on the open interval (a,b), and if $c \in (a,b)$, then $\lim_{x\to c+} f(x)$ and $\lim_{x\to c-} f(x)$ both exist.

PROOF: Suppose that f is nondecreasing. Choose $\delta > 0$ such that $(c-\delta, c+\delta)$ (bounded open interval) is contained in (a,b). Then the values of f on the open interval $(c-\delta, c)$ are bounded above by $f(c)$ and hence by 4.1H, $\lim_{x\to c-} f(x)$ exists. Similarly, the values of f on the open interval $(c, c+\delta)$ are bounded below by $f(c)$. Hence again by 4.1H, $\lim_{x\to c+} f(x)$ exists.

If f in nonincreasing we use 4.1I instead of 4.1H. This completes the proof. Note that we did not assume that (a,b) was bounded.

As long as we are on this subject we may as well define strictly increasing function and strictly decreasing function.

4.1K. DEFINITION. The real-valued function f on the interval $J \subset R$ is said to be strictly increasing if

$$f(x) < f(y) \qquad (x<y; \quad x,y \in J).$$

Similarly, f is said to be strictly decreasing if

$$f(x) > f(y) \qquad (x<y; \quad x,y \in J).$$

Thus if f is nonincreasing on J, then f is strictly increasing on J if and only if f is 1–1 on J.

Exercises 4.1

1. (a) If $|x-2|<1$, prove that $|x^2-4|<5$.
   (b) If $|x-3|<\frac{1}{10}$, prove that $|x^2-x-6|<0.51$.
   (c) If $|x+1|<\frac{1}{10}$, prove that $|x^3+1|<0.331$.
2. Let δ be any number such that $0<\delta<1$.
   (a) If $|x-2|<\delta$, prove that $|x^2-4|<5\delta$.
   (b) If $|x-3|<\delta$, prove that $|x^2-x-6|<6\delta$.
   (c) If $|x+1|<\delta$, prove that $|x^3+1|<7\delta$.
   (d) If $|x-2|<\delta$, prove that $|x-2)/(x+3)|<\delta/4$.
3. (a) Let $f(x)=x^2+4x$. Find $\delta>0$ such that

$$|f(x)-5|<\tfrac{1}{10} \qquad (0<|x-1|<\delta).$$

   (b) Prove directly from definition 4.1A that $\lim_{x\to 1}(x^2+4x)=5$.
4. For each of the functions in Figures 13–17, draw a diagram of the type in Figure 7. Relate the diagrams to definition 4.1A.
5. Prove, using only definition 4.1A, the truth of the following statements.
   (a) $\lim\limits_{x\to -2} x^2+3x=-2$.
   (b) $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1}=2$.
   (c) $\lim\limits_{x\to 0} \sqrt{4-x}=2$.
6. Prove that if $\lim_{x\to a} f(x)=L$ and $\lim_{x\to a} f(x)=M$, then $L=M$. (Compare with theorem 2.3B.)
7. If $\lim_{x\to a} f(x)=L$ and $c\in R$, prove that $\lim_{x\to a} cf(x)=cL$.

8. Prove

$$\lim_{x\to 1} \frac{x^7-2x^5+1}{x^3-3x^2+2}=1.$$

9. If $c\in R$ and $f(x)=c$ for all $x\in R$, prove that $\lim_{x\to a} f(x)=c$, where a is any point in R.
10. Let $[x]$ denote the greatest integer not exceeding x. (For example, $[-4]=-4$, $[-4.1]=-5$, $[15.4]=15$.) Prove that if $n\in I$, then

$$\lim_{x\to n^+}[x]=n, \qquad \lim_{x\to n^-}[x]=n-1.$$

11. Let

$$f(x)=[1-x^2] \qquad (-1\leqslant x\leqslant 1).$$

Does $\lim_{x\to 0} f(x)$ exist? If so, evaluate it.
12. For any $a\in R$ prove that $\lim_{x\to a} x=a$. Then, using theorems from this section, prove that $\lim_{x\to a} P(x)=P(a)$ where P is any polynomial function.
13. If $f(x)\geqslant 0$ for $|x-a|<h$ and if $\lim_{x\to a} f(x)=L$, prove that $L\geqslant 0$.
14. If $\lim_{x\to a} f(x)=L>0$, show that there exists $\delta>0$ such that

$$f(x)>0 \qquad (0<|x-a|<\delta).$$

(*Hint*: Take $\epsilon=L/2$.)
15. If $\lim_{x\to\infty} f(x)=A$ and $\lim_{x\to\infty} g(x)=B$, prove that $\lim_{x\to\infty}[f(x)+g(x)]=A+B$. Do the same with $\lim_{x\to\infty}$ replaced by $\lim_{x\to a+}$ and $\lim_{x\to b-}$.
16. Let f and g be nondecreasing functions on an interval (a,b) and let $h=f-g$. If $c\in(a,b)$ prove that $\lim_{x\to c} h(x)$ and $\lim_{x\to c-} h(x)$ exist.
17. If f is a real-valued function on $(0,\infty)$ and if

$$g(x)=f\left(\frac{1}{x}\right) \qquad (0<x<\infty),$$

prove that $\lim_{x\to\infty} f(x)=L$ if and only if $\lim_{x\to 0+} g(x)=L$.
18. Write out a definition of

$$\lim_{x\to-\infty} f(x)=L.$$

Prove, for this limit, a theorem corresponding to 4.1C.
19. Give an example of a 1–1 function on $(0,\infty)$ that is not monotone.
20. Give an example of a nondecreasing function on $[0,1]$ that is not strictly increasing.
21. Prove that if f is nondecreasing and bounded above on (a,∞), then $\lim_{x\to\infty} f(x)$ exists.
22. Let f be real-valued function on R and suppose $\lim_{x\to a} f(x)=L$. If $\{x_n\}_{n=1}^{\infty}$ is any sequence of real numbers which converges to a, and if $x_n\neq a (n\in I)$, prove that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.
23. Conversely, suppose $\lim_{n\to\infty} f(x_n)=L$ for every sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n\neq a$ $(n\in I)$ and $\lim_{n\to\infty} x_n=a$. Prove that $\lim_{x\to a} f(x)=L$.
24. Suppose only that $\lim_{n\to\infty} f(x_n)$ *exists* for every sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n\neq a$ and $\lim_{n\to\infty} x_n=a$. Prove that $\lim_{x\to a} f(x)$ exists.
25. Use exercises 22 and 23 to give a new proof of theorem 4.1C.

4.2 METRIC SPACES

4.2A. In the proofs of theorems 2.7A, 2.7D, 2.7G, and 2.7I, and their counterparts 4.1C and 4.1D, the following crucial properties of the absolute-value function were used.

$$|0|=0, \tag{1}$$

$$|a|>0 \qquad (a\in R, a\neq 0), \tag{2}$$

$$|a|=|-a| \qquad (a\in R), \tag{3}$$

$$|a+b|\leqslant |a|+|b| \qquad (a,b\in R). \tag{4}$$

Now, for $x,y\in R$, the geometric interpretation of $|x-y|$ is the distance from x to y. If we define the "distance function" ρ by

$$\rho(x,y)=|x-y| \qquad (x,y\in R),$$

then the properties (1)–(4) have the following consequences for any points $x,y,z\in R$:

$$\rho(x,x)=0. \tag{5}$$

(That is, the distance from a point to itself is 0.)

$$\rho(x,y)>0 \qquad (x\neq y). \tag{6}$$

(The distance between two distinct points is strictly positive.)

$$\rho(x,y)=\rho(y,x). \tag{7}$$

(The distance from x to y is equal to the distance from y to x.)

$$\rho(x,y)\leqslant \rho(x,z)+\rho(z,y) \qquad \text{(triangle inequality)}. \tag{8}$$

[This is proved by setting $a=x-z, b=z-y$ in (4). The inequality (8) says that going from x to y directly never takes longer than going from x to z and then to y.]

A satisfactory definition of limit can be constructed, not only for R, but for any set M which has a "distance function" ρ satisfying (5)–(8). A "distance function" is usually called a metric.

4.2B. DEFINITION. Let M be any set. A metric for M is a function ρ with domain $M\times M$ and range contained in $[0,\infty)$ such that

$$\rho(x,x)=0 \qquad (x\in M),$$

$$\rho(x,y)>0 \qquad (x,y\in M, x\neq y),$$

$$\rho(x,y)=\rho(y,x) \qquad (x,y\in M),$$

$$\rho(x,y)\leqslant \rho(x,z)+\rho(z,y) \qquad (x,y,z\in M) \qquad \text{(triangle inequality)}.$$

If ρ is a metric for M, then the ordered pair $\langle M,\rho\rangle$ is called a metric space. (In many cases we abuse language slightly and refer to the metric space $\langle M,\rho\rangle$ simply by M. Thus if we say "let M be a metric space," there is always a metric ρ for M lurking in the background.)

A metric for M thus has all the properties (5)–(8) of the distance function $|x-y|$ for R.

4.2C. Here are five examples of metric spaces.

1. The function ρ defined by $\rho(x,y)=|x-y|$ is obviously a metric for the set R of real numbers. We denote the resulting metric space $\langle R,\rho\rangle$ by R^1. We call this metric ρ the absolute value metric.

2. Here is another metric for the set R. Define $d: R\times R \to [0,\infty)$ by

$$d(x,x)=0 \qquad (x\in R),$$
$$d(x,y)=1 \qquad (x,y\in R;\quad x\neq y).$$

That is, the "distance" $d(x,y)$ between any two distinct points $x,y\in R$ is 1. The reader should verify that d is a metric for R. The metric d is called the *discrete metric*. We will henceforth denote the metric space $\langle R,d\rangle$ by R_d. The examples 1 and 2 show that a given set may have more than one metric.

3. Fix $n\in I$. If $x=\langle x_1,\ldots,x_n\rangle$ and $y=\langle y_1,\ldots,y_n\rangle$ are two ordered n-tuples of real numbers, define

$$\rho(x,y)=\left[\sum_{k=1}^{n}(x_k-y_k)^2\right]^{1/2}.$$

[For $n=2$, $\rho(x,y)$ is thus the usual distance formula for points in the Cartesian plane.] We will show that ρ satisfies the triangle inequality. Thus, if $z=\langle z_1,\ldots,z_n\rangle$, we must show $\rho(x,y)\leqslant\rho(x,z)+\rho(z,y)$. For $k=1,\ldots,n$ let $a_k=x_k-z_k, b_k=z_k-y_k$. Then

$$\rho(x,z)=\left(\sum_{k=1}^{n}a_k^2\right)^{1/2},$$

$$\rho(z,y)=\left(\sum_{k=1}^{n}b_k^2\right)^{1/2},$$

and

$$\rho(x,y)=\left[\sum_{k=1}^{n}(a_k+b_k)^2\right]^{1/2}.$$

We must thus show that

$$\left[\sum_{k=1}^{n}(a_k+b_k)^2\right]^{1/2}\leqslant\left(\sum_{k=1}^{n}a_k^2\right)^{1/2}+\left(\sum_{k=1}^{n}b_k^2\right)^{1/2}.$$

But this follows from 3.10C. It is trivial to verify that ρ satisfies the other requirements for a metric. We denote by R^n the metric space formed by the set of all n-tuples of real numbers with this metric ρ. The metric space R^n is called Euclidean n-space. (Note that for $n=1$, R^n becomes the R^1 of example 1 since

$$\left[\sum_{k=1}^{1}(x_k-y_k)^2\right]^{1/2}=|x_1-y_1|.$$

4. Let ℓ^∞ denote the set of all bounded sequences of real numbers. If $x=\{x_n\}_{n=1}^{\infty}$ and $y=\{y_n\}_{n=1}^{\infty}$ are points in ℓ^∞, define

$$\rho(x,y)=\underset{1\leqslant n<\infty}{\text{l.u.b.}}\;|x_n-y_n|.$$

For example, if $x=\{1+1/n\}_{n=1}^{\infty}$, $y=\{2-1/n\}_{n=1}^{\infty}$, then

$$\rho(x,y)=\underset{1\leqslant n<\infty}{\text{l.u.b.}}\left|\left(1+\frac{1}{n}\right)-\left(2-\frac{1}{n}\right)\right|=\underset{1\leqslant n<\infty}{\text{l.u.b.}}\left|-1+\frac{2}{n}\right|=1.$$

Again, it is easy to see that ρ satisfies the first three requirements of a metric. To demonstrate the triangle inequality, let $z=\{z_n\}_{n=1}^{\infty}$ also be a point in ℓ^∞. For any

$k \in I$ we have

$$|x_k - y_k| = |x_k - z_k + z_k - y_k| \leqslant |x_k - z_k| + |z_k - y_k|$$
$$\leqslant \underset{1 \leqslant n < \infty}{\text{l.u.b.}} |x_n - z_n| + \underset{1 \leqslant n < \infty}{\text{l.u.b.}} |z_n - y_n|,$$

and so

$$|x_k - y_k| \leqslant \rho(x,z) + \rho(z,y) \qquad (k \in I).$$

From this it follows that $\text{l.u.b.}_{1 \leqslant k < \infty} |x_k - y_k| \leqslant \rho(x,z) + \rho(z,y)$ (why?), and this is the triangle inequality for ρ.

It is customary to denote this metric space $\langle \ell^\infty, \rho \rangle$ simply by ℓ^∞. (The reason for the ∞ symbol on the ℓ is usually disclosed in the next course in analysis.)

5. For a final example of a metric space, consider the set ℓ^2 from Section 3.10. For $x,y \in \ell^2$ define $\rho(x,y) = \|x - y\|_2$. Then theorem 3.10E shows that ρ is a metric for ℓ^2. For example, using (3) of 3.10E, we have

$$\rho(x,y) = \|x - y\|_2 = \|(-1)(y - x)\|_2$$
$$= |-1| \cdot \|y - x\|_2 = \|y - x\|_2 = \rho(y,x).$$

Also, for $x,y,z \in \ell^2$ we have, using (4) of 3.10E,

$$\rho(x,y) = \|x - y\|_2 = \|x - z + z - y\|_2$$
$$\leqslant \|x - z\|_2 + \|z - y\|_2 = \rho(x,z) + \rho(z,y).$$

We denote the metric space $\langle \ell^2, \rho \rangle$ simply by ℓ^2.

We have thus listed R^1, R_d, R^n, ℓ^∞, and ℓ^2 as examples of metric spaces.

It is important to note that if ρ is a metric for the set M, then ρ defines a metric for any subset of M in an obvious way. For example, $\rho(x,y) = |x - y|$ defines a metric for any closed interval $[a,b]$ of real numbers.

4.2D. In the next section we will make use of the concept of cluster point.

DEFINITION. Let $\langle M, \rho \rangle$ be a metric space and suppose $A \subset M$. The point $a \in M$ is called a cluster point of A in M if, for every $h > 0$, there exists a point $x \in A$ such that $0 < \rho(x,a) < h$.

That is, a is a cluster point of A if there are points of A distinct from a but arbitrarily near a. Note that a need not belong to A.

For example, let $M = R^1$ and let $A = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$. Then 0 is the only cluster point of A in M. See Figure 18.

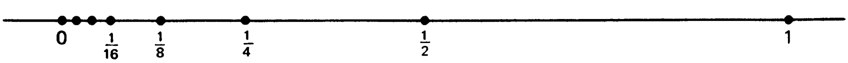

FIGURE 18. The only cluster point of $A = \{1, \frac{1}{2}, \frac{1}{4}, \cdots\}$ is 0.

Exercises 4.2

1. Show that if ρ is a metric for a set M, then so is 2ρ.
2. Show that if ρ and σ are both metrics for a set M, then $\rho + \sigma$ is also a metric for M.
3. Suppose ρ_1 and ρ_2 are metrics for the set M. Prove that $\max(\rho_1, \rho_2)$ is also a metric for M.
4. Let $\langle M, \rho \rangle$ be a metric space. Prove that $\min(1, \rho)$ is also a metric for M.

5. Let

$$\rho(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right| \qquad (x>0, y>0).$$

Prove that ρ is a metric for $(0,\infty)$.

6. Let $\langle M,\rho\rangle$ be a metric space. Prove that

$$|\rho(x,y) - \rho(x,y)| \leqslant \rho(y,z) \qquad (x,y,z \in M).$$

7. Let ℓ^1 be the class of all sequences $\{s_n\}_{n=1}^{\infty}$ of real numbers such that $\Sigma_{n=1}^{\infty}|s_n| < \infty$. Show that if $s = \{s_{n=1}$ and $t = \{t_n\}_{n=1}^{\infty}$ are in ℓ^1, then $\rho(s,t) = \Sigma_{n=1}^{\infty}|s_n - t_n|$ defines a metric for ℓ^1.

8. For $P\langle x_1,y_1\rangle$ and $Q\langle x_2,y_2\rangle$, define

$$\sigma(P,Q) = |x_1 - x_2| + |y_1 - y_2|.$$

Show that σ is a metric for the set of ordered pairs of real numbers.

Also, if

$$\tau(P,Q) = \max(|x_1 - x_2|, |y_1 - y_2|),$$

show that τ defines a metric for the same set.

9. Let 0 denote the point $\langle 0,0\rangle$ in R^2. For σ,τ as in Exercise 4, sketch the following subsets of R^2:

$$A = \{P \in R^2 | \sigma(0,P) < 1\},$$

$$B = \{P \in R^2 | \tau(0,P) < 1\}.$$

Compare with

$$C = \{P \in R^2 | \rho(0,P) < 1\},$$

where ρ is the metric for R^2.

10. If P, Q, R are points in R^3 and $\rho(P,R) = \rho(P,Q) + \rho(Q,R)$, what can you say about the relative position of P, Q, R?

Answer the same question with R_d in place of R^3.

11. Let A denote the open interval $(0,1)$. Show that the set of cluster points of A in R^1 is $[0,1]$.

12. If $A = (0,1)$, find the set of cluster points of A in R_d.

4.3 LIMITS IN METRIC SPACES

If we examine definition 4.1A we see that $\lim_{x\to a} f(x) = L$ means that given $\epsilon > 0$ there exists $\delta > 0$ such that the distance from $f(x)$ to L is less than ϵ provided that the distance from x to a is less than δ (but greater than 0). Now that we have stated this definition in terms of distances, it is not difficult to see how to formulate the corresponding definition for arbitrary metric spaces.

Suppose that $\langle M_1,\rho_1\rangle$ and $\langle M_2,\rho_2\rangle$ are metric spaces, that $a \in M_1$, and that f is a function whose range is contained in M_2 and whose domain contains all $x \in M_1$ such that $\rho_1(a,x) < h$ (for some $h > 0$) except possibly $x = a$. We also assume that a is a cluster point of the domain of f.

4.3A. DEFINITION. We say that $f(x)$ approaches L (where $L \in M_2$) as x approaches a if given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\rho_2(f(x), L) < \epsilon \qquad (0 < \rho_1(x,a) < \delta).$$

In this case we write $\lim_{x\to a} f(x)=L$, or $f(x)\to L$ as $x\to a$. If $\langle M_1,\rho_1\rangle=\langle M_2,\rho_2\rangle=R^1$, then $\rho_2(f(x),L)=|f(x)-L|$, $\rho_1(x,a)=|x-a|$, and 4.3A reduces to 4.1A.

In later chapters we very often consider functions f on the metric space $M=[a,b]$ (closed bounded interval with absolute-value metric). For this space the statement

$$\lim_{x\to a} f(x)=L \tag{*}$$

involves only points x to the right of a (since points in R^1 to the left of a are not in M). In 4.1F, L is referred to as the "right-hand limit of f." However, there is no need for us to use this terminology as long as we remember on what space f is defined. A similar remark applies to

$$\lim_{x\to b} f(x)=N.$$

These remarks are also relevant when we define the derivative of a real-valued function on $[a,b]$.

Here is an example illustrating 4.3A. Let $f: \ell^2\to R^1$ be defined as follows: If $x=\{x_n\}_{n=1}^{\infty}\in \ell^2$, let $f(x)=x_1$. That is, the image under f of any sequence in ℓ^2 is the first term of the sequence. Now let $a=\{a_n\}_{n=1}^{\infty}$ be any fixed element of ℓ^2. We will prove that $\lim_{x\to a} f(x)=a_1$. Given $\epsilon>0$ we must find $\delta>0$ such that the distance from $f(x)$ to a_1 (in the metric for R^1) is less than ϵ whenever the distance from x to a (in the metric for ℓ^2) is less than δ but greater than 0. That is, we must find $\delta>0$ such that

$$|f(x)-a_1|<\epsilon \qquad (0<\|x-a\|_2<\delta),$$

or

$$|x_1-a_1|<\epsilon \qquad (0<\|x-a\|_2<\delta). \tag{1}$$

But

$$\|x-a\|_2=\left[\sum_{n=1}^{\infty}(x_n-a_n)^2\right]^{1/2}\geqslant\left[(x_1-a_1)^2\right]^{1/2}=|x_1-a_1|,$$

and so $|x_1-a_1|\leqslant\|x-a\|_2$. If we thus choose $\delta=\epsilon$, then $\|x-a\|_2<\delta=\epsilon$ implies $|x_1-a_1|\leqslant\|x-a\|_2<\epsilon$ and (1) holds. This proves $\lim_{x\to a} f(x)=a_1$. [Note that $a_1=f(a)$ so that we have shown $\lim_{x\to a} f(x)=f(a)$.]

We most often apply definition 4.3A to real-valued functions—that is, when $\langle M_2,\rho_2\rangle=R_1$. The proof of the following theorem is then an exact duplicate of the proofs of 4.1C and 4.1D.

4.3B. THEOREM. Let $\langle M,\rho\rangle$ be a metric space and let a be a point in M. Let f and g be real-valued* functions whose domains are subsets of M. If† $\lim_{x\to a} f(x)=L$ and $\lim_{x\to a} g(x)=N$, then

$$\lim_{x\to a}[f(x)+g(x)]=L+N,$$

$$\lim_{x\to a}[f(x)-g(x)]=L-N,$$

$$\lim_{x\to a}[f(x)\,g(x)]=LN,$$

* Henceforth, whenever we use the phrase "real-valued function," we mean a function with range in R^1. That is, the metric in the range is the absolute-value metric.

†See footnote p. 113.

and, if $N \neq 0$,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{N}.$$

PROOF: We prove only that $\lim_{x \to a} f(x) g(x) = LN$. (Compare with the second proof of 2.7G.)

Since $\lim_{x \to a} g(x) = N$ we have, for some $\delta_1 > 0$,

$$|g(x) - N| < 1 \qquad (0 < \rho(x,a) < \delta_1).$$

Thus

$$|g(x)| < |N| + 1 = Q \qquad (0 < \rho(x,a) < \delta_1).$$

Now

$$\begin{aligned} f(x) g(x) - LN &= f(x) g(x) - Lg(x) + Lg(x) - LN \\ &= g(x)[f(x) - L] + L[g(x) - N], \end{aligned}$$

$$|f(x) g(x) - LN| \leqslant |g(x)| \cdot |f(x) - L| + |L| \cdot |g(x) - N|.$$

Hence if $0 < \rho(x,a) < \delta_1$,

$$|f(x) g(x) - LN| \leqslant Q \cdot |f(x) - L| + |L| \cdot |g(x) - N|. \tag{1}$$

Given $\epsilon > 0$ there exists $\delta_2 > 0$ such that

$$Q|f(x) - L| < \frac{\epsilon}{2} \qquad (0 < \rho(x,a) < \delta_2), \tag{2}$$

and there exists $\delta_3 > 0$ such that

$$|L||g(x) - N| < \frac{\epsilon}{2} \qquad (0 < \rho(x,a) < \delta_3). \tag{3}$$

If we let $\delta = \min(\delta_1, \delta_2, \delta_3)$, then from (1), (2), and (3) it follows that

$$|f(x) g(x) - LN| < \epsilon \qquad (0 < \rho(x,a) < \delta).$$

This proves $\lim_{x \to a} f(x) g(x) = LN$.

4.3C. A sequence of points in a metric space M is a function from I into M. As with sequences of real numbers, we will use the notation $\{a_n\}_{n=1}^{\infty}$ for a sequence of points M. For such sequences, convergence is defined as in 2.2A and 2.3A.

DEFINITION. Let $\langle M, \rho \rangle$ be a metric space and let $\{s_n\}_{n=1}^{\infty}$ be a sequence of points in M. We say that s_n approaches L (where $L \in M$) as n approaches infinity if given $\epsilon > 0$, there exists $N \in I$ such that

$$\rho(s_n, L) < \epsilon \qquad (n \geqslant N).$$

In this case we write $\lim_{n \to \infty} s_n = L$, or $s_n \to L$ as $n \to \infty$ and say that $\{s_n\}_{n=1}^{\infty}$ is convergent in M to the point L.

Cauchy sequences are defined as in 2.10A.

4.3D. DEFINITION. Let $\langle M, \rho \rangle$ be a metric space and let $\{s_n\}_{n=1}^{\infty}$ be a sequence of points in M. We say that $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence if given $\epsilon > 0$, there exists $N \in I$ such that

$$\rho(s_m, s_n) < \epsilon \qquad (m, n \geqslant N).$$

The proof of the following theorem is identical to that of 2.10B.

4.3E. THEOREM. Let $\langle M,\rho\rangle$ be a metric space. If $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence of points of M, then $\{s_n\}_{n=1}^{\infty}$ is Cauchy.

4.3F. Now comes a very important point. For some metric spaces there are Cauchy sequences which are *not* convergent. That is, theorem 2.10D cannot be extended to all metric spaces.

For example, let M be the set of all points $\langle x,y\rangle$ in the Euclidean plane R^2 such that $x^2+y^2<1$, with the R^2 metric used as metric for M. The sequence $A=\{\langle 0,n/n+1\rangle\}_{n=1}^{\infty}$ is a Cauchy sequence of points in M but there is no $L\in M$ such that A is convergent to L! (Draw a picture, then verify.) Hence the sequence of A points of M does not converge in M.

Of course, the sequence A considered as a sequence of points in R^2 does converge to the point $\langle 0,1\rangle$ in R^2. But the fact remains that A is not a convergent sequence in M (according to 4.3C) even though it is Cauchy.

The reader should carefully reexamine the proof of 2.10D to see where properties special to R^1 are used and thus why the proof does not immediately extend to cover all metric spaces as did the proof of 2.10B.

Exercises 4.3

1. Show that a sequence of points in any metric space cannot converge to two distinct limits.
2. For each $n\in I$ let $P_n=\langle x_n,y_n\rangle$ be a point in R^2. Show that $\{P_n\}_{n=1}^{\infty}$ converges to $P=\langle x,y\rangle$ in R^2 if and only if $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ converge in R^1 to x and y, respectively.
3. Let $s=\{1/k\}_{k=1}^{\infty}$. Find a sequence $\{s_n\}_{n=1}^{\infty}$ of points in ℓ^2 such that each s_n is distinct from s and such that $\{s_n\}_{n=1}^{\infty}$ converges to s in ℓ^2.
4. Suppose that ρ and σ are metrics for M such that

$$\lim_{n\to\infty} x_n = x \text{ in } \langle M,\rho\rangle$$

if and only if

$$\lim_{n\to\infty} x_n = x \text{ in } \langle M,\sigma\rangle.$$

(That is, a sequence converges in $\langle M,\rho\rangle$ if and only if it converges in $\langle M,\sigma\rangle$ and the limits are the same.) We then say that ρ and σ are equivalent.

Prove that the usual metric for R^2, and the metrics τ and σ of Exercise 8 of Section 4.2 are all equivalent to one another.
5. If ρ and σ are metrics for M, and if there exists $k>1$ such that

$$\frac{1}{k}\sigma(x,y)\leqslant\rho(x,y)\leqslant k\sigma(x,y)\qquad (x,y\in M),$$

prove that ρ and σ are equivalent.
6. Show that if $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence in a metric space $\langle M,\rho\rangle$, then the sequence of real numbers $\{\rho(s_1,s_n)\}_{n=1}^{\infty}$ is bounded.
7. If $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence of points in the metric space M, and if $\{x_n\}_{n=1}^{\infty}$ has a subsequence which converges to $x\in M$, prove that $\{x_n\}_{n=1}^{\infty}$ itself is convergent to x.
8. Show that if $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence in R_d, then there exists $N\in I$ such

that $x_N = x_{N+1} = x_{N+2} \cdots$. (That is, a sequence in R_d is convergent if and only if all the terms of the sequence are the same from some point on.)

9. Show that every Cauchy sequence in R_d *is* convergent.
10. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be convergent sequences in a metric space $\langle M,\rho\rangle$. Prove that $\{\rho(x_n,y_n)\}_{n=1}^{\infty}$ is convergent in R^1.

119 Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be Cauchy sequences in a metric space $\langle M,\rho\rangle$. Prove that $\{\rho(x_n,y_n)\}_{n=1}^{\infty}$ is Cauchy in R^1.

12. Explain why we do not want definition 4.3A to apply if a is not a cluster point of the domain of f.