Units 4 *

Series of real number's

Test for absolute Convergence :

Definition :

[$\ll$ notation]

$\Rightarrow$ Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of real numbers.

$\Rightarrow$ Then $\sum_{n=1}^{\infty} a_n$ is said to be dominated by $\sum_{n=1}^{\infty} b_n$ if $\exists\, n \in I \ni |a_n| \le |b_n| \; \forall n \ge N$

$\Rightarrow$ It can be writen as $\sum_{n=1}^{\infty} a_n \ll \sum_{n=1}^{\infty} b_n$

example :

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \ll \sum_{n=1}^{\infty} \frac{1}{2n+1}$$

Ans :-

P.T, $a_n = \frac{(-1)^n}{n^2}$, $b_n = \frac{1}{2n+1}$

$|a_1| = |-1| = 1$ | $|b_1| = \frac{1}{3}$

$|a_2| = \frac{1}{4}$ | $|b_2| = \frac{1}{5}$

$|a_3| = \frac{1}{9}$ | $|b_3| = \frac{1}{7}$

$|a_4| = \frac{1}{16}$ | $|b_4| = \frac{1}{9}$

$\vdots$ | $\vdots$

$|a_n| \le |b_n|, \; n \ge 3$

$$\therefore \sum_{n=1}^{\infty} \frac{(-1)^4}{n^2} \ll \sum_{n=1}^{\infty} \frac{1}{2n+1} \quad \forall n \ge 3$$

Thearem 4.1 [Comparsion Test for absolute Converge]

If $\sum_{n=1}^{\infty} a_n$ is dominated by $\sum_{n=1}^{\infty} b_n$ where $\sum_{n=1}^{\infty} b_n$ Converges absolutely, then $\sum_{n=1}^{\infty} a_n$ also Converges absolutely.

<u>Proof</u> :-

Since $\sum_{n=1}^{\infty} b_n$ Converges absolutely.

$$\text{Let } \sum_{n=1}^{\infty} b_n = M$$

since $\sum_{n=1}^{\infty} a_n \ll \sum_{n=1}^{\infty} b_n$, any $n \in I$

$$\Rightarrow |a_n| \leq |b_n|, \forall n \geq N$$

$$\text{let } S_n = |a_1| + |a_2| + \cdots\cdots + |a_n|$$

Then for $n \geq N$

$$S_n = |a_1| + |a_2| + \cdots\cdots + |a_N| + |a_{N+1}| + \cdots + |a_n|$$

$$\leq |a_1| + |a_2| + \cdots + |a_N| + |b_{N+1}| + \cdots + |b_n|$$

$$\leq |a_1| + |a_2| + \cdots + |a_N| + M$$

$\Rightarrow$ The sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ of $\sum_{n=1}^{\infty} |a_n|$ is bounded above

Hence, by the thearem,

If $\{S_n\}_{n=1}^{\infty}$ is bounded, then $\sum_{n=1}^{\infty} a_n$ Converges,

$\sum_{n=1}^{\infty}$ ~~KKK~~ a_n Converges absolutely.

Theorem 4.2 (Unit 3 theorem 3 model)

If $\sum_{n=1}^{\infty} a_n$ is dominated by $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} a_n = \infty$ then $\sum_{n=1}^{\infty} b_n = \infty$

(ie) if $\sum_{n=1}^{\infty} a_n \ll \sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} a_n = \infty$, then $\sum_{n=1}^{\infty} b_n = \infty$]

Proof:

Since $\sum_{n=1}^{\infty} a_n$ is dominated by $\sum_{n=1}^{\infty} b_n$ $\exists N \in I$

$\ni |a_n| \leq |b_n|, \forall n \geq N$

let, $t_n = |b_1| + |b_2| + \dots + |b_n|$

Then for $n \geq N$

$t_n = |b_1| + |b_2| + \dots + |b_{N_1}| + |b_{N_1+1}| + \dots + |b_n|$

$\geq |b_1| + |b_2| + \dots + |b_{N_1}| + |a_{N_1+1}| + \dots + |a_n| \rightarrow (1)$

Since $\sum_{n=1}^{\infty} |a_n| = \infty$

If $s_n = |a_1| + |a_2| + \dots + |a_n|$

then $s_n \rightarrow \infty$ as $n \rightarrow \infty$

hence for a real number $M \geq 0$

$\exists$ a $N_2 \in I$

$\ni s_n > M + (|a_1| + |a_2| + \dots + |a_{N_1}|), \forall n \geq N_2 \rightarrow (2)$

let $N = \max(N_1, N_2)$

Then for $n \geq N$.

From (2), $s_n > M + (|a_1| + |a_2| + \dots + |a_{N_1}|)$

$|a_1| + |a_2| + \dots + |a_{N_1}| + |a_{N_1+1}| + \dots + |a_n|$

$> M + (|a_1| + |a_2| + \dots + |a_{N_1}|)$

$\Rightarrow |a_{N_1+1}| + \dots\dots + |a_n| > M$

Sub in ①

$t_n = |b_1| + |b_2| + \dots\dots + |b_N| + M, \ \forall n \geq N$

$\Rightarrow t_n \to \infty$ as $n \to \infty$

hence $\sum_{n=1}^{\infty} |b_n| = \infty$

Theorem 4.3

a) If $\sum_{n=1}^{\infty} b_n$ Converges absolutely and if

$\lim_{n \to \infty} \frac{|a_n|}{|b_n|}$ exist, then $\sum_{n=1}^{\infty} a_n$ Converges absolutely

b) $\sum_{n=1}^{\infty} |a_n| = \infty$ and if $\lim_{n \to \infty} \frac{|a_n|}{|b_n|}$ exists.

then $\sum_{n=1}^{\infty} |b_n| = \infty$

Proof :

Since $\lim_{n \to \infty} \frac{|a_n|}{|b_n|}$ exists, $\left\{ \frac{|a_n|}{|b_n|} \right\}_{n=1}^{\infty}$ Converges.

hence $\left\{ \left|\frac{a_n}{b_n}\right| \right\}_{n=1}^{\infty}$ is bounded.

[Since a convergent sequence is bounded]

$\Rightarrow \exists$ a $M > 0 \ni \left|\frac{a_n}{b_n}\right| \leq M, \forall n \in I$

$\Rightarrow |a_n| \leq M |b_n|, \forall n \in I$ ——— ①

$\Rightarrow \sum_{n=1}^{\infty} a_n$ is dominated by the series $\sum_{n=1}^{\infty} Mb_n$

a) Since $\sum_{n=1}^{\infty} b_n$ Converges absolutely.

$\sum_{n=1}^{\infty} Mb_n$ Converges absolutely.

hence $\sum_{n=1}^{\infty} a_n \ll \sum_{n=1}^{\infty} b_n$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ Converges absolutely [by the theorem 4.2]

b) ① $\Rightarrow \frac{1}{M} |a_n| \leq |b_n|, \; \forall n \in I$.

$\sum_{n=1}^{\infty} \frac{1}{M} (a_n) \leq \sum_{n=1}^{\infty} (b_n)$

$\therefore$ If $\sum_{n=1}^{\infty} |a_n| = \infty$, then $\sum_{n=1}^{\infty} \frac{1}{M} |a_n| = \infty$

$\therefore \sum_{n=1}^{\infty} |b| = \infty$ [by therom 4.2]

Theorem 4.4:

[Ratio Test]

Let $\sum_{n=1}^{\infty} a_n$ be a series of non zero real numbers and let $a = \lim_{n \to \infty} \inf \left| \frac{a_{n+1}}{a_n} \right|$.

$A = \lim_{n \to \infty} \sup \left| \frac{a_{n+1}}{a_n} \right|$,

Then, a) If $A < 1$, then $\sum_{n=1}^{\infty} a_n < \infty$

b) If $a > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

c) If $a \leq 1 \leq A$, Then the test fails.

Proof:

a) If $A < 1$, choose $B \ni A < B < 1$

Let $\varepsilon = B - A > 0 \Rightarrow B = A + \varepsilon$

By the theorem,

"If $M = \lim_{n \to \infty} \sup . S_n$,

then $S_n \leq M + \varepsilon$ for all but a finite number of values of n."

$\exists$ an $N \in I \ni \left| \frac{a_{n+1}}{a_n} \right| \leq A + \varepsilon, \; \forall n \geq N$

$\left[\because A = \lim_{n \to \infty} \sup \left| \frac{a_{n+1}}{a_n} \right|\right]$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| < B, \ \forall \ n \geq N \quad \text{Where } B < 1$$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| \leq B.$$

$$\left|\frac{a_{N+1}}{a_N}\right| \leq B, \ \left|\frac{a_{N+2}}{a_{N+1}}\right| \leq B \ \text{ \& so on.}$$

$$\therefore \left|\frac{a_{N+2}}{a_N}\right| = \left|\frac{a_{N+2}}{a_{N+1}}\right| \left|\frac{a_{N+1}}{a_N}\right| < B^2$$

for any $k \geq 0$

$$\left|\frac{a_{N+k}}{a_N}\right| \leq B^k$$

$$\Rightarrow |a_{N+k}| \leq |a_N| B^k \quad [k = 0, 1, 2, 3 \ldots\ldots]$$

$$\Rightarrow \sum_{K=0}^{\infty} a_{N+k} << \sum_{K=0}^{\infty} |a_N| B^k \longrightarrow (1)$$

Since $0 < B < 1$, $\sum_{K=0}^{\infty} B^k$ Converges

hence $\sum_{K=0}^{\infty} |a_N| B^k$ converges.

$\therefore \sum_{K=0}^{\infty} |a_{N+1}|$ Converges. [By theorem 4.1]

i.e. $|a_N| + |a_{N+1}| + |a_{N+2}| + \ldots\ldots$ Converges

$|a_1| + |a_2| + |a_3| + \ldots\ldots\ldots + |a_{N-1}| + |a_N| + \ldots$ Converge

$$\Rightarrow \sum_{n=1}^{\infty} a_n \quad \text{Converges}$$

Hence if $A < 1$, then $\sum_{n=1}^{\infty} a_n < \infty$

b) If $a > 1$, choose $b \ni a > b > 1$

let $\varepsilon = a - b > 0 \Rightarrow b = a - \varepsilon$

By the theorem :-

" If $m = \lim_{n\to\infty} \inf S_n$, then $S_n > m - \varepsilon$

for all but a finite values of n ".

$$\lim_{n\to\infty} \inf \left|\frac{a_{n+1}}{a_n}\right| = a$$

$$\Rightarrow \exists \text{ an } n \in I \ni \left|\frac{a_{n+1}}{a_n}\right| > a - \varepsilon = b > 1, \forall n \geq N$$

$$\Rightarrow |a_{n+1}| > |a_n|, \forall n \geq N$$

$$\Rightarrow |a_N| \leq |a_{N+1}| \leq |a_{N+2}| \leq \ldots$$

hence $\{a_n\}_{n=1}^{\infty}$ cannot converge to 0.

(ie) $\lim_{n\to\infty} a_n \neq 0$

hence, $\sum_{n=1}^{\infty} a_n$ diverges

[$\because \sum_{n=1}^{\infty} a_n$ doesn't satisfies the necessary condition for convergence $\lim_{n\to\infty} a_n = 0$]

c) If $a \leq 1 \leq A$, then the test fails.

To prove this consider the example

$\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$, for $\sum_{n=1}^{\infty} \frac{1}{n}$, $a_n = \frac{1}{n}$

$$\therefore \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1}{n+1} \times \frac{n}{1}\right| = \frac{n}{n+1} = \frac{1}{(1 + 1/n)}$$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$$

$$\lim_{n\to\infty} \sup \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \inf \left|\frac{a_{n+1}}{a_n}\right| = 1$$

$$\therefore A = a = 1$$

For $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $a_n = \frac{1}{n^2}$: $a_{n+1} = \frac{1}{(n+1)^2}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1}{(n+1)^2} \cdot n^2\right| = \frac{1}{(1+\frac{1}{n})^2}$$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$$

$$\lim_{n \to \infty} \sup \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \inf \left| \frac{a_{n+1}}{a_n} \right| = 1$$

$\therefore A = a = 1$

Hence for both the series $A = a = 1$

But $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ~~diverges~~ Converges

$\therefore$ Hence the test fails.

Theorem 4.5 :-

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of

Root test :-

If $\lim_{n \to \infty} \sup \sqrt[n]{|a|} = A$, then the series of real number's

$\sum_{n=1}^{\infty} a_n$,

a) Converges absolutely if $A < 1$

b) diverges if $A > 1$

c) If $A = 1$, the test fails.

Proof :

a) If $A < 1$, choose $B \ni A < B < 1$

Let $\epsilon = B - A > 0 \Rightarrow B = A + \epsilon$

Since $A = \underset{n \to \infty}{\text{lt}} \sup \sqrt[n]{|a|}$, By the theorem

If $M = \underset{n \to \infty}{\text{lt}} \sup s_n$, then $s_n < M + \epsilon$ & $s_n > M - \epsilon$

$\therefore \exists$ exist $N \in I \ni \sqrt[n]{|a_n|} < A + \epsilon \;\; \forall n \geq N$

$\Rightarrow \sqrt[n]{|a_n|} < B \Rightarrow |a_n| < B^n \;\; \forall n \geq N$

$\Rightarrow \sum_{n=1}^{\infty} a_n << \sum_{n=1}^{\infty} B^n$

But $\sum_{n=1}^{\infty} B^n$ is a Convergent series.

$\therefore$ By theorem (4.1)

$\sum_{n=1}^{\infty} a_n$ is a Convergent series.

b) If $A > 1$, choose $B \ni A > B > 1$

let $\epsilon = A - B > 0 \Rightarrow B = A - \epsilon$

$\therefore$ By the theorem,

$\sqrt[n]{|a|} > A - \epsilon$

$\Rightarrow \sqrt[n]{|a_n|} > B > 1$

$\therefore |a_n| > 1$

$\Rightarrow \{a_n\}_{n=1}^{\infty}$ does't Converge to '0'

$\sum_{n=1}^{\infty} a_n$ is a divergent series

c) If $A = 1$, then, P.T

the test fails.

Consider two series $\sum_{n=1}^{\infty} \frac{1}{n}$ (diverges series) and

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ (convergent series)

For $\sum_{n=1}^{\infty} \frac{1}{n}$; $a_n = \frac{1}{n}$

$\sqrt[n]{|a|} = \left(\frac{1}{n}\right)^{\frac{1}{n}}$; $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}}$

$$= \lim_{n\to\infty} e^{\log\left(\frac{1}{n}\right)^{\frac{1}{n}}}$$

$$= \lim_{n\to\infty} e^{\frac{1}{n}\log(n)^{-1}} = \lim_{n\to\infty} e^{-\frac{\log n}{n}}$$

$$= e^{0} = 1$$

For $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $a_n = \frac{1}{n^2}$, $\sqrt[n]{|a_n|} = \left(\frac{1}{n^2}\right)^{\frac{1}{n}}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{1}{n^2}\right)^{\frac{1}{n}} = \lim_{n\to\infty} e^{\log\left(\frac{1}{n^2}\right)^{1/n}}$$

$$= \lim_{n\to\infty} e^{-\frac{2\log n}{n}} = e^{0} = 1$$

$\therefore A = \lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, for both the series

But $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series and

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a Convergent series.

$\therefore$ Hence the test fails.

Theorem 4.6

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers, then,

a) if $\lim_{n\to\infty} \sup \sqrt[n]{|a|} = 0$; the series $\sum_{n=0}^{\infty} a_n x^n$ Converge absolutely $\forall x$.

b) if $\lim_{n\to\infty} \sup \sqrt[n]{|a_n|} = L > 0$, then $\sum_{n=0}^{\infty} a_n x^n$ Converges absolutely for $|x| < \frac{1}{L}$ and diverges for $|x| > \frac{1}{L}$

c) if $\lim_{n\to\infty} \sup \sqrt[n]{|a|} = \infty$, then $\sum_{n=1}^{\infty} a_n x^n$ Converges only for $x = 0$ and diverges for all other x.

Proof :

Let, Consider $\sqrt[n]{|a_n x^n|} = |x| \sqrt[n]{|a_n|}$

let $A = \lim_{n\to\infty} \sup \sqrt[n]{|a_n x^n|}$

$= |x| \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$

(a) Take $\lim_{n\to\infty} \sup \sqrt[n]{|a_n|} = 0$

$A = |x| . 0 = 0 < 1$

By theorem (4.5), $\sum_{n=0}^{\infty} a_n x^n$ Converges absolutely.

(b) Take $\lim_{n\to\infty} \sup \sqrt[n]{|a_n|} = L$

$A = |x| . L$

By root test, $A < 1$, the series converges
$A > 1$, the series diverges

(i) $|x| L < 1$, the series converges.
$|x| L > 1$. the series diverges

(ie) $|x| < \frac{1}{L}$, the series converges

$\times$ $|x| > \frac{1}{L}$, the series diverges.

(C) If $\lim_{n\to\infty} \sup \sqrt[n]{|a_n|} = \infty$

$A = |x| \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$

$= |x| . \infty = \infty$ except for $x = 0$

If $x = 0$, $A = 0 < 1$

Hence by root test, $\sum_{n=0}^{\infty} a_n x^n$ Converges

and $\sum_{n=0}^{\infty} a_n x^n$ diverges. for except $x = 0$

Corollary 4.7 :-

If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = x_0$ then it Converges absolutely $\forall \ni |x| < |x_0|$ given.

Proof :-

$|x| < |x_0|$

$\Rightarrow |x|^n < |x_0|^n$

$\Rightarrow |x^n| < |x_0^n|$

$\Rightarrow |a_n x^n| < |a_n x_0|^n \quad \forall n \in I$

$\Rightarrow \sum_{n=0}^{\infty} a_n x^n << \sum_{n=0}^{\infty} a_n x_0^n$

By theorem (4.1), $\sum_{n=0}^{\infty} a_n x_0^n$ Converges absolutely

$\Rightarrow \sum_{n=0}^{\infty} a_n x^n$ Converges absolutely.

Series whose terms form a nonincreasing sequence.

Cauchy Condensation test :

Theorem 4.8 :-

If $\{a_n\}_{n=1}^{\infty}$ is a nonincreasing sequence of positive number's and if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ Converges. then $\sum_{n=1}^{\infty} a_n$ Converges.

Proof :

Given (1) $\{a_n\}_{n=1}^{\infty}$ is a nonincreasing sequence.

(ie) $a_1 \geq a_2 \geq a_3 \ldots\ldots \geq a_n \geq a_{n+1} \geq$

(2) $\sum_{n=0}^{\infty} 2^n a_{2^n}$ Converges

Since, $a_1 \geq a_2 \geq a_3 \geq \ldots\ldots$

$a_1 = a_1$

$a_2 + a_3 \leq a_2 + a_2 = 2a_2$

$$\begin{aligned} a_4 + a_5 + a_6 + a_7 &\leq a_4 + a_4 + a_4 + a_4 \\ &= 4a_4 \\ &= 2^2 a_{2^2} \end{aligned}$$

$a_{2^n} + a_{2^n+1} + \ldots\ldots a_{2^{n+1}-1} \leq 2^n a_{2^n}$

$$\therefore \sum_{k=1}^{2^{n+1}-1} a_k \leq \sum_{k=0}^{n} 2^k a_{2^k} \leq \sum_{k=0}^{\infty} 2^k a_{2^k} \quad \forall n \in I$$

$$\Rightarrow \sum_{n=1}^{\infty} ak_1 \leq \sum_{k=0}^{\infty} 2^k a_{2^k} \quad \forall m \in I$$

$\therefore \sum_{n=0}^{\infty} 2^n a_{2^n}$ Converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ Converges

[Since the sequence of n^{th} partial sum's of $\sum_{n=1}^{\infty} a_n$ is bounded and hence $\sum_{n=1}^{\infty} a_n$ Converges]

Theorem 4.9 :

If $\{a_n\}_{n=1}^{\infty}$ is a nonincreasing sequence of positive number's if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges

Proof :

Given, $a_1 \geq a_2 \geq a_3 \geq a_4 \ldots\ldots\ldots$

Given, $\sum_{n=0}^{\infty} 2^n a_{2^n}$ diverges.

$a_3 + a_4 \geq a_4 + a_4 = 2a_4 = 2a_{2^2}$

$a_5 + a_6 + a_7 + a_8 \geq a_8 + a_8 + a_8 + a_8 = 4a_8$

In general,

$a_{2^n+1} + \ldots\ldots\ldots + a_{2^{n+1}} \geq 2^n a_{2^{n+1}}$

$\Rightarrow \frac{1}{2} 2^{n+1} a_{2^{n+1}}$

$$\sum_{k=3}^{2^{n+1}} a_k \geq \frac{1}{2} \sum_{k=1}^{} 2^{k+1} a_{2^{k+1}} = \frac{1}{2} \sum_{k=2}^{n+1} 2^k a_{2^k} \quad \text{--- (1)}$$

Since, $\sum_{n=0}^{\infty} 2^n a_{2^n}$ diverges, (1) $\Rightarrow$ The sequence of n^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$ diverges.

Corollary 4.10 :

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges.

Proof :

Let $a_n = \frac{1}{n^2}$

$$\sum_{n=1}^{\infty} 2^n a_{2^n} = \sum_{n=1}^{\infty} 2^n \frac{1}{(2^n)^2} = \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$$

Hence by theorem (4.8),

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

Hence $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Thearem 4.11

If $\{a_n\}_{n=1}^{\infty}$ is a nonincreasing sequence of positive number's and $\sum_{n=1}^{\infty} a_n$ Converges then

$$\lim_{n\to\infty} na_n = 0$$

Proof: Let $s_n = a_1 + a_2 + \dots\dots + a_n$

If $\sum_{n=1}^{\infty} a_n = A$

$$\lim_{n\to\infty} s_n = A = \lim_{n\to\infty} s_{2n}$$

Thus $\lim_{n\to\infty} s_{2n} - \lim_{n\to\infty} s_n = \lim_{n\to\infty} (s_{2n} - s_n)$

$$= A - A = 0$$

Now, $s_{2n} - s_n = a_{n+1} + a_{n+2} + \dots\dots + a_{2n} \geq$

$$a_{2n} + a_{2n} + \dots\dots + a_{2n} = na_{2n}$$

$$\lim_{n\to\infty} na_{2n} = 0$$

$$\lim_{n\to\infty} 2na_{2n} = 0 \quad \text{——} \; (1)$$

But $a_{2n+1} \leq a_{2n}$. Thus $(2n+1)a_{2n+1} \leq (2n+1)a_{2n}$

$2n \to$ even

$2n+1 \to$ odd

$$\therefore \lim_{n\to\infty} (2n+1)a_{2n+1} = 0 \quad \text{——} \; (2)$$

From (1) & (2), $\lim_{n\to\infty} na_n = 0$, When n is odd (or) even

Example :-

If we drop the hypothesis that $\{a_n\}_{n=1}^{\infty}$ is non-increasing consider the series $\sum_{n=1}^{\infty} a_n$ Where

$$a_n = \begin{cases} \frac{1}{n}, & n = 1, 4, 9, 16 \ldots\ldots \text{ (a perfect square)} \\ \frac{1}{n^2}, & \text{if } n \text{ is not a perfect square.} \end{cases}$$

Then. $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \ldots\ldots$

$$= \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9} + \ldots$$

$$= \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right) + \left(1 + \frac{1}{4} + \frac{1}{9} + \ldots\right)$$

$$= 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent.

$\sum_{n=1}^{\infty} a_n$ is convergent.

But $\lim_{n \to \infty} na_n = n\left(\frac{1}{n}\right) = 1$. When n is a perfect sequare.

$\therefore$ If $\{a_n\}_{n=1}^{\infty}$ is not an non-increasing sequare

then $\lim_{n \to \infty} na_n \neq 0$

Summation by Part's :-

Theorem 4.12

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequares of real number's and let $s_n = a_1 + a_2 + \ldots\ldots + a_n$

Then for each $n \in I$, $\sum_{k=1}^{n} a_k . b_k = s_n b_{n+1} - \sum_{k=1}^{n} s_k (b_{k+1} - b_k)$

Proof:- Define $s_0 = 0$ and $a_k = s_k - s_{k-1}$

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} (s_k - s_{k-1}) b_k$$

$$\Rightarrow b_1(s_1 - s_0) + b_2(s_2 - s_1) + b_3(s_3 - s_2) + \dots + b_{n-1}(s_{n-1} - s_{n-2}) + b_n(s_n - s_{n-1})$$

$$\Rightarrow s_1(b_1 - b_2) + s_2(b_2 - b_3) + \dots + s_{n-1}(b_{n-1} - b_n) + s_n(b_n - b_{n+1}) + s_n b_{n+1}$$

$$\Rightarrow s_n b_{n+1} + \sum_{k=1}^{n} s_k (b_k - b_{k+1})$$

$$\Rightarrow s_n b_{n+1} - \sum_{k=1}^{n} s_k (b_{k-1} - b_k)$$

Abel's Theorem :- 4.13

If $\{a_n\}_{n=1}^{\infty}$ is a sequence of real number's whose partial sums $s_n = \sum_{k=1}^{n} a_k$ satisfy $m \le s_n \le M, n \in I$ for some $m, M \in R$ and if $\{b_n\}_{n=1}^{\infty}$ is a non-increasing sequence, then $mb_1 \le \sum_{k=1}^{n} a_k b_k \le Mb_1$ $(n \in I)$

Proof:

From Theorem (4.12)

$$\sum_{k=1}^{n} a_k b_k = s_n b_{n+1} - \sum_{k=1}^{n} s_k (b_{k-1} - b_k)$$

since $\{b_n\}$ is a non-increasing sequence,

$b_k \ge b_{k+1}$

$\Rightarrow b_k - b_{k+1} \ge 0$,

$$\sum_{k=1}^{n} a_k b_k = s_n b_{n+1} + \sum_{k=1}^{n} s_k (b_k - b_{k+1})$$

since, $s_k \le M$, $k = 1$ to n

$$\sum_{k=1}^{n} a_k b_k \le M b_{n+1} + M \sum_{k=1}^{n} (b_k - b_{k-1})$$

$$= M b_{n+1} + M(b_1 - b_2) + M(b_2 - b_3) + \dots + M(b_n - b_{n+1})$$

$= Mb_1$

$\Rightarrow \sum_{k=1}^{n} a_k b_k \leq Mb_1$

$||^{rly}$, if $m \leq s_k \ \forall \ k = 1$ to n

$$\sum_{k=1}^{n} a_k b_k \geq m\, b_{n+1} + m \sum_{k=1}^{n} (b_k - b_{k+1})$$

$$= m\, b_{n+1} + m(b_1 - b_2) + m(b_2 - b_3) + \ldots \ldots + m(b_n - b_{n+1})$$

$= mb_1$

$\Rightarrow mb_1 \leq \sum_{k=1}^{n} a_k b_k$ —— ②

From ① & ②. $mb_1 \leq \sum_{k=1}^{n} a_k b_k \leq Mb_1$

NOTE:

Thus in Abel's lemma

if $\left| \sum_{k=1}^{n} a_n \right| \leq M \ \forall n$, then $\left| \sum_{k=1}^{n} a_k b_k \right| \leq Mb_1$.

Definition:

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers

Then $\{s_n\}_{n=1}^{\infty}$ is called a cauchy sequence if for any $\varepsilon >$

$\exists N \in I \ni |s_m - s_n| < \varepsilon \ \forall \ m, n \in I$.

Theorem : 4.14 :-

If the sequence of real numbers $\{s_n\}_{n=1}^{\infty}$ converges, then $\{s_n\}_{n=1}^{\infty}$ is a cauchy sequence.

Theorem : 4.15 :-

If $\{s_n\}_{n=1}^{\infty}$ is a cauchy sequence of real numbers, then $\{s_n\}_{n=1}^{\infty}$ is convergent.

Theorem : 4.16 (Dirichlet's test)

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers whose partical sums $s_n = \sum_{k=1}^{n} a_k$ form a bounded sequence and let $\{b_n\}_{n=1}^{\infty}$ be a non increasing sequence of ~~nonnegative~~ negative number's which converges to 0. Then $\sum_{k=1}^{\infty} a_k b_k$ converges.

Proof :-

Since cauchy sequence is convergent sequence it is sufficient to prove that the partial sums of $\sum_{k=1}^{\infty} a_k b_k$ form a cauchy sequence. That is given $\varepsilon > 0$ $\exists$ $N \in I$, $\left|\sum_{k=m}^{n} a_k b_k\right| < \varepsilon$ $(n \geq m \geq N)$

Since s_n is a bounded sequence. $\exists$ $M > 0$ $\ni |s_n| \leq M$ $(n \in I)$

Hence for any $m, n \in I$

$$\left|\sum_{k=m}^{n} a_k\right| = |s_n - s_{m-1}| \leq |s_n| + |s_{m-1}|$$

$$= M + M = 2M$$

By Abel's lemma for the sequence $\{a_k\}_{k=m}^{\infty}$ and $\{b_k\}_{k=m}^{\infty}$ $\left|\sum_{k=m}^{n} a_k b_k\right| \leq 2M b_m$.

Since $\{b_n\}_{n=1}^{\infty}$ be a non-increasing sequence of non-negative number's. $\exists$ $N \in I$ $\ni$ $b_n < \frac{\varepsilon}{2M}$ $(n \geq N)$

Hence $2Mb_m < \varepsilon$ $(m \geq N)$.

So that, $\left|\sum_{k=m}^{n} a_k b_k\right| < \varepsilon$.

Hence the partial sums of $\sum_{k=1}^{\infty} a_k b_k$ is a cauchy sequence. There fore, $\sum_{k=1}^{\infty} a_k b_k$ converges.

example :-

P.T $\sum_{k=1}^{\infty} \frac{\sin(k)}{k}$ Converges using dirichlet's test :

1) let $a_k = \sin(k)$ & $b_k = \frac{1}{k}$

$b_k = \frac{1}{k} \to 0$ as $k \to \infty$ $\left(\lim_{k\to\infty} b_k = 0\right)$

2) Consider $\sum_{k=1}^{m} \sin k$

Then $2\sin(1) \sum_{k=1}^{m} \sin k$

$= \sum_{k=1}^{m} \cos(1-k) - \cos(1+k)$

$= (\cos 0 - \cos 2) + (\cos 1 - \cos 3) +$

$(\cos 2 - \cos 4) + \dots + (\cos(2m) - \cos m)$

$+ (\cos(1-m) - \cos(m+1))$

$= \cos 0 + \cos 1 - \cos m - \cos(m+1)$

$$\sum_{k=1}^{m} \sin k = \frac{\cos 0 + \cos 1 - \cos m - \cos(m+1)}{2\sin(1)}$$

$$\left| \sum_{k=1}^{m} \sin k \right| \le \left| \frac{4}{2\sin(1)} \right| = \left| \frac{2}{\sin(1)} \right| = M$$

$\cos 0 = 1$

$\cos 1 = 1$

$\cos m =$

$\cos(m+1) =$

The partial sum of $\{\sin k\}_{k=1}^{\infty}$ form a bounded sequence.

3) $\therefore$ By Dirichlet's test

$\sum_{k=1}^{\infty} \frac{\sin(k)}{k}$ convergent.

Theorem 4.17 :

If $\sum_{n=1}^{\infty} a_n$ is a convergent series of real number's and if the sequence $\{b_n\}_{n=1}^{\infty}$ is monotone and convergent, then $\sum_{n=1}^{\infty} a_n b_n$ also Converges.

Proof :

Let us suppose that $\{b_n\}_{n=1}^{\infty}$ is non-decreasing and let $c_n = b - b_n$.

Where, $b = \lim_{n \to \infty} b_n$

Then $c_n \geq 0$ $(b - b_n \geq 0)$, $\lim_{n \to \infty} c_n = 0$

and $\{c_n\}_{n=1}^{\infty}$ is non increasing.

$\therefore$ by theorem (4.16), the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

By theorem " If $\sum_{n=1}^{\infty} a_n$ Converges to A and $\sum_{n=1}^{\infty} b_n$ Converges to B, then $\sum_{n=1}^{\infty} (a_n + b_n)$ converges to $A+B$. Also, if $c \in \mathbb{R}$, then $\sum_{n=1}^{\infty} c\, a_n$ Converges to cA",

The series $\sum_{n=1}^{\infty} b a_n$ Converges.

But $a_n b_n = b a_n - a_n c_n$

$((c_n = b - b_n) \Rightarrow c_n a_n = b a_n - a_n b_n \Rightarrow a_n b_n = b a_n - c_n a_n)$

Thus $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} (b a_n - c_n a_n)$ Converges.

example's :-

Consider the series $\sum_{n=1}^{\infty} a_n = 1-1+\frac{1}{2}-\frac{1}{2}+\frac{1}{3}-\frac{1}{3}$

is a convergent series.

Consider the sequence $\{b_n\}_{n=1}^{\infty} = \{0, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{2}{3}, \frac{3}{4}, \frac{3}{4}, \dots\}$ is monotone and convergent

sequence.

$\therefore$ By theorem (4.17), $\sum_{n=1}^{\infty} a_n b_n = 0 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{3} + \frac{2}{3}^2 - \frac{1}{4} + \frac{3}{4}^2 - \dots$

must convergent

(C,1) summability of series

Definition :-

Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers that partial sums $S_n = a_1 + a_2 + \dots + a_n$ we shall say that $\sum_{n=1}^{\infty} a_n$ is (C,1) summable to 'A' if $\lim_{n\to\infty} S_n = A$ (C,1)

Lemma :-

Theorem 4.18 :

Let $\sum_{n=1}^{\infty} a_n$ be (C,1) summable and let $t_n = a_1 + 2a_2 + \dots\dots n a_n$ then if $\lim_{n\to\infty} \frac{t_n}{n} = 0$ the series $\sum_{n=1}^{\infty} a_n$ converges.

Proof:

we first ST $t_n = (n+1) S_n - n\sigma_n, n \in I$ — ①

Where $\sigma_n = n^{-1}(S_1 + S_2 + \dots + S_n)$

For $n=1$ $t_1 = a_1 = S_1 = \sigma_1$

The result is true for $n=1$

Let us assume the result is false for $n+1$

We have,

$$t_{n+1} = t_n + (n+1)a_{n+1}$$

$$= (n+1)s_n - n\sigma_n + (n+1)a_{n+1}$$

$$= (n+1)(s_n + a_{n+1}) - n\sigma_n$$

$$= (n+1)(s_{n+1}) - n\sigma_n$$

$$= (n+1)(s_{n+1}) - n(n^{-1})(s_1 + s_2 + \dots + s_n)$$

$$= (n+1)(s_{n+1}) - (s_1 + s_2 + \dots + s_n)$$

$$= (n+1)(s_{n+1}) - (s_1 + s_2 + \dots + s_n + s_{n+1} - s_{n+1})$$

$$= (n+2)(s_{n+1}) - (s_1 + s_2 + \dots + s_{n+1})$$

$$= (n+2)s_{n+1} - (n+1)\sigma_{n+1}$$

hence, $t_n = (n+1)s_n - n\sigma_n$.

is true for all n given that $\sum_{n=1}^{\infty} a_n$ is $(C,1)$ summable. Then $\lim_{n\to\infty} \sigma_n = A$.

From ①,

$$s_n = \frac{t_n + n\sigma_n}{n+1}$$

$$s_n = \frac{n}{n+1}\left(\frac{t_n}{n} + \sigma_n\right)$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n}{n+1}\left(\frac{t_n}{n} + \sigma_n\right)$$

$$= \lim_{n\to\infty} \frac{n}{n+1}(0+A) = 1(0+A) = A$$

$\therefore \{s_n\}_{n=1}^{\infty}$ converges.

hence $\sum_{n=1}^{\infty} a_n$ converges.

Theorem : 4.19

If $\sum_{n=1}^{\infty} a_n$ is (C, 1) summable and

$$\lim_{n\to\infty} na_n = 0$$

then $\sum_{n=1}^{\infty} a_n$ is convergent.

Proof : given $\sum_{n=1}^{\infty} a_n$ is (C, 1) summable, and $\{na_n\}_{n=1}^{\infty}$

Converges to '0'

P.T $\sum_{n=1}^{\infty} a_n$ is converges (by the theorem)

$\{na_n\}_{n=1}^{\infty}$ converges to '0'.

$$\Rightarrow \lim_{n\to\infty} \frac{a_1 + 2a_2 + \dots + na_n}{n} = 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{t_n}{n} = 0$$

$\therefore$ by lemma (4.18) $\sum_{n=1}^{\infty} a_n$ Converges.