2 - Unit

SEQUENCES

Definition :

A sequences $S = \{s_i\}_{i=1}^{\infty}$ of real numbers is a function from I (the set of positive integers) in to R (set of real numbers).

Note :

1) Here $S = \{s_i\}_{i=1}^{\infty}$ (or) $s_1, s_2, s_3, \ldots$

The number s_i = ($i = 1, 2, \ldots$) is the i^{th} term of the sequence.

2) Sequences such as $S = \{s_i\}_{i=1}^{M}$ (or) $S = \{s_i\}_{i=-\infty}^{\infty}$ are also defined.

Example's :-

1) $1, 2, 3, 4, \ldots$ (ie) $S = \{n\}_{n=1}^{\infty}$

2) $3, 6, 9, 12, \ldots$ (ie) $S = \{3n\}_{n=1}^{\infty}$

3) $1, \frac{1}{2}, \frac{1}{3}, \ldots$ (ie) $S = \{\frac{1}{n}\}_{n=1}^{\infty}$

4) $\pm 1, \pm 3, \pm 5, \pm 7, \ldots$ (ie) $S = \{2n+1\}_{n=-\infty}^{\infty}$

Subsequence :

A subsequence N of $\{n\}_{n=1}^{\infty}$ (the sequence of positive integers) is a function from I (the set of positive integers) into I such that,

$$N(i) < N(j) \text{ if } i < j \quad (i, j \in I)$$

since $N : I \to I$ it follows that $N : I \to R$.

Therefore N is a sequence.

(ie) a subsequence of $\{n\}_{n=1}^{\infty}$ is a sequence of integers whose terms get larger and larger

Example :-

Let the prime sequence be 2, 3, 5, 7, 11 ...... is the subsequence of $\{n\}_{n=1}^{\infty}$.

Definiton ::

If $S = \{S_n\}_{n=1}^{\infty}$ is a sequence of real number and $N = \{n_i\}_{i=1}^{\infty}$ is a subsequence of the sequence of positive integers, then the composite function $S \circ N$ is called a subsequence of S.

Note that for $i \in I$ we have $N(i) = n_i$.

$$S \circ N(i) = S[N(i)] = S(n_i) = S_{n_i}$$

$$(ie) \quad S \circ N = \{S_{n_i}\}_{i=1}^{\infty}$$

example 1 :

B is the sequence 1, 0, 1, 0, ...

and $N = \{n_i\}_{i=1}^{\infty} = \{2i - 1\}_{i=1}^{\infty}$

Here $N = 1, 3, 5, 7, \ldots\ldots = n_1, n_2, n_3, \ldots\ldots$

Thus $B \circ N = B(N(i))$

$= B(n_i)$

$= \{B_{n_i}\}_{i=1}^{\infty} = B_1, B_3, B_5, B_7, \ldots$

$B \circ N = 1, 1, 1, \ldots\ldots$

Example 2 :

If $c = \{c_n\}_{n=1}^{\infty} = \{\sqrt{n}\}_{n=1}^{\infty}$

Answer (ie) $c_1, c_2, c_3, c_4, \ldots = \sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots$

$N = \{n_i\}_{i=1}^{\infty} = \{i^4\}_{i=1}^{\infty}$

$n_1, n_3, n_3, \ldots = 1^4, 2^4, 3^4, \ldots$

$C \circ N = C(N(i)) = C(n_i)$

$= C_{n_i} = C_{n_1}, C_{n_2}, C_{n_3}$

$= C_{1^4}, C_{2^4}, C_{3^4}, \ldots$

$= \sqrt{1^4}, \sqrt{2^4}, \sqrt{3^4}, \ldots$

$= 1^2, 2^2, 3^2, \ldots$

Problem's :-

1) Find S_8 in the Fibonacci Series.

$S = \{S_i\}_{i=1}^{\infty}$ where $S_{n+1} = S_n + S_{n-1}$

$S = 1, 1, 2, 3, 5, 8, 13, 21, \ldots$

$S_8 = 21$

2) Write a formula for each sequences

a) $1, 0, 1, 0, 1, 0, \ldots$

b) $1, 3, 6, 10, 15, \ldots$

c) $1, -4, 9, -16, 25, -36, \ldots$

d) $1, 1, 1, 2, 1, 3, 1, 4, 1, 5, 1, 6, \ldots$

a) $1, 0, 1, 0, 1, 0, \ldots$

$S = 1, 0, 1, 0, 1, 0, \ldots$

$$S_i = \begin{cases} 1 & \text{if } i \text{ is odd number} \\ 0 & \text{if } i \text{ is even number} \end{cases}$$

$S = \{s_i\}_{i=1}^{\infty}$

b) $S = 1, 3, 6, 10, 15, \ldots$

$= \{ S_{n+1} = n + (n+1) \} \Rightarrow \{ S_{n+1} = n+n+1 \}_{n=0}^{\infty}$

c) $S = 1, -4, 9, -16, 25, -36, \ldots$

$S = \{ (-1)^{n+1} n^2 \}_{n=1}^{\infty}$

d) $S = 1, 1, 1, 2, 1, 3, 1, 4, 1, 5, 1, 6, \ldots$

$$S_i = \begin{cases} 1 & i, \text{odd number} \\ \frac{n}{2} & i, \text{even number} \end{cases}$$

Limit of a sequence:

Definiation:

Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of real numbers we say that S_n approaches the limit L (as $n \to \infty$) if for every $\varepsilon > 0$, there is a positive integer N such that $|S_n - L| < \varepsilon$ $(n \geq N)$

If S_n Approaches the limit L we write

$$\lim_{n \to \infty} S_n = L$$

Example:

1. Consider the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$

(ie) $S = \{S_n\}_{n=1}^{\infty} = \{\frac{1}{n}\}_{n=1}^{\infty}$

Naturally we would guess that the limit of the sequence is 0.

(ie) $\left|\frac{1}{n} - 0\right| < \varepsilon \quad (n \geq N)$

$\frac{1}{n} < \varepsilon$ for $(n \geq N)$

$n \geq N \Rightarrow \frac{1}{N} \leq \frac{1}{N}$

$\therefore \frac{1}{N} < \varepsilon$ if and only if $N > \frac{1}{\varepsilon}$

This $N > \frac{1}{\varepsilon}$

Thus $\lim_{n \to \infty} \frac{1}{n} = 0$

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2) Consider the sequence $S = \{1\}_{n=1}^{\infty}$

Ans: $S = 1, 1, 1, 1, \ldots\ldots$

$\lim_{n \to \infty} S_n = 1$ for $(S_n = 1)$

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3) Consider the sequence $1, 2, 3 \ldots\ldots \infty$

Ans: (ie) $S = \{n\}_{n=1}^{\infty}$

This sequence does not have a limit

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4) Consider the sequence $S = \{(-1)^n\}_{n=1}^{\infty}$

Ans: $S = -1, 1, -1, 1, \ldots\ldots$

suppose this sequence has a limit L,

then suppose this sequence has a limit L, then

$\lim_{n \to \infty} S_n = L$

Taking $\varepsilon = \frac{1}{2}$

(ie) $|(-1)^n - L| < \frac{1}{2} \quad (n \geq N)$

for even $|1 - L| < \frac{1}{2}$ (1)

and $|-1 - L| < \frac{1}{2}$ (2)

Both ① & ② Cannot be passible

② $\Rightarrow |1+L| < \frac{1}{2}$

$2 = |2| = |1+1| = |1+L+1-L| \leq |1+L| + |1-L|$

$< \frac{1}{2} + \frac{1}{2} = 1$

When is a contradiction

Thus the sequence $S = \{(-1)^n\}$ has no limit

5) find the limit of the sequence $S = \left\{\frac{2n}{n+4n^{1/2}}\right\}_{n=1}^{\infty}$

Ans :-

Let $S_n = \frac{2n}{n+4n^{1/2}}$

$\lim_{n\to\infty} S_n = L \Rightarrow \lim_{n\to\infty} \frac{2n}{n+4\sqrt{n}}$

$\boxed{\frac{1}{\infty} = 0 \quad \frac{B}{\infty} = 0}$

$\lim_{n\to\infty} \frac{2n}{n+4\sqrt{n}} = \lim_{n\to\infty} \frac{2n}{n(1+4\frac{1}{\sqrt{n}})} = \lim_{n\to\infty} \frac{2}{1+4\frac{1}{\sqrt{n}}}$

$= \frac{2}{1+4\left(\frac{1}{\infty}\right)} = \frac{2}{1} = 2$

Theorem :-

If $\{S_n\}_{n=1}^{\infty}$ is a sequence of nonnegative numbers and if $\lim_{n\to\infty} S_n = L$ then $L \geq 0$ proof.

Suppose, on contrary let $L < 0$

Then $\varepsilon = -\frac{L}{2}$ there exists $N \in I$

Such that $|S_n - L| < -\frac{L}{2}$ $(n \geq N)$

$|S_n - L| < -\frac{L}{2}$

$S_n - L < -\frac{L}{2}$ (since S_n are the +ve of L is -ve)

$$S_n < -\frac{L}{2} + L$$

$$S_n < \frac{L}{2}$$

Which is a contradiction

since all $S_n \geq 0$

Thus $L < 0$ contradicting the hypothesis

Therefore $L \geq 0$

Hence proved.

Problem's

1. Prove that $\lim_{n \to \infty} \frac{2n}{n+3} = 2$

Ans:

$$\lim_{n \to \infty} \frac{2n}{n+3} = \lim_{n \to \infty} \frac{2n}{n(1+3/n)}$$

$$= \lim_{n \to \infty} \frac{2}{1+3/n} = \frac{2}{1+0} = 2$$

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2. prove that $\lim_{n \to \infty} \frac{1}{\sqrt{n+1}} = 0$

Ans:

$$\lim_{n \to \infty} \frac{1}{\sqrt{n+1}} = \lim_{n \to \infty} \frac{1}{\sqrt{n}\sqrt{1+\frac{1}{n}}}$$

$$\frac{1}{\sqrt{n(1+\frac{1}{n})}} = \frac{1}{\sqrt{n}\sqrt{1+\frac{1}{n}}}$$

$$= \lim_{n \to \infty} \frac{1/\sqrt{n}}{\sqrt{1+1/n}}$$

$$= \frac{0}{\sqrt{1+0}} = 0$$

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3) find the limit of the following sequences

a) $\left\{\frac{n^2}{n+5}\right\}_{n=1}^{\infty}$

Ans:

$$\lim_{n \to \infty} \frac{n^2}{n+5} = \lim_{n \to \infty} \frac{n^2}{n(1+\frac{5}{n})} \Rightarrow \lim_{n \to \infty} \frac{n}{1+5/n}$$

Cannot Apply $n \to \infty$

Therefore the sequence $\left\{ \frac{n^2}{n+5} \right\}_{n=1}^{\infty}$ does not have a limit

b) $\left\{ \frac{3n}{n+7\sqrt{n}} \right\}_{n=1}^{\infty}$

$$\lim_{n \to \infty} \frac{3n}{n+7\sqrt{n}} = \lim_{n \to \infty} \frac{3n}{n\left(1+ 7/\sqrt{n}\right)}$$

$$\Rightarrow \lim_{n \to \infty} \frac{3}{1+ 7/\sqrt{n}} = \frac{3}{1+0} = 3$$

c) $\left\{ \frac{3n}{n+7n^2} \right\}_{n=1}^{\infty}$

$$\lim_{n \to \infty} \frac{3n}{n+4n^2} = \lim_{n \to \infty} \frac{3n}{n^2\left(\frac{1}{n}+7\right)}$$

$$= \lim_{n \to \infty} \frac{3/n}{\left(1/n + 7\right)} = \frac{0}{0+7} = 0$$

Convergent Sequence :-

Definition:

If the sequence of real numbers $\{S_n\}_{n=1}^{\infty}$ has the limit L, we say that $\{S_n\}_{n=1}^{\infty}$ is convergent to L.

If $\{S_n\}_{n=1}^{\infty}$ does not have a limit then $\{S_n\}_{n=1}^{\infty}$ is divergent sequence.

Example's :

1) Sequence $1, 1, 1, 1, \ldots$ is convergent and limit is 1.

2) Sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$ is convergent to limit 0,

3) sequence $0, 1, 0, 1, 0, 1, \ldots$ is a divergent sequence.

Therom :

If the sequence of real numbers $\{S_n\}_{n=1}^{\infty}$ is convergent to L, then $\{S_n\}_{n=1}^{\infty}$ cannot be convergent to a limit distinct from L.

(ie) if $\lim\limits_{n\to\infty} S_n = L$ and $\lim\limits_{n\to\infty} S_n = M$

then $L = M$

Proof :- Let $\lim\limits_{n\to\infty} S_n = L$ and $\lim\limits_{n\to\infty} S_n = M$

On contrary let us assume $L \neq M$

since $\lim\limits_{n\to\infty} S_n = L$, $|S_n - L| < \varepsilon \quad (n \geq N_1)$

$S_n = L$, $|S_n - L| < \varepsilon \quad (n \geq N_1)$

Ily since $\lim\limits_{n\to\infty} S_n = M$, $|S_n - M| < \varepsilon \quad (n \geq N_2)$

$S_n = M$, $|S_n - M| < \varepsilon \quad (n \geq N_2)$

for some $N_1, N_2 \in I$.

Let $N = \max(N_1, N_2)$

Then $N \geq N_1$ and $N \geq N_2$

So both $|S_N - L| < \varepsilon$ and

$|S_N - M| < \varepsilon$

$|M-L| = |(S_n - L) - (S_n - M)| \leq |S_n - L| + |S_n - M|$

$\Rightarrow \quad < \varepsilon + \varepsilon$

$|M-L| < 2\varepsilon = |M-L|$

Which is a countradiction

$\therefore L = M$ // (Proved).

Thearem :-

If the sequence of real number's $\{S_n\}_{n=1}^{\infty}$ is convergent to L, them any subsequence of $\{S_n\}_{n=1}^{\infty}$ is also convergent to the same limit.

Carollary:

All subsequence's of a convergent sequence of real number's converge to the same limit.

Examples :

1) $\{(-1)^n\}_{n=1}^{\infty}$ is divergent sequence but its subsequence's $1, 1, 1, \ldots$ Converges to 1 and $-1, -1, -1, \ldots$ Converges to -1.

Thus disvergent sequence may have a convergent subsequence.

* Divergent Sequence :-

Definition :-

Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of real numbers we say that S_n approaches infinity an n approachen infinity if for any real number

$M > 0$ there is a positive integar N such that

$$S_n \geq M \quad (n \geq N)$$

In this case we write $S_n \to \infty$ as $n \to \infty$ (or)

we can say $\{S_n\}_{n=1}^{\infty}$ divergents infinity.

Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of real number's

we say that.

S_n approaches minus infinity as n approaches infity if for any real number

$M > 0$ there is a positive interger N such

that $S_n < -M \quad (n \geq N)$

Then $S_n \to -\infty$ as $n \to \infty$.

If the sequence $\{S_n\}_{n=1}^{\infty}$ of real number's

deverger but not diverge to infinity and does not diverge to minus infitiity, we say that $\{S_n\}_{n=1}^{\infty}$ oscillates. (oscillating Sequence).

<u>Note</u>:-

1) $\{1, 2, 3, \ldots\}$ is the sequence divergen to infinity.

2) $\{-1, -2, -3, \ldots\}$ is the sequence divergen to minus infinity

3) $\{\log \frac{1}{n}\}_{n=1}^{\infty}$ is the sequence divergen to minus infinity

4) $\{1, -1, 1, -1, 1, -1\}\ldots$ is a oscillating sequence (divergent)

5) Oscillating sequence is a divergent sequence.

Bounded Sequence:

Definition:

⇒ The sequence $\{s_n\}_{n=1}^{\infty}$ is bounded above if the range of $\{s_n\}_{n=1}^{\infty}$ is bounded above.

(ie) if there is a number N such that $s_n \leq N$ for all s_n.

then, $\{s_n\}$ is bounded above.

⇒ The sequence $\{s_n\}_{n=1}^{\infty}$ is bounded below if the range of $\{s_n\}_{n=1}^{\infty}$ is bounded below.

(ie) if there is a number N such that $s_n \geq N$ for all s_n then $\{s_n\}$ is below bounded.

Example:

1) $\{1, 2, 3, \ldots\}$ is bounded below since $0 \leq s_n \; \forall s_n$.

2) $\{-1, -2, -3, \ldots\}$ is bounded above since $s_n \leq 0 \; \forall s_n$.

Thus $\{s_n\}_{n=1}^{\infty}$ is bounded if and only if there exists $M \in R$.

such that $|s_n| \leq M \quad (\forall n \in I)$

$|s_n| \leq M$

(ie) $-M \leq s_n \leq M$

Note:

Divergent Sequence is not a bounded.

Therom:

If the sequence of real numbers $\{s_n\}_{n=1}^{\infty}$ is Convergent.

then $\{s_n\}_{n=1}^{\infty}$ is bounded.

Proof: Suppose $L = \lim_{n\to\infty} s_n$. Then given $\varepsilon = 1$

there exists $N \in I$ such that

$$|s_n - L| < 1 \qquad (n \geq N)$$

$$\Rightarrow |s_n - L| \leq |s_n| - |L| < 1$$

$$|s_n| < L + 1$$

Let $M = \max\{|s_1|, |s_2|, |s_3|, \ldots\ldots |s_{N-1}|\}$

then we have

$$|s_n| < M + L + 1$$

This show's that

$\{s_n\}_{n=1}^{\infty}$ is bounded.

Monotone Sequences:

Defn:

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

If $s_1 \leq s_2 \leq \ldots\ldots \leq s_n \leq s_{n+1} \leq \ldots$ then $\{s_n\}_{n=1}^{\infty}$ its called nondecreasing.

Similarly, if $s_1 \geq s_2 \geq s_3 \geq \ldots \geq s_n \geq s_{n+1} \geq \ldots$

then $\{s_n\}_{n=1}^{\infty}$ is called nonincreasing.

Thus a monotone sequence is a sequence which is either nonincreasing or nondecreasing or both.

Example:

1) $\{1, 2, 3, \ldots\}$ is a non decreasing sequence.

2) $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is a nonincreasing sequence

3) Find whether the given sequence is nonincreasing or non decreasing.

a) $\left\{2 - \frac{1}{2^{n-1}}\right\}_{n=1}^{\infty}$ b) $\left\{2n + (-1)^n\right\}_{n=1}^{\infty}$

c = $\left\{(-1)^n n^2\right\}_{n=1}^{\infty}$

Ans:

a) $\left\{2 - \frac{1}{2^{n-1}}\right\}_{n=1}^{\infty}$

$$s_n = 2 - \frac{1}{2^{n-1}} = \frac{2.2^{n-1} - 1}{2^{n-1}} = \frac{2^n - 1}{2^{n-1}}$$

$$s_1 = \frac{2-1}{2^0} = 1 \qquad s_2 = \frac{2^2-1}{2^1} = \frac{3}{2} = 1.5$$

$$s_3 = \frac{2^3-1}{2^1} = \frac{3}{2} = 1.5 \qquad s_4 = \frac{2^4-1}{2^3} = \frac{15}{8} = 1.875$$

b) $\left\{2n + (-1)^n\right\}_{n=1}^{\infty}$

$$s_n = 2n + (-1)^n$$

$$s_1 = 2 + (-1) = 2 - 1 = 1$$

$$s_2 = 4 + (-1)^2 = 4 + 1 = 9$$

$$s_3 = 6 + (-)^3 = 6 - 1 = 5$$

$$s_4 = 8 + (-1)^4 = 8 + 1 = 9$$

$S_1 \leq S_2 \leq S_3 \leq S_4 \leq \ldots\ldots$

$\therefore \{2_n + (-1)^n\}_{n=1}^{\infty}$ is a non decreasing sequence

c) $\{(-1)^n \, n^2\}_{n=1}^{\infty}$

$S_n = (-1)^n \, n^2$

$S_1 = (-1)^1 \, 1^2 = -1$

$S_2 = (-1)^2 \, 2^2 = 4$

$S_3 = (-1)^3 \, 3^2 = -9$

$S_4 = (-1)^4 \, 4^2 = 16$

$S_1 < S_2 > S_3 < S_4$

$\therefore \{(-1)^n \, n^2\}_{n=1}^{\infty}$

is not a monotone sequence.

Theorem :-

A nondecreasing sequence which is bounded above is convergent.

Proof :-

Suppose $\{s_n\}_{n=1}^{\infty}$ is non decreasing and bounded above. Then the set $A = \{s_1, s_2, \ldots\ldots\}$ is a nonempty subset of R, which is bounded above.

There fore it has a lower upper bound (l.u.

let $M = \text{l.u.b}\{s_1, s_2 \ldots\ldots\} = \text{l.u.b of } A$

We have to prove $s_n \to M$ as $n \to \infty$

Given $\varepsilon > 0$ the number $M - \varepsilon$ is not an u.b of

Hence for some $N \in I$,

$S_N > M - \varepsilon$. But since

$\{S_n\}_{n=1}^{\infty}$ is non decreasing

$S_n > M - \varepsilon$, $(n \geq N) \rightarrow (1)$

Also since M is an U. b of A

$M \geq S_n$ $(n \in I) \rightarrow (2)$

From (1) & (2) we conclude $|S_n - M| < \varepsilon \ (n \geq N)$

$\therefore \{S_n\}_{n=1}^{\infty}$ is a convergent sequence

$\therefore$ Proved.

Note:

1) A non increasing sequence $\{S_n\}_{n=1}^{\infty}$ is always bounded above.

2) A non decreasing sequence $\{S_n\}_{n=1}^{\infty}$ is always bounded below.

Promblem:

1) Prove that the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}_{n=1}^{\infty}$ is Convergent.

Proof:

Let $S_n = \left(1+\frac{1}{n}\right)^n$

By Binomial Expansion,

$$(x+a)^n = x^n + n\,x^{n-1}a + \frac{n(n-1)}{1.2}x^{n-2}a^2 + \frac{n(n-1)(n-2)}{1.2.3}x^{n-3}a^3 + \dots + \frac{n(n-1)\dots\dots 1.2}{1.2.3,\dots\dots n}a^n$$

$$\therefore \left(1+\frac{1}{n}\right)^n = 1^n + n\,1^{n-1}\frac{1}{n} + \frac{n(n+1)}{1.2}1^{n-2}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{1.2.3}1^{n-3}\frac{1}{n^3} + \dots \frac{n(n-1)(n-2)\dots 2.1}{1,2,3\dots\dots n}\frac{1}{n^n}$$

Thus $(k+1)^{th}$ term is

$$\frac{n(n-1)(n-2)\ldots\ldots n-(k-1)}{1,2\ldots\ldots k}\;\frac{1}{n^k}$$

$$= \frac{1}{1.2\ldots\ldots k}\;1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\left(n-\frac{k-1}{n}\right)$$

If we expand for $\{S_{n+1}\}$

we get $(n+2)$ term's its $(k+1)^{th}$ term

is $$\frac{1}{1.2.3\ldots k}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\ldots\left(1-\frac{k-1}{n+1}\right) - ②$$

$$② \geq ①$$

Thus $S_n \leq S_{n+1}$ (ie) non decreasing sequence

Also $S_n < 1+1+\frac{1}{2}+\frac{1}{1,2,3}+\ldots+\frac{1}{1.2\ldots n}$

$$< 1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}$$

$$= 1+\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} < 1+\frac{1}{1-\frac{1}{2}} = 3$$

this $\{S_n\}_{n=1}^{\infty}$ is bounded above

By above theorem & since

$\{S_n\}_{n=1}^{\infty}$ is non decreasing and bounded

above $\{S_n\}_{n=1}^{\infty}$ is convergnent.

Operations on convergent sequences Theorem:-

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequences of real numbers, if $\lim_{n\to\infty} s_n = L$ and if $\lim_{n\to\infty} t_n = M$,

then $\lim_{n\to\infty} (s_n + t_n) = L+M$

ie) the limits of sum of two convergent sequences is the sum of the limit.

Proof:

Given $\varepsilon > 0$ we must find $N \in I$ such that

$|(s_n + t_n) - (L+M)| < \varepsilon \quad (n \geq N)$ — ①

let, $|s_n + t_n) - (L+M)| = |(s_n - L) + (t_n - M)|$

$\leq |s_n - L| + |t_n - M|$ — ②

Since $\lim_{n\to\infty} s_n = L$ and $\lim_{n\to\infty} t_n = M$

$|s_n - L| < \frac{\varepsilon}{2}$ and $|t_n - M| < \frac{\varepsilon}{2}$

$|s_n - L| + |t_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ $\frac{\cancel{2}\varepsilon}{\cancel{2}}$

$|s_n - L| + |t_n - M| < \varepsilon$

From ② $|(s_n + t_n) - (L+M)| < \varepsilon$ for $(n \geq N)$

where $N = \max(N_1, N_2)$

Thus $\lim_{n\to\infty} (s_n + t_n) = L+M$ (proved)

Theorem:

If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real number if $c \in R$ and if $\lim_{n\to\infty} s_n = L$ then $\lim_{n\to\infty} cs_n = cL$

Proof:

If $c = 0$, then theorem is obvious

i.e.) $\lim_{n\to\infty} (0 s_n) = 0$

If $c \neq 0$, Then we have to prove

$$|c s_n - cL| < \varepsilon \quad (n \geq N)$$

Since $\lim_{n\to\infty} s_n = L$

There exists $N \in I$ such that

$$|s_n - L| < \frac{\varepsilon}{|c|} \quad (n \geq N)$$

Then $|c|\, |s_n - L| < \varepsilon$

$$|c s_n - cL| < \varepsilon \text{ for } n \geq N$$

Thus $\lim_{n\to\infty} c s_n = cL$ (Proved)

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3. **Theorem**:

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequences of real numbers, if $\lim_{n\to\infty} s_n = L$ and if $\lim_{n\to\infty} t_n = M$ then $\lim_{n\to\infty} s_n t_n = LM$.

Proof: We know that $ab = \frac{1}{4}[(a+b)^2 + (a-b)^2]$ for $a, b \in R$

Also $\lim_{n\to\infty} (s_n + t_n) = L + M$.

& $\lim_{n\to\infty} (s_n + t_n)^2 = (L+M)^2$

& $\lim_{L\to\infty} (s_n - t_n) = L - M$

& $\lim_{L\to\infty} (s_n - t_n)^2 = (L-M)^2$

$\therefore \lim_{n\to\infty} [(s_n + t_n)^2 - (s_n - t_n)^2] = (L+M)^2 - (L-M)^2$
$= 4LM$.

Also $\lim_{n\to\infty} \frac{1}{4}\left[(s_n+t_n)^2-(s_n-t_n)^2\right] = \frac{1}{4} 4LM = LM$

$\therefore \lim_{n\to\infty} (s_n t_n)$ $\therefore$ proved //

Thearem:

a) If $0<x<1$, then $\{x^n\}_{n=1}^{\infty}$ convergent to 0.

b) If $1<x<\infty$, then $\{x^n\}_{n=1}^{\infty}$ diverge to infinity.

Proof:

a) If $0<x<1$ then $x^{n+1} = x.x^n < x^n$

hence $\{x^n\}_{n=1}^{\infty}$ is non increasing Also since $x^2>0$

for $n \in I$, $\{x_n\}_{n=1}^{\infty}$ is bounded below.

since $\{x_n\}_{n=1}^{\infty}$ for $0<x<1$ is nonincreasing and

bounded below.

It is a convergent sequence.

$\therefore \lim_{n\to\infty} x^n = 0$

To prove $\lim_{n\to 0} x^n = 0$

let $\lim_{n\to\infty} x^n = L$

Them $\lim_{n\to\infty} x^{n+1} = xL$

ie) seq $\{x^{n+1}\}_{n=1}^{\infty}$ convergent to xL

seq $\{x^{n+1}\}_{n=1}^{\infty}$ is a sus sequence of $\{x^n\}_{n=1}^{\infty}$

Thus $L = xL$

$L - xL = 0 \Rightarrow L(1-x) = 0$

$1-x \neq 0$ $\quad 0<x<1$

$\therefore L = 0$

$\lim_{n\to\infty} x^n = 0$ //

$\therefore$ proved.

b) Proof :-

If $x > 1$ then $x^{n+1} = x\,x^{n} > x^{n}$

Thus $\{x^{n+1}\}_{n=1}^{\infty}$ is a non decreasing sequence

To prove that :- $\{x^{n}\}_{n=1}^{\infty}$ is not bounded above

If $\{x^{n}\}_{n=1}^{\infty}$ is bounded above.

$\{x^{n}\}_{n=1}^{\infty}$ convergent to some limit L in R

Also $\{x^{n+1}\}_{n=1}^{\infty}$ coverges to some limit $x L$ in R

$L =$ in R

$L = xL \Rightarrow (1-x) L = 0$

$(1-x) \neq 0$

$\therefore L = 0$

$$\lim_{n \to \infty} x^{n} = 0$$

but $x^{n} \geq 1$

Which is a contradiction

$\therefore \{x^{n}\}$ is not bounded above

$\therefore \{x^{n}\}$ is diverges to infinity.

Theorem:

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequences of real numbers and $\lim_{n\to\infty} s_n = L$ and $\lim_{n\to\infty} t_n = M$

then $\lim_{n\to\infty} (s_n - t_n) = L - M$

Proof:

We know that $\lim_{n\to\infty} (s_n + t_n) = L + M$

and $\lim_{n\to\infty} c s_n = cL$

if $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are convergent sequences with Limit L and M.

Here $\lim_{n\to\infty} t_n = M$

then $\lim_{n\to\infty} (-1) t_n = (-1) M$

$\lim_{n\to\infty} (-t_n) = -M$

then $\lim_{n\to\infty} [s_n + (-t_n)] = L + (-M)$

$\therefore \lim_{n\to\infty} (s_n - t_n) = L - M$.

$\therefore$ proved//

Theorem:

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are convergent sequences of real numbers, if $s_n \le t_n$ $(n \in I)$ and if $\lim_{n\to\infty} s_n = L$, $\lim_{n\to\infty} t_n = M$, then $L \le M$

Proof:

We know that,

$$\lim_{n\to\infty} (t_n - s_n) = M - L$$

since $s_n \leq t_n \Rightarrow t_n - s_n \geq 0$

We have to prove that $L \leq M$ (ie) $M - L \geq 0$

since $t_n - s_n \geq 0$

$$\lim_{n\to\infty} (t_n - s_n) \geq 0 \Rightarrow M - L \geq 0$$

$$\Rightarrow M \geq L \text{ (or) } L \leq M.$$

$\therefore$ Proved.

Lemma:

If $\{s_n\}_{n=1}^{\infty}$ is a sequences of real numbers which convergent to L, then $\{s_n^2\}_{n=1}^{\infty}$ convergent to L^2

Proof:

We have to prove $\lim_{n\to\infty} s_n^2 = L^2$

(ie) $|s_n^2 - L^2| < \varepsilon \quad (n \geq N)$

(or) $|s_n - L|\,|s_n + L| < \varepsilon \quad (n \geq N)$

Now, since $\{s_n\}_{n=1}^{\infty}$ is convergent, it is bounded.

$\therefore |s_n| \leq M \quad (n \in I)$

Also, $|s_n + L| \leq |s_n| + |L| \leq M + |L| . \; (n \in I)$ — ①

Since $\lim_{n\to\infty} s_n = L$

$$|s_n - L| < \frac{\varepsilon}{M + |L|} \quad (n \geq N) \text{ — ②}$$

from ① & ②

$$|s_n - L|\,|s_n + L| < \frac{\varepsilon}{M + |L|} \times M + |L| \quad (n \in I) \;\&\; n \geq N)$$

$|s_n - L| \, |s_n + L| < \epsilon$

$|s_n^2 - L^2| < \epsilon \qquad (n \geq N)$

$\therefore \lim_{n \to \infty} s_n^2 = L^2$

$\therefore$ Proved /

Lemma:-

If $\{t_n\}_{n=1}^{\infty}$ is a sequence of real numbers, if $\lim_{n \to \infty} t_n = M$ where $M \neq 0$ then $\lim_{n \to \infty} \frac{1}{t_n} = \frac{1}{M}$

Proof:-

Since $M \neq 0$

Either $M < 0$ (or) $M > 0$

Case 1:

If $M > 0$

We have to prove $\left| \frac{1}{t_n} - \frac{1}{M} \right| < \epsilon$ for $(n \geq N)$

ie) $\left| \frac{M - t_n}{t_n M} \right| < \epsilon$

(or) $\left| \frac{t_n - M}{t_n M} \right| < \epsilon$ ie) $\frac{|t_n - M|}{|t_n M|} < \epsilon$

Since $\{t_n\}$ is a convergent sequence with limit M

$|t_n - M| < \epsilon$, for $(n \geq N)$

if $\epsilon = \frac{M}{2}$

$|t_n - M| < \frac{M}{2}$

ie) $t_n > \frac{M}{2} \quad (n \geq N_1)$

Also, there exists $N_2 \in I$ such that,

$|t_n - M| < \frac{M^2 \epsilon}{2} \quad (n \geq N_2)$

If $N = \max(N_1, N_2)$ then for $n \geq N$.

$$\frac{|t_n - M|}{|t_n M|} = \frac{1}{|t_n M|} \cdot |t_n - M| < \frac{1}{M^2/2} \cdot \frac{M^2 E}{2} = E$$

Thus $\left|\frac{1}{t_n} - \frac{1}{M}\right| < E \quad (n \geq N)$

$\therefore \lim_{n\to\infty} \frac{1}{t_n} = \frac{1}{M}$ for $m \neq 0$.

Thearem :

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}^{\infty}$ are sequences of real number, if $\lim_{n\to\infty} s_n = L$ and if $\lim_{n\to\infty} t_n = M$

Where $M \neq 0$ than $\lim_{n\to\infty} \frac{s_n}{t_n} = \frac{L}{M}$.

Proof :

since $\{t_n\}_{n=1}^{\infty}$ is a convergent seq.

and $\lim_{n\to\infty} t_n = M$

$\Rightarrow \lim_{n\to\infty} \frac{1}{t_n} = \frac{1}{M} \quad (M \neq 0)$

Also $\lim_{n\to\infty} s_n = L$

$\therefore \lim_{n\to\infty} s_n \cdot \frac{1}{t_n} = L \cdot \frac{1}{M}$

$\lim_{n\to\infty} \frac{s_n}{t_n} = \frac{L}{M}$

Proved //

Problem's :

1) Find $\lim_{n\to\infty} \frac{2n}{n + 4n^{1/2}}$

Answer :

$$\lim_{n\to\infty} \frac{2n}{n+4n^{1/2}} = \lim_{n\to\infty} \frac{2n}{n\left(1+\frac{4}{\sqrt{n}}\right)}$$

$$= \lim_{n\to\infty} \frac{2}{1+\frac{4}{\sqrt{n}}} = \frac{2}{1+\frac{4}{\infty}} = 2$$

2) Prove that $\lim_{n\to\infty} \frac{3n^2-6n}{5n^2+4} = \frac{3}{5}$

$$\lim_{n\to\infty} \frac{n^2(3-6/n)}{n^2(5-4/n)} = \lim_{n\to\infty} \frac{3-6/n}{5+4/n}$$

$$= \frac{3-6/\infty}{5+4/\infty}$$

$$= \frac{3-0}{5+0} = \frac{3}{5}$$

Proved //

Operation's on Divergent sequences,

Theorem :-

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequence of real numbers that diverge to infinity, then so do their sum and product.

(ie). $\{s_n + t_n\}_{n=1}^{\infty}$ and $\{s_n t_n\}_{n=1}^{\infty}$ diverge to infinity.

Proof :-

Given $M>0$ choose $N_1 \in I$ such that

$s_n > M \quad (n \geq N_1)$

and choose N_2 such that $t_n > 1 \quad (n \geq N_2)$

then for $N = \text{Max}(N_1, N_2)$

We have,

$$s_n + t_n > M + 1 > M \quad (n \geq N)$$

$$s_n t_n > M.1 = M \quad (n \geq N)$$

Thus $\{s_n + t_n\}_{n=1}^{\infty}$ and $\{s_n . t_n\}_{n=1}^{\infty}$ diverge to infinity.

Corollary :-

If $\{s_n\}_{n=1}^{\infty}$ diverges to infinity and if $\{t_n\}_{n=1}^{\infty}$ converges then $\{s_n + t_n\}_{n=1}^{\infty}$ diverges to infinity.

Theorem :-

If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequence of real numbers if $\{t_n\}_{n=1}^{\infty}$ is bounded then $\{s_n + t_n\}_{n=1}^{\infty}$ diverges to infinity.

Proof :-

Given $Q > 0$ such that

$$|t_n| \leq Q \quad (n \in I) \quad (\text{since } t_n \text{ is bounded}).$$

Given $M > 0$ choose $N \in I$ such that

$$s_n > M + Q \quad (n \geq N)$$

Then for $n \geq N$

$$s_n + t_n > s_n - |t_n| > M + Q - Q$$

$$s_n + t_n > M \quad (n \geq N)$$

$\therefore \{s_n + t_n\}_{n=1}^{\infty}$ divergent to infinity

$\therefore$ Proved/