Unit - II

Matrices

1. Definition Of Matrix

Matrix is an arrangement of elements (numbers) in rows and columns. The numbers are enclosed by parentheses or brackets or double bars.

For Example:

1. $\begin{pmatrix} 1 & 5 & 9 \\ 3 & 7 & 6 \\ 5 & 14 & 19 \end{pmatrix}$, 2. $\begin{bmatrix} 0 & 15 \\ -3 & 32 \\ 7 & 41 \end{bmatrix}$ 3. $\begin{Vmatrix} 13 & 16 \\ 23 & 50 \end{Vmatrix}$

are all matrices.

2. Importance:

The common operations of addition, multiplication, transposition, inversion, etc, are possible and simple in matrix algebra.

3. Notation:

Capital letters A, B, C, ... are used to denote matrices. Small letters a, b, c .... with two subscripts denote the elements.

The General form of a matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ a_{23} & a_{23} & \dots & a_{3j} & \dots & a_{3n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix} \quad -(1)$$

a_{ij} is the element at i th row and j th column and is usually referred as (i,j) the element.

4. Order of a Matrix

Order of a Matrix indicates the number of rows and the number of columns of a matrix.

The order of equation is 'm' by 'n' or 'm x n'.

$$A = \begin{bmatrix} 5 & 10 & 19 \\ 41 & 49 & 50 \end{bmatrix}$$

The above matrix A is of order 2 x 3.

5. Types of Matrices:

1) Square Matrix
2) Row Matrix
3) Column Matrix
4) Zero or Null Matrix
5) Equal Matrices
6) Equivalent Matrices
7) Diagonal Matrices
8) Scalar Matrix
9) Unit Matrix or Identity Matrix
10) Symmetric Matrix
11) Skew Symmetric Matrix
12) Triangular Matrix
13) Sub-matrix
14) Orthogonal Matrix
15) Non-Singular

1) Square Matrix:

When the number of rows and the number of columns of a matrix are equal, the matrix is called a Square matrix.

If,
The number of rows = the numbers of Columns $= n$
The matrix is called a square matrix of order n or an n-Square matrix.

For example,

$$A = \begin{bmatrix} 3 & 4 & 7 \\ 0 & 1 & 1 \\ 5 & 9 & 10 \end{bmatrix}$$ is a square matrix of order 3.

2) Row Matrix:

If there is only one row in a matrix, it is called a row matrix.

It is otherwise called as row vector.

Example: $A = [10, 32, 50]$

It is of order of 1×3

3. Column Matrix:

If there is only one column in a matrix, it is called a column matrix.

It is otherwise called as column vector.

Example: $A = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

It is of order 2×1

4. Zero or Null Matrix:

If all the elements of a matrix are zeros, it is called a zero matrix and it is denoted by O.

It is otherwise called as null matrix.

Example 1:

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_{2 \times 2}$$

Example 2:

$$O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{2 \times 3}$$

5. Equal Matrices:

Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are equal (that is $A = B$) if and only if

i) They have same order, that is $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$ and

ii) The elements at the corresponding places are equal, that is,

$a_{ij} = b_{ij}$ for every i and j

Example 1:

If $A = \begin{pmatrix} 2 & 4 & 6 \\ 5 & 7 & 9 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 4 & 6 \\ 5 & 7 & 9 \end{pmatrix}$

$A = B$

Example 2:

If $A = \begin{pmatrix} 1 & 6 & 9 \\ 3 & 10 & 14 \\ 9 & 13 & 15 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 6 & 9 \\ 3 & x & y \\ 9 & 12 & 15 \end{pmatrix}$

$x = 10$ $y = 14$

if $A = B$

6. Equivalent Matrices:

Two matrices A and B of the same order are said to be equivalent if one of them can be obtained from the other by elementary transformations

Written as $A \sim B$

Read as A equivalent to B.

7. Diagonal Matrix:

A square matrix all of whose elements except those in the principal or leading or main diagonal are zero is called a diagonal matrix.

Matrix $A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$

It is written also as

$$A = \text{diag.}(a_{11}\ a_{22} \ldots a_{nn})$$

ie, A is a square matrix such that

$a_{ij} = 0$ when $i \neq j$

Example, $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 10 \end{bmatrix}$

ie, $A = \text{diag}(5\ 6\ 10)$ is a diagonal matrix.

8. Scalar Matrix :

A diagonal matrix in which all the elements in the diagonal are equal is called a scalar matrix.

$A = \begin{bmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & a \end{bmatrix}$ is a scalar matrix

It is written also as
$$A = diag(a\ a\ a \ldots a).$$

A is a square matrix such that $a_{ij} = a$ when $i = j$ and $a_{ij} = 0$ when $i \neq j$.

It is to be seen later that

$A = aI$ where I is a unit matrix.

9. Unit Matrix or Identity Matrix:

A square matrix whose diagonal elements are 1 (unity) each and non diagonal elements are zeros is called a unit matrix or an identity matrix and is denoted by the alphabet.

A diagonal matrix in which all the elements in the principal diagonal are 1 each is called a unit Matrix. A unit matrix is a particular case of a scalar matrix. The order of the unit matrix may be written as the subscript of the alphabet I.

Hence, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

I is a square matrix such

ie. $a_{ij} = 1$ for all $i = j$

$a_{ij} = 0$ for all $i \neq j$

10. Symmetric Matrix:

A square matrix is called a symmetric matrix

$a_{ij} = a_{ji}$ for all i and j

That is, $A' = A$ where A' is the transpose of A and it has been defined under 'Matrix Operations I'.

Example:

$$A = \begin{bmatrix} 5 & 7 & 9 \\ 7 & 0 & -3 \\ 9 & -3 & -1 \end{bmatrix}$$

11. Skew Symmetric Matrix:

A square matrix such that $a_{ij} = -a_{ji}$ for all i and j is called a skew symmetric matrix.

That is $A' = -A$

Example:

$$A = \begin{bmatrix} 0 & 7 & 9 \\ -7 & 0 & -3 \\ -9 & 3 & 0 \end{bmatrix}$$

12. Triangular Matrix:

Triangular matrices are of two kinds, viz. upper triangular matrix and lower triangular matrix. A matrix $A = (a_{ij})_{m \times n}$ is an upper triangular matrix if $a_{ij} = 0$ for $i > j$.

A matrix $A = (a_{ij})_{m \times n}$ is a lower triangular matrix if $a_{ij} = 0$ for $i < j$.

$$A = \begin{bmatrix} 5 & 7 & 9 \\ 0 & 14 & 15 \\ 0 & 0 & 32 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 3 & 9 & 0 & 0 \\ 19 & 23 & 33 & 0 \end{bmatrix}$$

are upper and lower triangular matrices respectively. Triangular matrix need not be a square matrix.

13. Sub - Matrix :

A matrix obtained by deleting one or more rows or one or more columns (or both) is a sub-matrix of a given matrix. Any matrix is said to be a sub - matrix of itself.

Example : $A = \begin{pmatrix} 9 & 10 & 15 \\ 6 & 19 & 41 \end{pmatrix}$

$\begin{pmatrix} 9 & 10 & 15 \\ 6 & 19 & 41 \end{pmatrix}, \begin{pmatrix} 9 & 10 \\ 6 & 19 \end{pmatrix}, (9\ 10\ 15), \begin{pmatrix} 15 \\ 41 \end{pmatrix}$ and (41)

are a few sub-matrices of A.

14. Orthogonal Matrix.

A Square matrix A is said to be an orthogonal matrix if

$$A'A = AA' = 1$$

15. Non - Singular Matrix :

A square matrix A is said to be non-singular if $|A| \neq 0$.

$|A|$ denotes determinant A and the definition of a determinant is to be seen later.

A square matrix A is said to be singular if $|A| = 0$.

6. Matrix Operations - I

Addition, Subtraction, Scalar multiplication and transposition are consider under this section.

Inversion is considered after determinants.

1. (i) Addition :

$A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$

Sum, $A + B = (a_{ij} + b_{ij})_{m \times n}$.

Two matrices of the same order are said to be conformable for addition.

Two matrices can be added if they are only same order.

Elements at identical positions are to be added.

Example 1 : If $A = \begin{bmatrix} 4 & 6 & 9 \\ 3 & 5 & 10 \end{bmatrix}$ and

$B = \begin{bmatrix} 5 & 0 & 1 \\ 4 & -7 & -3 \end{bmatrix}$.

$$A + B = \begin{bmatrix} 4+5 & 6+0 & 9+1 \\ 3+4 & 5+(-7) & 10+(-3) \end{bmatrix} = \begin{bmatrix} 9 & 6 & 10 \\ 7 & -2 & 7 \end{bmatrix}$$

$$B + A = \begin{bmatrix} 5+4 & 0+6 & 1+9 \\ 4+3 & -7+5 & -3+10 \end{bmatrix} = \begin{bmatrix} 9 & 6 & 10 \\ 7 & -2 & 7 \end{bmatrix}$$

$$= A + B$$

ii) Subtraction : If $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$

$A - B = (a_{ij} - b_{ij})_{m\times n}$.

Two matrices of the same order are said to be conformable for subtraction.

Two matrices can be subtracted if they are only same order.

Example 2: If $A = \begin{bmatrix} 4 & 6 & 9 \\ 3 & 5 & 10 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 & 1 \\ 4 & -7 & -3 \end{bmatrix}$

$$A - B = \begin{bmatrix} 4-5 & 6-0 & 9-1 \\ 3-4 & 5-(-7) & 10-(-3) \end{bmatrix} = \begin{bmatrix} -1 & 6 & 8 \\ -1 & 12 & 13 \end{bmatrix}$$

$$B - A = \begin{bmatrix} 5-4 & 0-6 & 1-9 \\ 4-3 & -7-5 & -3-10 \end{bmatrix} = \begin{bmatrix} 1 & -6 & -8 \\ 1 & -12 & -13 \end{bmatrix} \neq A-B$$

Example 3 : If $A = \begin{bmatrix} 4 & -1 & 0 \\ -3 & 5 & -6 \\ 2 & -7 & 8 \end{bmatrix}$ and $B = \cancel{\begin{bmatrix} 5 & -1 & -1 \\ -8 & 7 & -8 \\ -1 & -11 & 5 \end{bmatrix}}$

Find A+B and A-B. $B = \begin{bmatrix} -1 & 0 & 1 \\ 5 & -2 & 2 \\ 3 & 4 & 3 \end{bmatrix}$

Solution :

$$A + B = \begin{bmatrix} 4+(-1) & -1+0 & 0+1 \\ -3+5 & 5+(-2) & -6+2 \\ 2+3 & -7+(4) & 8+3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 3 & -4 \\ 5 & -3 & 11 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 4-(-1) & -1-0 & 0-1 \\ -3-5 & 5-(-2) & -6-2 \\ 2-3 & -7-4 & 8-3 \end{bmatrix} = \begin{bmatrix} 5 & -1 & -1 \\ -8 & 7 & -8 \\ -1 & -11 & 5 \end{bmatrix}$$

2. Scalar multiplication:

Scalar is a real number in the context of matrix operations.

To get a scalar multiple of a matrix, every element of a matrix is to be multiplied by the scalar.

K is a scalar,

A is a matrix.

Product is KA which is a scalar multiple of A.

Example: $A = \begin{bmatrix} 0 & 4 \\ 2 & 5 \\ -7 & 9 \end{bmatrix}$, $3A = \begin{bmatrix} 0 & 12 \\ 6 & 15 \\ -21 & 27 \end{bmatrix}$,

$-4A = \begin{bmatrix} 0 & -16 \\ -8 & -20 \\ 28 & -36 \end{bmatrix}$

Example 4: $A = \begin{bmatrix} 2 & 3 & 5 \\ 4 & 7 & 9 \\ 1 & 6 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 & 2 \\ 4 & 2 & 5 \\ 6 & -2 & 7 \end{bmatrix}$

$5(A+B) = 5A + 5B$

Solution:

$$A+B = \begin{bmatrix} 2+3 & 3+1 & 5+2 \\ 4+4 & 7+2 & 9+5 \\ 1+6 & 6+(-2) & 4+7 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 7 \\ 8 & 9 & 14 \\ 7 & 4 & 11 \end{bmatrix}$$

$$5(A+B) = 5\begin{bmatrix} 5 & 4 & 7 \\ 8 & 9 & 14 \\ 7 & 4 & 11 \end{bmatrix} = \begin{bmatrix} 25 & 20 & 35 \\ 40 & 45 & 70 \\ 35 & 20 & 55 \end{bmatrix} \text{-①}$$

$$5A = 5\begin{bmatrix} 2 & 3 & 5 \\ 4 & 7 & 9 \\ 1 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 15 & 25 \\ 20 & 35 & 45 \\ 5 & 30 & 20 \end{bmatrix}$$

$$5B = 5\begin{bmatrix} 3 & 1 & 2 \\ 4 & 2 & 5 \\ 6 & -2 & 7 \end{bmatrix} = \begin{bmatrix} 15 & 5 & 10 \\ 20 & 10 & 25 \\ 30 & -10 & 35 \end{bmatrix}$$

$$5A+5B = \begin{bmatrix} 10+15 & 15+5 & 25+10 \\ 20+20 & 35+10 & 45+25 \\ 5+30 & 30+(-10) & 20+35 \end{bmatrix} = \begin{bmatrix} 25 & 20 & 35 \\ 40 & 45 & 70 \\ 35 & 20 & 55 \end{bmatrix} \quad -(2)$$

Hence from (1) and (2)

$$5(A+B) = 5A+5B$$

Example 5: $A = \begin{bmatrix} 3 & 5 \\ 2 & a \end{bmatrix}$, $B = \begin{bmatrix} 4 & b \\ 2 & 9 \end{bmatrix}$ and $C = \begin{bmatrix} 26 & a \\ 14 & 45 \end{bmatrix}$

Find a and b when $2A+5B = C$

Solution:

$$2A = 2\begin{bmatrix} 3 & 5 \\ 2 & a \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ 4 & 2a \end{bmatrix}$$

$$5B = 5\begin{bmatrix} 4 & b \\ 2 & 9 \end{bmatrix} = \begin{bmatrix} 20 & 5b \\ 10 & 45 \end{bmatrix}$$

$$2A+5B = \begin{bmatrix} 26 & 10+5b \\ 14 & 45+2a \end{bmatrix}$$

$2A+5B = C$ gives $\begin{bmatrix} 26 & 10+5b \\ 14 & 45+2a \end{bmatrix} = \begin{bmatrix} 26 & a \\ 14 & 45 \end{bmatrix}$

$\therefore 10+5b = a \to (1)$ and $45+2a = 45 \to (2)$

$\therefore$ From (2), $a = 0$ and (1), $b = -2$

Example 6: $A = \begin{pmatrix} 9 & 1 \\ 4 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 5 \\ 7 & 12 \end{pmatrix}$, find the matrix X such that $3A + 5B + 2X = 0$

Solution:

$$3A = 3\begin{pmatrix} 9 & 1 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 27 & 3 \\ 12 & 9 \end{pmatrix}$$

$$5B = 5\begin{pmatrix} 1 & 5 \\ 7 & 12 \end{pmatrix} = \begin{pmatrix} 5 & 25 \\ 35 & 60 \end{pmatrix}$$

$$\therefore X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \therefore 2X = \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix},$$

$$3A + 5B + 2X = 0$$

$$= \begin{pmatrix} 27 & 3 \\ 12 & 9 \end{pmatrix} + \begin{pmatrix} 5 & 25 \\ 35 & 60 \end{pmatrix} + \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix}$$

$$= \begin{pmatrix} 27+5 & 3+25 \\ 12+35 & 9+60 \end{pmatrix} + \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix} = \begin{pmatrix} 32 & 28 \\ 47 & 69 \end{pmatrix} + \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix}$$

$$= \begin{pmatrix} 32+2a & 28+2b \\ 47+2c & 69+2d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\therefore 32+2a = 0$; $28+2b = 0$; $47+2c = 0$;

$69+2d = 0$

$\therefore a = -16$; $b = -14$; $c = -23.5$; $d = -34.5$

$$\therefore X = \begin{pmatrix} -16.0 & -14.0 \\ -23.5 & -34.5 \end{pmatrix}$$

Note: X may also be determined as $X = -\frac{1}{2}(3A + 5B)$

3. Multiplication : $A=(a_{ij})_{m\times p}$ and $B=(b_{ij})_{n\times p}$ there product AB is a matrix $C=(c_{ij})_{m\times n}$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{ip}b_{pj} \text{ for all } i \text{ and } j$$

c_{ij} of C is the sum of the products of the pairs of elements of i th row of A and j th column of B.

The every element of rows of A and column of B are to be multiplied in pairs.

The product AB is defined or A is conformable to B. The product of A by itself is A^2. $\therefore A^2 = A.A$.

$$A^3 = A.A^2 = A^2.A.$$

Example 7: $A = \begin{pmatrix}3 & 5 & 6\end{pmatrix}_{1\times 3}$; $B = \begin{pmatrix}4\\1\\2\end{pmatrix}_{3\times 1}$

$$AB = \begin{pmatrix}3\times 4 + 5\times 1 + 6\times 2\end{pmatrix}_{1\times 1} = \begin{pmatrix}29\end{pmatrix}$$

Example 8: $A = \begin{pmatrix}2 & 3 & 5\end{pmatrix}_{1\times 3}$; $B = \begin{pmatrix}4 & 7\\0 & 1\\-6 & 9\end{pmatrix}_{3\times 2}$

$$\underset{1\times 2}{AB} = \begin{pmatrix}2\times 4 + 3\times 0 + 5\times(-6) & 2\times 7 + 3\times 1 + 5\times 9\end{pmatrix}$$

$$= \begin{pmatrix}-22 & 62\end{pmatrix}_{1\times 2}$$

Example 9: AB = B and BA = A. $A = \begin{bmatrix}4 & -2\\3 & -1\end{bmatrix}$

$$B = \begin{bmatrix}2 & 4\\3 & 6\end{bmatrix}$$

Solution : $AB = \begin{bmatrix}4\times 2 - 2\times 3 & 4\times 4 - 2\times 6\\3\times 2 - 1\times 3 & 3\times 4 - 1\times 6\end{bmatrix} = \begin{bmatrix}2 & 4\\3 & 6\end{bmatrix} = B$

$$BA = \begin{bmatrix}2\times 4 + 4\times 3 & 2\times(-2) + 4\times(-1)\\3\times 4 + 6\times 3 & 3\times(-2) + 6\times(-1)\end{bmatrix} = \begin{bmatrix}20 & -8\\30 & -12\end{bmatrix} \neq A$$

Example 10: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$; $B = \begin{bmatrix} -1 & -2 & -4 \\ -1 & -2 & -4 \\ 1 & 2 & 4 \end{bmatrix}$

Find AB.

Solution:

$$AB = \begin{bmatrix} 1\times-1+2\times-1+3\times1 & 1\times-2+2\times-2+3\times2 & 1\times-4+2\times-4+3\times4 \\ 2\times-1+4\times-1+6\times1 & 2\times-2+4\times-2+6\times2 & 2\times-4+4\times-4+6\times4 \\ 3\times-1+6\times-1+9\times1 & 3\times-2+6\times-2+9\times2 & 3\times-4+6\times-4+9\times4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} -1\times1-2\times2-4\times3 & -1\times2-2\times4-4\times6 & -1\times3-2\times6-4\times9 \\ -1\times1-2\times2-4\times3 & -1\times2-2\times4-4\times6 & -1\times3-2\times6-4\times9 \\ 1\times1\times2\times2+4\times3 & 1\times2\times2\times4+4\times6 & 1\times3+2\times6+4\times9 \end{bmatrix}$$

$$= \begin{bmatrix} -17 & -34 & -51 \\ -17 & -34 & -51 \\ 17 & 34 & 51 \end{bmatrix}$$

$BA \neq AB$

Another method

$$AB = \begin{bmatrix} -1-2+3 & -2-4+6 & -4-8+12 \\ -2-4+6 & -4-8+12 & -8-16+24 \\ -3-6+9 & -6-12-18 & -12-24+36 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 11:

Products AB and BA of two matrices A and B. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$, $B \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

Solution: $A_{1\times 4}$ and $B_{4\times 1}$

$$\underset{1\times 1}{AB} = \begin{bmatrix} 1\times 1 + 2\times 2 + 3\times 3 + 4\times 4 \end{bmatrix} = \begin{bmatrix} 30 \end{bmatrix}$$

$$\underset{4\times 4}{BA} = \begin{bmatrix} 1\times 1 & 1\times 2 & 1\times 3 & 1\times 4 \\ 2\times 1 & 2\times 2 & 2\times 3 & 2\times 4 \\ 3\times 1 & 3\times 2 & 3\times 3 & 3\times 4 \\ 4\times 1 & 4\times 2 & 4\times 3 & 4\times 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix}$$

Note: $\underset{1\times 4}{A} \; \underset{4\times 1}{B}$, there are 4 products in the element. In $\underset{4\times 1}{B} \; \underset{1\times 4}{A}$, there is one product in each element.

Example: 12. Find $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 2 & 0 \\ 1 & -1 & 3 \\ 7 & 2 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \\ 4 \end{bmatrix}$

Solution:

$$\underset{1\times 3}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} 4 & 2 & 0 \\ 1 & -1 & 3 \\ 7 & 2 & 1 \end{bmatrix}} \begin{bmatrix} 8 \\ 6 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2+21 & 2-2+6 & 0+6+3 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 27 & 6 & 9 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 216+36+36 \end{bmatrix}$$

$$= \begin{bmatrix} 288 \end{bmatrix}$$

Example 13:

	Single Band	Two Band	Three Band
Shop A	23	20	15
Shop B	40	10	8

i) The Initial stock ii) The order

iii) The supply iv) Final stock

v) Cost of individual items (column matrix)

vi) Total cost of stock in the shops.

Solution:

i) The initial stock matrix,

$$A = \begin{bmatrix} 23 & 20 & 15 \\ 40 & 10 & 8 \end{bmatrix}$$

ii) The order matrix, $B = \begin{bmatrix} 40 & 40 & 20 \\ 26 & 30 & 20 \end{bmatrix}$

iii) The supply matrix, $C = \frac{1}{2}$

$$B = \begin{bmatrix} 20 & 20 & 10 \\ 13 & 15 & 10 \end{bmatrix}$$

iv) The final stock matrix, $D = A + C$

$$= \begin{bmatrix} 43 & 40 & 25 \\ 53 & 25 & 18 \end{bmatrix}$$

v) The cost vector, $E = \begin{bmatrix} 100 \\ 220 \\ 300 \end{bmatrix}$

vi) The total cost of stock in the shops

$$F = DE = \begin{bmatrix} 43 & 40 & 25 \\ 53 & 25 & 18 \end{bmatrix} \begin{bmatrix} 100 \\ 220 \\ 300 \end{bmatrix} = \begin{bmatrix} 20600 \\ 16200 \end{bmatrix}$$

4. Transpose: Let A be a matrix of order $m \times n$. The transpose of A is denoted by A' or A^t (or even A^T). It is of order $n \times m$.

This process in turn gives the first column of A as the first row of A', the second column of A as the second row of A'.

Example 14: If $A = \begin{bmatrix} 1 & 5 & 9 \\ 10 & 14 & 19 \end{bmatrix}$, $A' = \begin{bmatrix} 1 & 10 \\ 5 & 14 \\ 9 & 19 \end{bmatrix}$

Note: $(A')' = A$

Example 15: Verify that $B^T A^T = (AB)^T$ when

$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$

$\underset{2\times 3}{B^T} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix}$, $\underset{3\times 2}{A^T} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$ and $\underset{2\times 2}{B^T A^T} = \begin{bmatrix} 1 & 4 \\ 4 & 4 \end{bmatrix}$ - (1)

$\underset{2\times 2}{AB} = \begin{bmatrix} 1 & 4 \\ 4 & 4 \end{bmatrix}$ and $(AB)^T = \begin{bmatrix} 1 & 4 \\ 4 & 4 \end{bmatrix}$ - (2)

From (1) and (2) $B^T A^T = (AB)^T$

The properties of transpose is also dealt with in the next subsection.

5. Properties:

a) Addition of Matrices.

i) Commutative:

If A and B are matrices of the same order.

$$A+B = B+A.$$

ii) Associative:

If A, B and C are matrices of the same order.

$$(A+B)+C = A+(B+C)$$

iii) Distributive with respect to scalar:

$$K(A+B) = KA + KB$$

iv) Existence of identity:

The null matrix O of the order of A is the additive identity of A.

$$A+O = A = O+A$$

v) Existence of inverse:

A is the additive inverse of A.

$$A+(-A) = O = (-A)+A$$

vi) Cancellation law:

If A, B and C are matrices of the same order.

$A+C = B+C$ implies $A=B$.

b) Multiplication of Matrices

i) Not Commutative : Two matrices A and B, AB and BA might have been defined but they not be equal.

$$AB \neq BA$$

ii) Associative : If ABC is defined

$$ABC = (AB)C = A(BC)$$

iii) Multiplication is distributive with respect to addition :

A, B and C are matrices of order $m \times n$, $n \times p$ and $n \times p$.

$$A(B+C) = AB + AC$$

or

$$(A+B)C = AC + BC$$

iv) Existence of identity :

If A is a square matrix of order n, I_n is the Identity matrix.

$$AI_n = A = I_n A$$

Rectangular matrix

$$AI_n = A \text{ and } I_m A = A$$

v) Existence of inverse :

A is a square matrix of order n and non singular, there exists a square matrix B of order n.

$$AB = I_n = BA$$

B is the inverse of A.

Note : 1. There exists no inverse for a singular matrix.

2. $B = A^{-1}$ & $A = B^{-1}$

vi) $AB = 0$ (Null matrix) does not imply $A = 0$ or $B = 0$

$$A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}, B = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = BA$$

Neither A nor B is a null matrix.

vii) If A is of order $m \times n$, the null matrix O of order $n \times m$

$$AO = O_{m \times n} \quad \text{and} \quad OA = O_{m \times n}$$

viii) $AB = AC$ does not imply $B = C$

c) Transpose :

i) The transpose of the transpose of a matrix is the original matrix itself.

$$(A')' = A$$

ii) The transpose of the sum of matrices is the sum of matrices is the sum of the transposes of the individual matrices

$$(A+B)' = A' + B'$$

iii) $(KA)' = KA'$

iv) $(AB)' = B'A', (ABC)' = C'B'A', \ldots$

2. If $A = \begin{bmatrix} 3 & 1 & 5 \\ 2 & 6 & 4 \\ 8 & 7 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 5 & 3 \\ 2 & 4 & 0 \end{bmatrix}$, Find $5A$, A' and $-2A+3B$.

Solution

$$5A = 5\begin{bmatrix} 3 & 1 & 5 \\ 2 & 6 & 4 \\ 8 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 15 & 5 & 25 \\ 10 & 30 & 20 \\ 40 & 35 & 45 \end{bmatrix}$$

$$A' = \begin{bmatrix} 3 & 2 & 8 \\ 1 & 6 & 7 \\ 5 & 4 & 9 \end{bmatrix}$$

$$-2A + 3B = -2\begin{bmatrix} 3 & 1 & 5 \\ 2 & 6 & 4 \\ 8 & 7 & 9 \end{bmatrix} + 3\begin{bmatrix} 1 & 2 & 4 \\ 1 & 5 & 3 \\ 2 & 4 & 0 \end{bmatrix}$$

$$= (-)\begin{bmatrix} +6 & +2 & +10 \\ +4 & +12 & +8 \\ +16 & +14 & +18 \end{bmatrix} + \begin{bmatrix} 3 & 6 & 8 \\ 3 & 15 & 9 \\ 6 & 12 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 4 & 2 \\ -1 & 3 & 1 \\ -10 & -2 & -18 \end{bmatrix}$$

21. Examine whether the matrix $A = \begin{bmatrix} 7 & 4 & 3 \\ 3 & 2 & 1 \\ 5 & 3 & 2 \end{bmatrix}$ is non-singular.

$$A = \begin{bmatrix} 7 & 4 & 3 \\ 3 & 2 & 1 \\ 5 & 3 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 7 & 4 & 3 \\ 3 & 2 & 1 \\ 5 & 3 & 2 \end{vmatrix}$$

$$= 7(4-3) - 4(6-5) + 3(9-10)$$

$$= 7 \times 1 - 4 \times 1 + 3 \times -1$$

$$= 7 - 4 - 3$$

$$= 0$$

$$|A| = 0$$

Given matrix is singular matrix.