Dr. P.K. SIVAKUMARAN, ASSISTANT PROFESSOR OF STATISTICS
PROBABILITY AND STATISTICS-20MCA25C

Mathematical Expectation:

Expectation is a very basic concept and is employed widely in decision theory, management science, system analysis, theory of games and many other fields. Some of these applications will be discussed in the chapter on Decision Theory.

The expected value or mathematical expectation of a random variable X is the weighted average of the values that X can assume with probabilities of its various values as weights.

Thus the expected value of a random variable is obtained by considering the various values that the variable can take multiplying these by their corresponding probabilities and summing these products. Expectation of X is denoted by E(X)

Expectation of a discrete random variable:

Let X be a discrete random variable which can assume any of the values of x_1, x_2, x_3 .... x_n with respective probabilities p_1, p_2, p_3......p_n. Then the mathematical expectation of X is given by

$$E(x) = x_1p_1 + x_2p_2 + x_3p_3 + \dots \dots\dots x_np_n$$
$$= \sum_{i=1}^{n} x_ip_i, \quad \text{where } \sum_{i=1}^{n} p_i = 1$$

Note:

Mathematical expectation of a random variable is also known as its arithmetic mean. We shall give some useful theorems on expectation without proof.

Theorems on Expectation:

1. For two random variable X and Y if E(X) and E(Y) exist, E(X + Y) = E(X) + E(Y). This is known as addition theorem on expectation.
2. For two independent random variable X and Y, E(XY) = E(X).E(Y) provided all expectation exist. This is known as multiplication theorem on expectation.
3. The expectation of a constant is the constant it self. ie E(C) = C
4. E(cX) = cE(X)
5. E (aX + b) = aE(X) +b
6. Variance of constant is zero. ie Var(c) = 0
7. Var (X + c) = Var X

   Note: This theorem gives that variance is independent of change of origin.

8. $\text{Var}(aX) = a^2 \text{var}(X)$

   Note: This theorem gives that change of scale affects the variance.

9. $\text{Var}(aX + b) = a^2 \text{Var}(X)$

10. $\text{Var}(b - ax) = a^2 \text{Var}(x)$

Definition:

Let f (x) be a function of random variable X. Then expectation of f (x) is given by $E(f(x)) = \sum f(x) P(X = x)$, where $P(X = x)$ is the probability function of x.

Particular cases:

1. If we take $f(x) = X^r$, then $E(X^r) = \sum x^r p(x)$ is defined as the **r^{th} moment about origin** or r^{th} raw moment of the probability distribution. It is denoted by μ'_r

   Thus $\mu'_r = E(X^r)$

   $\mu'_1 = E(X)$

   $\mu'_2 = E(X^2)$

   Hence mean $= \overline{X} = \mu'_1 = E(X)$

   $$\text{Variance} = \frac{\sum x^2}{N} - \left[\frac{\sum x}{N}\right]^2$$

$$= E(x^2) - (E(x))^2$$
$$= \mu'_2 - (\mu'_1)^2$$

Variance is denoted by μ_2

2. If we take $f(x) = (X - \overline{X})^r$ then $E(X - \overline{X})^r = \sum (X - \overline{X})^r p(x)$ which is μ_r, **the rth moment about mean or rth central moment**. In particular if r = 2, we get

$$\mu_2 = E(X - \overline{X})^2$$
$$= \sum (X - \overline{X})^2 p(X)$$
$$= E[X - E(X)]^2$$

These two formulae give the variance of probability distribution in terms of expectations.

Example :

Find the expected value of x, where x represents the outcome when a die is thrown.

Solution:

Here each of the outcome (ie., number) 1, 2, 3, 4, 5 and 6 occurs with probability $\frac{1}{6}$. Thus the probability distribution of X will be

x	1	2	3	4	5	6
P(x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Thus the expected value of X is

$$E(X) = \sum x_i p_i$$
$$= x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4 + x_5p_5 + x_6p_6$$
$$E(X) = \left[1 \times \frac{1}{6}\right] + \left[2 \times \frac{1}{6}\right] + \left[3 \times \frac{1}{6}\right] + \left[4 \times \frac{1}{6}\right] + \left[5 \times \frac{1}{6}\right] + \left[6 \times \frac{1}{6}\right]$$
$$= \frac{7}{2}$$
$$E(X) = 3.5$$

Remark:

In the games of chance, the expected value of the game is defined as the value of the game to the player.

The game is said to be favourable to the player if the expected value of the game is positive, and unfavourable, if value of the game is negative. The game is called a fair game if the expected value of the game is zero.

Example :

A player throws a fair die. If a prime number occurs he wins that number of rupees but if a non-prime number occurs he loses that number of rupees. Find the expected gain of the player and conclude.

Solution:

Here each of the six outcomes in throwing a die have been assigned certain amount of loss or gain. So to find the expected gain of the player, these assigned gains (loss is considered as negative gain) will be denoted as X.

These can be written as follows:

Outcome on a die	1	2	3	4	5	6
Associated gain to the outcome (x_i)	– 1	2	3	– 4	5	– 6
P (x_i)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Note that 2, 3 and 5 prime numbers now the expected gain is

$$E(x) = \sum_{E=1}^{6} x\ p_i$$

$$= (-1)\left[\frac{1}{6}\right] + (2)\left[\frac{1}{6}\right] + (3)\left[\frac{1}{6}\right] + (-4)\left[\frac{1}{6}\right] + (5)\left[\frac{1}{6}\right] + (-6)\left[\frac{1}{6}\right]$$

$$= -\left[\frac{1}{6}\right]$$

Since the expected value of the game is negative, the game is unfavourable to the player.

Example :

An urn contains 7 white and 3 red balls. Two balls are drawn together at random from the urn. Find the expected number of white balls drawn.

Solution:

From the urn containing 7 white and 3 red balls, two balls can be drawn in $10C_2$ ways. Let X denote the number of white balls drawn, X can take the values 0, 1 and 2.

The probability distribution of X is obtained as follows:

P(0) = Probability that neither of two balls is white.

= Probability that both balls drawn are red.

$$= \frac{3C_2}{10C_2} = \frac{3 \times 2}{10 \times 9} = \frac{1}{15}$$

P(1) = Probability of getting 1 white and 1 red ball.

$$= \frac{7C_1 \times 3C_1}{10C_2} = \frac{7 \times 3 \times 2}{10 \times 9} = \frac{7}{15}$$

P(2) = Probability of getting two white balls

$$= \frac{7C_2}{10C_2} = \frac{7 \times 6}{10 \times 9} = \frac{7}{15}$$

Hence expected number of white balls drawn is

$$E(x) = \Sigma x_i p(x_i) = \left[0 \times \frac{1}{15}\right] + \left[1 \times \frac{7}{15}\right] + \left[2 \times \frac{7}{15}\right]$$

$$= \frac{7}{5} = 1.4$$

Example :

A dealer in television sets estimates from his past experience the probabilities of his selling television sets in a day is given below. Find the expected number of sales in a day.

Number of TV sold in a day	0	1	2	3	4	5	6
Probability	0.02	0.10	0.21	0.32	0.20	0.09	0.06

Solution :

We observe that the number of television sets sold in a day is a random variable which can assume the values 0, 1, 2, 3, 4, 5, 6 with the respective probabilities given in the table.

Now the expectation of x = E(X) = $\Sigma x_i p_i$

$= x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4 + x_5p_5 + x_6p_6$

$= (0)\ (0.02) + (1)\ (0.010) + 2(0.21) + (3)\ (0.32) + 4(0.20)$

$+(5)\ (0.09) + (6)\ (0.06)$

E(X) = 3.09

The expected number of sales per day is 3

Example :

Let x be a discrete random variable with the following probability distribution

X	– 3	6	9
P (X = x)	1/6	1/2	1/3

Find the mean and variance.

Solution :

$$E(x) = \sum x_i\, p_i$$
$$= (-3)\left[\frac{1}{6}\right] + (6)\left[\frac{1}{2}\right] + (9)\left[\frac{1}{3}\right]$$
$$= \left[\frac{11}{2}\right]$$

$$E(x^2) = \sum x_i^2\, p_i$$
$$= (-3)^2\left[\frac{1}{6}\right] + (6)^2\left[\frac{1}{2}\right] + (9)^2\left[\frac{1}{3}\right] = \left[\frac{93}{2}\right]$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$
$$= \left[\frac{93}{2}\right] - \left[\frac{11}{2}\right]^2$$
$$= \left[\frac{93}{2}\right] - \left[\frac{121}{4}\right]$$
$$= \frac{186 - 121}{4}$$
$$= \frac{65}{4}$$

Expectation of a continuous random variable:

Let X be a continuous random variable with probability density function f(x), then the mathematical expectation of x is defined as

$$E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx, \text{ provided the integral exists.}$$

Remark:

If g(x) is function of a random variable and E[g(x)] exists,

$$\text{then } E[(g(x)] = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$$

Example :

Let X be a continuous random variable with p.d.f given by $f(x) = 4x^3$, $0 < x < 1$. Find the expected value of X.

Solution:

We know that $E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

In this problem $E(X) = \int_0^1 x\,(4x^3)\,dx$

$$= 4\int_0^1 x\,(x^3)\,dx$$

$$= 4\left[\frac{x^5}{5}\right]_0^1$$

$$= \frac{4}{5}\left[\ \ ^5\right]_0^1$$

$$= \frac{4}{5}[1^5 - 0^5]$$

$$= \frac{4}{5}[1]$$

$$= \frac{4}{5}$$

Example :

Let x be a continuous random variable with pdf. given by $f(x) = 3x^2,\ 0 < x < 1$ Find mean and variance

Solution :

$$E(x) = \int_{-\infty}^{\infty} xf(x)dx$$

$$E(x) = \int_0^1 x(3x^2)dx$$

$$= 3\int_0^1 (x^3)dx$$

$$= 3\left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{3}{4}\left[\ \ ^4\right]_0^1$$

$$= \frac{3}{4}[1^4 - 0]$$

$$= \frac{3}{4}$$

$$E(x)^2 = \int_{-\infty}^{\infty} x^2 f(x)dx$$

$$= \int_0^1 x^2(3x^2)dx$$

$$= \int_0^1 3(x^4)dx$$

$$= 3\left[\frac{x^5}{5}\right]_0^1$$

$$= \frac{3}{5}\left[\ {}^{5}\ \right]_0^1$$

$$= \frac{3}{5}[1^5 - 0]$$

$$= \frac{3}{5}$$

$$\text{Variance} = E(x^2) - [E(x)]^2$$

$$\text{Var}(x) = \frac{3}{5} - \left(\frac{3}{4}\right)^2$$

$$= \frac{3}{5} - \frac{9}{16}$$

$$= \frac{48-45}{80} = \frac{3}{80}$$

Moment generating function (M.G.F) (concepts only):

To find out the moments, the moment generating function is a good device. The moment generating function is a special form of mathematical expectation and is very useful in deriving the moments of a probability distribution.

Definition:

If X is a random variable, then the expected value of e^{tx} is known as the moment generating functions, provided the expected value exists for every value of t in an interval, $-h < t < h$, where h is some positive real value.

The moment generating function is denoted as $M_x(t)$

For discrete random variable

$$M_x(t) = E(e^{tx})$$

$$= \sum e^{tx} p(x)$$

$$= \sum \left(1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \ldots\ldots\ldots \right) p_x(x)$$

$$M_x(t) = \left(1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \ldots\ldots\ldots \right) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

In the above expression, the r^{th} raw moment is the coefficient of $\frac{t^r}{r!}$ in the above expanded sum. To find out the moments differentiate the moment generating function with respect to t once, twice, thrice…… and put $t = 0$ in the first, second, third, ….. derivatives toobtain the first, second, third,…… moments.

From the resulting expression, we get the raw moments about the origin. The centralmoments are obtained by using the relationship between raw moments and central moments.

Characteristic function:

The moment generating function does not exist for every distribution. Hence anotherfunction, which always exists for all the distributions is known as characteristic function.

It is the expected value of e^{itx}, where $i = \sqrt{-1}$ and t has a real value and the characteristic function of a random variable X is denoted by $\phi_x(t)$

For a discrete variable X having the probability function p(x), the characteristic function is $\phi_x(t) = \sum e^{itx}\, p(x)$

For a continuous variable X having density function f(x), such that $a < x < b$, the characteristic function $\phi_x(t) = \int^{b} e^{itx} f(x)\, dx$.

UNIT III

Mathematical Expectations

Expectations of a random variable

If $x_1, x_2 \dots x_n$ are the discrete random variables with their Probabilities $P(x_1)\ P(x_2) \dots P(x_n)$ then their mathematical expectation can be defined as

$$E(x) = \sum x_i P(x_i)$$

For the continuous random variable x and its probability density function is $f(x)$ then their expectation is defined as

$$E(x) = \int x\, f(x)\, dx$$

Expectations of mean, variance and moments

Consider the random variable x and the mean is nothing but the expectation

$$\text{mean} = E(X)$$

$$\text{Variance} = E(X - E(X))^2$$

$$= E(X^2) - [E(X)]^2$$

moments :

The r^{th} raw moment is defined as

$$\mu_r' = E(X^r)$$

The r^{th} central moment is defined as

$$\mu_r = E[X - E(X)]^r$$

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Addition theorem on expectations
Statement:
The mathematical expectation of the sum of random variables is equal to the sum of the expectations provided all the expectation exists

Symbolically If $x, y, z, \ldots T$ are n random variables then

$$E(x+y+z+\cdots+T) = E(x)+E(y)+E(z)+\cdots+E(T)$$

If all the expectation exists

Proof:
Let us consider two random variables x and y. The random variables x assume the values $x_1, x_2 \ldots x_m$ and their respective probabilities are $p_1, p_2 \ldots p_m$ where

$$p_i = P[x = x_i] \text{ where } i = 1, 2, \ldots m$$

The random variable y assume the values $y_1, y_2 \ldots y_n$ and their respective probabilities are $p_1', p_2', \ldots p_n'$ where

$$p_j' = P[y = y_j] \text{ where } j = 1, 2, \ldots n$$

By the definition of expectations

$$\left.\begin{aligned} E(x) &= \Sigma x_i p_i \\ E(y) &= \Sigma y_j p_j' \end{aligned}\right\} \text{ — (1).}$$

Since any one of the values of m values of x_i can be associated with any n values of y_j where $i = 1, 2 \ldots m$ and $j = 1, 2 \ldots n$

By def

$$E(x+y) = \sum_i \sum_j (x_i + y_j) p_{ij}$$

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$$= \sum_i \sum_j x_i p_{ij} + \sum_i \sum_j y_j p_{ij}$$

$$= \sum_i x_i \sum_j p_{ij} + \sum_j y_j \sum_i p_{ij}$$

$$= \sum_i x_i p_i + \sum_j y_j p_j$$

where $p_i = \sum_{j=1}^{m} p_{ij}$; $p_j = \sum_{i=1}^{n} p_{ij}$

hence

$$E(X+Y) = E(X) + E(Y) \quad \text{——— (2)}$$

Now let $K = X+Y$.

$$E(K+Z) = E(K) + E(Z) \quad \text{from (2)}$$
$$= E(X+Y) + E(Z)$$
$$= E(X) + E(Y) + E(Z)$$

hence by mathematical induction

$$E(X+Y+Z+\cdots T) = E(X) + E(Y) + \cdots + E(Z)$$

Multiplication theorem on expectation

Statement:

The mathematical expectation of the product of no of independent random variable is equal to the product of their expectation. Symbolically if $X, Y, Z \ldots T$ are independent random variable then

$$E(X, Y, Z \ldots T) = E(X)\, E(Y) \ldots E(Z)$$

Proof:

Let us prove the theorem for two random variables X, Y

Let the random variables x assume the values $x_1, x_2 \dots x_m$ with their respective probabilities $p_1, p_2, \dots p_m$ where $p_i = p[X = x_i]$ $i = 1, 2 \dots m$

The random variable y assume the values $y_1 y_2 \dots y_n$ with their respective probabilities $p_1' p_2', \dots p_n'$ where
$p_j' = p[y = y_j]$ $j = 1, 2 \dots n$

Then by the definition of expectation

$$\left.\begin{aligned} E(x) &= \Sigma x_i p_i \\ E(y) &= \Sigma y_j p_j' \end{aligned}\right\} \text{——} (1)$$

The product x, y is a random variable which can assume $m \times n$ values $x_i y_j$ ($i = 1, 2 \dots m$, $j = 1, 2 \dots n$)

$$P_{ij} = P[X = x_i] \cap P[y = y_j]$$
$$= p_i p_j' \quad \text{as } x, y \text{ are independent}$$

By Def

$$E(xy) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j p_{ij}$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j p_i p_j'$$
$$E(xy) = \sum_{i=1}^{m} x_i p_i \cdot \sum_{j=1}^{n} y_j p_j'$$

$$E(xy) = E(x) \; E(y) \text{ ——— } (2)$$

consider the 3 r.v's x, y, z
where $k = xy$
$E(kz) = E(k) \cdot E(y)$ from (2)

$= E(xy)\ E(z)$

$E(xyz) = E(x)\,E(y)\,E(z)$

By the method of mathematical induction the result holds for $x, y, z \ldots T$ also

So

$E(x, y, z \ldots T) = E(x)\,E(y) \ldots E(T)$

Addition theorem of Mathematical Expectation for continous r.v's

Statement:

If x and y are continous random variables with joint density function $f(x,y)$

Then

$E(x+y) = E(x) + E(y)$

Proof:

Given x and y are two random variables with joint probability density function $f(x,y)$

$$E(x+y) = \iint (x+y)\,f(x,y)\,dx,dy$$

$$= \int x\left[\int f(x,y)\,dy\right] dx + \int y\left[\int f(x,y)\,dx\right] dy$$

$$= \int x\,f(x)\,dx + \int y\,f(y)\,dy$$

as $f(x,y)\,dx = f(y)$

$f(x,y)\,dy = f(x)$

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$= E(x) + E(y)$

Multiplication theorem of Mathematical ~~expression~~ expectation for continous r.v's

Statement:

If x and y are continous random variables with joint density function f(x,y) Then

$E(xy) = E(x) \cdot E(y)$

Proof:

Given x and y are two independent random variables with joint density fn f(x,y) then

$E(xy) = \iint xy\, f(x,y)\, dx\, dy$

$= \iint xy\, f(x)\, f(y)\, dx\, dy$

since x and y are independent

$f(xy) = f(x)\, f(y)$

$= \int x\, f(x)\, dx \int y\, f(y)\, dy$

$E(xy) = E(x)\, E(y)$

Hence the proof.

Properties of Mathematical Expectation

1. If x is a r.v and a is constant then

i) $E[a\psi(x)] = a\, E[\psi(x)]$

ii) $E(\psi(x)+a] = E[\psi(x)] + a$

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where $\psi(x)$, a function of x is a random variable and all expectation exists

i) $E[a\psi(x)] = \int_{-\infty}^{\infty} a\,\psi(x) f(x)\,dx$

$= a\int_{-\infty}^{\infty} \psi(x) f(x)\,dx$

$= a\ E[\psi(x)]$

ii) $E[\psi(x)+a] = \int_{-\infty}^{\infty} [\psi(x)+a]\, f(x)\,dx$

$= \int_{-\infty}^{\infty} \psi(x) f(x)\,dx + a\int_{-\infty}^{\infty} f(x)\,dx$

$= E[\psi(x)] + a \int_{-\infty}^{\infty} f(x)\,d$

corollary: If $\psi(x) = x$ then

$E[ax] = a\,E(x)$ and
$E[x+a] = E(x) + a$

2. If x is a r.v. and a and b are constant then
$$E[ax+b] = a\,E(x) + b.$$
provided all the expectation exists

Proof:

By def

$E[ax+b] = \int_{-\infty}^{\infty} (ax+b)\, f(x)\,dx$

$= a\int_{-\infty}^{\infty} x\, f(x)\,dx + b\int_{-\infty}^{\infty} f(x)\,dx$

$= a\,E(x) + b(1)$ as $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$= a\,E(x)$

Properties of Variance.

1. If x is a r.v then $V(ax+b) = a^2 V(x)$ where a and b are constants

Proof: Let $y = ax+b$

Then

$$E(y) = aE(x) + b.$$

subtracting the above two

$$y - E(y) = ax + \cancel{b} - aE(x) - \cancel{b}$$

$$= a[x - E(x)]$$

squaring and taking expectations on Both sides

$$E[y - E(y)]^2 = a^2\,E[x - E(x)]^2$$

$$\Rightarrow V(y) = a^2 V(x)$$

or

$$V(ax+b) = a^2 V(x)$$

Covariance.

If x and y are two r-v's then co-variance between them is defined as

$$\mathrm{cov}(xy) = E\{[x - E(x)][y - E(y)]\}$$

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$$\mathrm{cov}(x,y) = E[(x - E(x))(y - E(y))]$$

$$= E[xy - xE(y) - yE(x) + E(x)E(y)]$$

$$= E[xy) - E(y)E(x) - \cancel{E(x)E(y)} + \cancel{E(x)E(y)}$$

$$= E(xy) - E(x)E(y)$$

$\therefore$

$$\mathrm{cov}(x,y) = E(xy) - E(x)E(y)$$

Expectation of a liniar combination of

Statement:

Let $x_1, x_2 \ldots x_n$ be n random variables and if $a_1, a_2, \ldots a_n$ be any constants then

$$E\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i E(x_i)$$

Proof: We know that

The mathematicl expectation of sum of n random variables is equal to the sum of their expectation

ie

$$E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n)$$

also $E(aX) = aE(X)$

The theorem is proved by the principle of mathematical induction for

Let us assume the theorem is true for $n = k$

$$y = a_1x_1 + a_2x_2 + \cdots + a_kx_k .$$

consider 2 variables x_1 and x_2 and constants x_1, x_2

$$y = a_1x_1 + a_2x_2$$

$$E(y) = E[a_1x_1 + a_2x_2)$$

$$= E[a_1x_1] + E(a_2x_2]$$

$$= a_1E(x_1) + a_2E(x_2)$$ it is true for $n=2$

now suppose it is true for $n = k$

$$y = a_1x_1 + a_2x_2 + \cdots + a_kx_k$$

Taking expectation

$$E(y) = a_1E(x_1) + a_2E(x_2) + \cdots + a_kE(x_k)$$

Then let $n = k+1$

$$y = a_1x_1 + a_2x_2 + \cdots + a_kx_k + a_{k+1}x_{k+1}$$

Taking expectation

$$E(y) = a_1E(x_1) + a_2E(x_2) + \cdots + a_{k+1}E(x_{k+1})$$

The theorem is true for $n=2$, $n=k$, $n=k+1$ hence it is true for positive values of n

$\therefore$

$$E\left(\sum_{i=1}^{n} ax_i\right) = \sum_{i=1}^{n} a_iE(x_i)$$

Variance of a linear combination of R.V's.

Statement:

Let $x_1, x_2 \dots x_n$ be n r.v's

$$V\left[\sum_{i=1}^{n} a_i x_i\right] = \sum_{i=1}^{n} a_i^2 V(x_i) + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \operatorname{cov}(x_i x_j)$$

Proof:

Let $U = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$

Take expectation

$$E(U) = a_1 E(x_1) + a_2 E(x_2) + \dots + a_n x_n$$

subtracting the above

$$U - E(U) = a_1 (x_1 - E(x_1)) + a_2 (x_2 - E(x_2)) + \dots + \dots + a_n (x_n - E(x_n))$$

squaring the above and taking expectation

$$E[U - E(U)]^2 = E[a_1 (x_1 - E(x_1)) + a_2 (x_2 - E(x_2)) + \dots + a_n (x_n - E(x_n))]^2$$

$$= a_1^2 E(x_1 - E(x_1))^2 + a_2^2 E(x_2 - E(x_2))^2 + \dots + a_n^2 E(x_n - E(x_n))^2 + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j E[x_i - E(x_i)][x_j - E(x_j)]$$

$$\Rightarrow V(U) = a_1^2 V(x_1) + a_2^2 V(x_2) + \dots a_n^2 V(x_n) + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \operatorname{cov}(x_i x_j)$$

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$$V\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i^2 V(x_i) + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \, cov(x_i x_j)$$

Remarks

1. If $a_i = 1 \qquad i = 1, 2 \ldots n$

$$V(x_1 + x_2 + \cdots + x_n) = V(x_1) + V(x_2) + \cdots + V(x_n) + 2 \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i<j}}^{n} cov(x_i x_j)$$

2. If x and y are independent

$$V(x_1 + x_2 \cdots + x_n) = V(x_1) + V(x_2) + \cdots + V(x_n)$$

3. If c is a constant.

$$E(c) = \int_{-\infty}^{\infty} c \, f(x) \, dx$$

$$= c \int_{-\infty}^{\infty} f(x) \, dx$$

$$= c$$

$\therefore \quad E(c) = c$

4 If x and y are 2 r.vs such that $y \leq x$ then

$$E(y) \leq E(x)$$

Proof:

Since $y \leq x \Rightarrow y - x \leq 0$

or $x - y \geq 0$

$$E(x-y) \geq 0 \Rightarrow E(x) - E(y) \geq 0$$

$$E(x) \geq E(y)$$

$$\Rightarrow E(y) \leq E(y)$$

Examples

1. Let x be discrete r.v with the following p.m.f

x :	-2	-1	0	1	2	3
$P(x)$:	0.1	K	0.2	$2K$	0.3	K

i) Determine K.

ii) Find mean and variance

i) $\Sigma P_i = 1$

$$\Rightarrow 0.1 + K + 0.2 + 2K + 0.3 + K = 1$$

$$4K + 0.6 = 1$$

$$4K = 1 - 0.6$$

$$K = 0.4/4$$

$$K = 0.1$$

Now

X :	-2	-1	0	1	2	3
$P(X)$:	0.1	0.1	0.2	0.2	0.3	0.1

Mean : $E(x)$

$$E(X) = \Sigma x P(x)$$

$$= (-2)(0.1) + (-1)(0.1) + 0 + 1(0.2) + (2)(0.3) + 3(0.1)$$

$$= -\cancel{0.2} - 0.1 + \cancel{0.2} + 0.6 + 0.3$$

$$= -0.1 + 0.9$$

$$= 0.8$$

<u>Variance</u> :

$$V(X) = E(X^2) - E(X)^2$$

X^2 :	4	1	0	1	4	9
$P(X)$:	0.1	0.1	0.2	0.2	0.3	0.1

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$E(x^2) = \Sigma x^2 P(x)$

$= (4)(0.1) + (1 \times 0.1) + (0 \times 0.2) + (1 \times 0.2) + (4 \times 0.3) + (9 \times 0.1)$

$= 2.8$

$V(X) = E(x^2) - E(x)^2$

$= 2.8 - (0.8)^2$

$= 2.8 - 0.64$

$= 2.16$

2. Given the following data

x :	-3	-2	-1	0	1	2	3
$P(x)$	0.05	0.10	0.30	0	0.30	0.15	0.10

compute

i) $E(X)$ ii) $E(2X \pm 3)$ iii) $E(4X+3)$

iv) $E(X^2)$ v) $V(X)$ vi) $V(2X \pm 3)$

$E(X) = (-3 \times 0.05) + (-2 \times 0.10) + (-1 \times 0.30) + 0 + (1 \times 0.30) + (2 \times 0.15) + (3 \times 0.10)$

$= -0.15 - 0.20 - 0.30 + 0.30 + 0.3 + 0.3$

$= 0.25$

$E(2X \pm 3) = E(2x+3)\ E(2X-3)$

$E(2X+3) = 2E(X) + 3$

$= 2(0.25 + 3)$

$= 3.5$

$E(2X-3) = 2.E(X) - 3$

$= 2(0.25) - 3$

$= 0.5 - 3 = \cancel{0.28}$

$= -2.5$

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$E(2x \pm 3) = 3.5, \ -2.5$

iii) $E(4x+5) = 4 \cdot E(x) + 5$

$= 4 \times 0.25 + 5$

$= 6$

iv) $E(x^2) = 9 \times 0.05 + 4 \times 0.10 + 1 \times 0.30 + 0 + 1 \times 0.30$

$+ 4 \times 0.15 + 9 \times 0.10$

$= 2.95$

v) $V(x) = E(x^2) - E(x)^2$

$= 2.95 - (0.25)^2$

$= 2.95 - 0.0625$

$= 2.8875$

vi) $V(2x \pm 3) = V(2x+3) \cdot V(2x-3)$

$V(2x+3) = 2^2 V(x) + 3$

$= 4 \times 2.8875 + 3$

$= 11.55 + 3$

$= 14.55$

$V(2x-3) = 2^2 V(x) - 3$

$= 4 \times 775 - 3$

$= 11.55 - 3$

$= 8.55$

vii) $\therefore V(2x \pm 3) = 8.55, \ 14.55$

3. Let x be r.v with the foll p.distn

X :	-3	6	9
P(X=x)	1/6	1/2	1/3

Find $E(x)$ $E(x^2)$ $E(2x+1)^2$ $V(x)$

4 For the foll p.d.f find $E(x)$ and $V(x)$

i) $f(x) = \frac{1}{2}(x+1) \qquad -1 \le x \le 1$

ii) $f(x) = y_0 (x - x^2) \qquad 0 \le x \le 1$

i) $f(x) = \frac{1}{2}(x+1)$

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{-1}^{1} \frac{1}{2}(x+1)\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} [(x+1)\, dx]$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + x\right]_{-1}^{1}$$

$$= \frac{1}{2}\left\{\left(\frac{1}{2}+1\right) - \left(\frac{1}{2} - 1\right)\right\}$$

$$= \frac{1}{2}\left[\frac{1}{2} + 1 - \frac{1}{2} + 1\right]$$

$$= \cancel{\tfrac{1}{2}} \times \cancel{2}$$

$$= 1$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} x^2 (x+1)\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} (x^3 + x^2)\, dx$$

$$= \frac{1}{2}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^{1}$$

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$$= \frac{1}{2}\left[\frac{1}{4}+\frac{1}{3}-\left(\frac{1}{4}-\frac{1}{3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{3}+\frac{1}{3}\right]$$

$$= 1/3$$

$$V(X) = E(X^2) - E(X)^2$$

$$= \frac{1}{3} - (1)^2$$

$$= \frac{1}{3} - 1$$

$$= -2/3$$

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UNIT V

Concept of bivariat Distribution

There is a possibility of assigning defining more than one random variable on the same sample space. eg. considering height and weight of every person. To describe such experiment mathematically we study two dimensional random variable

Definition

Let x and y be two random variables defined on the same sample space S then the function (X,Y) that assign a point in R^2 (= RxR) is called two-dimensional random variable.

Let (X,Y) be a two-dimensional random variable defined on the sample space S and $w \in S$. The value of (X,Y) at w is given by the pair of real numbers [X(w), Y(w)]

$$X(w) = (X_1, X_2, \dots X_n)$$

$$Y(w) = (Y_1, Y_2, \dots Y_n)$$

~~[illegible]~~ and the product set

$$X(w)\,Y(w) = (X_1, X_2 \dots X_n)\,(Y_1, Y_2 \dots Y_n)$$

into a probability space by defining the probability of the ordered pair (x_i, y_j) to be
$P(X = x_i, Y = y_j)$ which we write
$p(x_i, y_j)$

The function p on $X(S) \times Y(S)$ defined by
$p_{ij} = P(X = x_i \cap Y = y_j) = p(x_i, y_j)$
is called the joint probability function of X and Y is represented as

X \ Y	y_1	y_2	...	y_j	y_m	Total
x_1	p_{11}	p_{12}		p_{1j}	p_{1m}	$P_{1\cdot}$
x_2	p_{21}	p_{22}		p_{2j}	p_{2m}	$P_{2\cdot}$
⋮						
x_i	p_{i1}	p_{i2}		p_{i3}	p_{im}	$P_{i\cdot}$
⋮						
x_n	p_{n1}	p_{n2}		p_{n3}	p_{nm}	p_{ni}
Total	$p_{\cdot 1}$	$p_{2\cdot}$		$p_{\cdot 3}$	$p_{\cdot m}$	1

Joint probability mass function

If (X, Y) is a two dimensional discrete random variable, then joint discrete function of X, Y are also called the joint probability mass function of X, Y is denoted by p_{XY} and defined as

$P_{X,Y}(x_i, y_j) = P(X = x_i, Y = y_j)$ of (X, Y)

$P_{XY}(x_i, y_j) = 0$

Marginal probability Function:

Let (X, Y) be a discrete two-dimensional r.v which takes up countable no of values (x_i, y_j) then the prob distribution of X is determined as

$p_X(x_i) = P(X = x_i)$

$= P(X = x_i \cap Y = y_1) + P(X = x_i \cap Y = y_2) + \dots + P(X = x_i \cap Y = y_m)$

$= P_{i1} + P_{i2} + \dots + p_{im}$

$= \sum_{j=1}^{m} p_{ij}$

$= \sum_{j=1}^{m} (x_i, y_j)$

$= p_{i.}$

is known as marginal probability mass function or discrete marginal density function X.

also. $\sum_{i=1}^{n} p_{i.} = p_{1.} + p_{2.} + \dots + p_{n.}$

$= \sum_{i=1}^{n} \sum_{j=1}^{m} p_{ij} = 1$

similarly

$$P_y(y_j) = \sum_{i=1}^{n} p_{ij}$$

$$= \sum_{i=1}^{n} p(x_i, y_j)$$

$$= p_{\cdot j}$$

which is the probability mass function of y.

Conditional probability function.

Let (X, Y) be a discrete two-dimensional random variable. Then the conditional discrete density function or conditional probability mass function of X given Y = y denoted by

$$p_{X/Y}(x,y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

lly the conditional probability mass function $p_{Y/X}$) is similarly defined

as $$p_{Y/X}(y/x) = \frac{P(X = x, Y = y)}{P(X = x)}$$

Two dimensional distribution function

The distribution function of the two dimensional random variable (X, Y) is a real valued function F defined for all real x and y be

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the relation:

$$F_{XY}(x,y) = P(X \le x, Y \le y)$$

Marginal distribution function

From the joint distribution function $F_{XY}(x,y)$ it is possible to obtain the individual distribution functions $F_X(x)$ and $F_Y(y)$ which are termed as marginal distribution function of X and Y respectively with respect to the joint distribution function $F_{XY}(x,y)$

$$F_X(x) = P(X \le x) = P(X \le x, Y < \infty)$$

$$= \lim_{y \to \infty} F_{XY}(x,y)$$

$$= F_{XY}(x,\infty)$$

III^ly

$$F_Y(x) = P(Y \le y) = P(X < \infty, Y \le \infty)$$

$$= \lim_{x \to \infty} F_{XY}(x,y)$$

$$= F_{XY}(\infty, y)$$

$F_X(x)$ is termed as marginal distribution of X corresponding to the joint distribution function $F_{XY}(x,y)$

III^ly

$F_Y(y)$ is termed as marginal distribution of the random variable Y corresponding to joint distribution function $F_{XY}(x,y)$

Discrete r.v's :

$$F_X(x) = \sum_{y} P(X \leq x, Y = y)$$

$$F_Y(y) = \sum_{x} P(X = x, Y \leq y)$$

continous r.v's

$$F_X(x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx.$$

$$F_Y(y) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy.$$

condition probability density function

Let (X, Y) be two jointly distributed continous random variable with joint density function $f(x, y)$

The conditional density function of Y is defined as

$$f(x/y) = \frac{f(x, y)}{f(y)}$$

The conditional density function of X is given by

$$f(y/x) = \frac{f(x, y)}{f(x)}$$

Independent random variable :

Two random variable X and Y are said to be independent if

i) $P(X = x_i, Y = y_j) = P(X = x_i) \cdot P(Y = y_j)$
if x and y are discret

ii) $f(x, y) = f(x) \cdot f(y)$
if x and y are continious

Discrete random Variables

1. The joint probability distn of two random variables X and y is given by $P(X=0, Y=1) = 1/3$, $P(X=1, Y=-1) = 1/3$ and $P(X=1, Y=1) = 1/3$.
Find i) Marginal distributions of X and Y
ii) conditional probability distribution of X given $Y=1$

Solution :

X / Y	-1	0	1	Marginal Y
-1	0	0	$1/3$	$1/3$
0	0	0	0	0
1	0	$1/3$	$1/3$	$2/3$
Marginal X	0	$1/3$	$2/3$	1

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Marginal distribution of x is.

Values of x	-1	0	1
$P(X = x)$	0	1/3	2/3

Marginal distribution of y is.

Values of y	-1	0	1
$P(y = y)$	1/3	0	2/3

Conditional probability distribution of x given y is.

$$P(X = x / y = y) = \frac{P(X = x, y = y)}{P(y = y)}$$

$$P(X = -1 / y = 1) = \frac{P[(X = -1) \cap P(y = 1)]}{P(y = 1)}$$

$$= 0$$

$$P(X = 0 / y = 1) = \frac{P[X = 0] \cap P(y = 1)}{P(y = 1)}$$

$$= \frac{1/3}{2/3} = 1/2$$

$$P(X = 1 / y = 1) = \frac{P(X = 1] \cap P(y = 1)}{P(y = 1)}$$

$$= \frac{1/3}{2/3} = 1/2$$

2. The joint probability distribution of 2 r.v's are given below

$y \backslash x$	1	2	3	Total
1	0.1	0.1	0.2	0.4
2	0.2	0.3	0.1	0.6
Total	0.3	0.4	0.3	

i) Find the marginal prob of x, y.

ii) Find the conditional probabilities of x given y = 1

iii) Find the conditional probabilities of y given x = 2.

i.) Marginal prob of x

$P[X = x_i] = p_{i.} = \sum_j p_{ij}$

$P[X=1] = 0.1 + 0.2 = 0.3$

$P[X=2] = 0.1 + 0.3 = 0.4$

$P[X=3] = 0.2 + 0.1 = 0.3$

ii) Marginal prob of y

$P[Y = y_j] = p_{.j} = \sum_i p_{ij}$

$P[y=1] = 0.1 + 0.1 + 0.2 = 0.4$

$P[y=2] = 0.2 + 0.3 + 0.1 = 0.6$

2. conditional prob of y given $x = 2$

$$p(y_i / x=1) = \frac{p[y = y_i \cap x=2]}{p(x=2]}$$

$$p(y=1/x=1) = \frac{P(y=1 \cap x=2)}{P(x=2)}$$

$$= 0.1/0.4$$

$$= 0.25$$

$$p(y=2/x=1) = \frac{P(y=2 \cap x=1)}{P(x=1)}$$

$$= 0.3/0.4$$

$$= 0.75$$

3. The joint prob of two random variable are given below

	X	-1	0	1
y	0	1/15	2/15	1/15
	1	3/15	2/15	1/15
	2	2/15	1/15	2/15

1. Find the marginal prob of x and y

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2. Find the conditional prob of x given $y = 1$

3. Find the conditional prob of y given $x = 0$

i) Marginal prob of x

$P[X = x_i] = p_{i.} = \sum_j p_{ij}$

$P[X = -1] \quad \frac{1}{15} + \frac{3}{15} + \frac{2}{15}$

$= \frac{1+3+2}{15}$

$= 6/15$

$P[X = 0] = 2/15 + \frac{2}{15} + \frac{1}{15}$

$= \frac{2+2+1}{15}$

$= 5/15$

$P[X = 1] = 1/15 + 1/15 + 2/15$

$= 4/15$

Marginal prop of y

$P[Y = y_j] = p_{.j} = \sum_i p_{ij}$

$P[Y = 0] = \frac{1}{15} + \frac{2}{15} + \frac{1}{15}$

$= 4/15$

$p[y=1] = 3/15 + 2/15 + 1/15$

$= 6/15$

$p[y=2] = 2/15 + 1/15 + 2/15$

$= 5/15$

conditional prob of x given y = 1

$$p(x/y=1) = \frac{p(x = x_i \cap y = 1)}{p(y=1)}$$

$$p\left(\frac{x=-1}{y=1}\right) = \frac{p(x=-1 \cap y=1)}{p(y=1)}$$

$$= \frac{3/15}{6/15}$$

$$= 3/6 = 1/2$$

$$p(x=0/y=1) = \frac{p(x=0 \cap y=1)}{p(y=1)}$$

$$= \frac{2/15}{6/15} = 2/6 = 1/3$$

$$p(x=1/y=1) = \frac{p(x=1 \cap y=1)}{p(y=1)}$$

$$= \frac{2/15}{6/15} = 1/6$$

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conditional prob of y given $x=0$

$$p(y/x=0) = \frac{p(y=y_j \cap x=0)}{p(x=0)}$$

$$p(y=0/x=0) = \frac{p(y=0 \cap x=0)}{p(x=0)}$$

$$= \frac{2/15}{5/15}$$

$$= 2/5$$

$$p(y=1/x=0) = \frac{p(y=1 \cap x=0)}{p(x=0)}$$

$$= \frac{2/15}{5/15}$$

$$= 2/15$$

$$p(y=2/x=0) = \frac{p(y=1 \cap x=0)}{p(x=0)}$$

$$= \frac{1/15}{5/15}$$

$$= 1/5$$

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A two dimensional r.v (X, Y) have bivariate distribution given by

$$P(X=x, Y=y) = \frac{x^2+y}{32}$$

for $x = 0, 1, 2, 3$

$y = 0, 1$

Find the marginal distribution of x and y.

Solution:

y \ x	0	1	2	3	Marginal distn of y
0	0	$1/32$	$4/32$	$9/32$	$14/32$
1	$1/32$	$2/32$	$5/32$	$10/32$	$18/32$
Marginal distn of x	$1/32$	$3/32$	$9/32$	$19/32$	

Continuous Random Variable.

1. If x and y are two random variables having joint density function

$$f(x,y) = \begin{cases} \frac{1}{8}(6-x-y) & 0 \leq x < 2, \; 2 \leq y \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

Find

i) $P(X<1 \cap Y<3)$

ii) $P(X+Y<3)$

iii) $P(X<1 / Y<3)$

i) $P(X < 1 \cap Y < 3) = \int_{-\infty}^{1} \int_{-\infty}^{3} f(x,y)\, dx\, dy$

$= \int_{0}^{1} \int_{2}^{3} \frac{1}{8}(6 - x - y)\, dx\, dy$

$= 3/8$

ii) $P(X + Y < 3) = \int_{0}^{1} \int_{2}^{3-x} \frac{1}{8}(6 - x - y)\, dx\, dy$

$= 5/24$

iii) $P(X < 1 / Y < 3) = \frac{P(X < 1 \cap Y < 3)}{P(Y < 3)}$

$= \frac{3/8}{5/8}$

$= 3/5$

2. The joint probability density function of two random variables (X, Y) is given by

$$f(x,y) = \frac{9(1 + x + y)}{2(1+x)^4 (1+y)^4} \qquad \begin{array}{l} 0 < x < \infty \\ 0 < y < \infty \end{array}$$

1) Find the marginal density function of x and y.

2. Find the conditional density function of x and y

Solution:

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \frac{9}{2}\int_0^{\infty} \frac{(1+x+y)}{(1+x)^4 (1+y)^4}\,dy$$

$$= \frac{9}{2}\int_0^{\infty} \frac{(1+y)+x}{(1+x)^4 (1+y)^4}\,dy$$

$$= \frac{9}{2(1+x)^4}\int_0^{\infty} \frac{(1+y)}{(1+y)^4}\,dy + x\int_0^{\infty} \frac{1}{(1+y)^4}\,dy$$

$$= \frac{9}{2(1+x)^4}\left\{\int_0^{\infty}\left[\frac{1+y}{(1+y)^4}\right]dy + x\int_0^{\infty}\frac{1}{(1+y)^4}\,dy\right.$$

$$= \frac{9}{2(1+x)^4}\left[\frac{1}{-2(1+y)^2}\right]_0^{\infty} + \left[\frac{x}{-3(1+y}\right]_0^{\infty}$$

$$= \frac{9}{2(1+x)^4}\left[\frac{1}{2} + \frac{x}{3}\right]$$

$$= \frac{9}{2(1+x)^4}\,\frac{3+2x}{6}$$

$$f(x) = \frac{\cancel{9}\,3}{2(1+x)^4}\,\frac{3+2x}{\cancel{6}\,2}$$

$$= \frac{3}{4}\,\frac{3+2x}{(1+x)^4}$$

$$f(y) = \int f(x,y)\,dx$$

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$$= \frac{9}{2}\int_0^\infty \frac{(1+x+y)}{(1+x)^4\,(1+y)^4}\,dx$$

$$= \frac{9}{2(1+x)^4}\int_0^\infty \frac{(1+x)+y}{(1+x)^4\,(1+y)^4}\,dx$$

$$= \frac{9}{2(1+y)^4}\int_0^\infty \frac{1}{(1+x)^3}\,dx + y\int \frac{1}{(1+x)^4}\,dx$$

$$= \frac{9}{2(1+y)^4}\left[\frac{1}{-2(1+x)^3}\right]_0^\infty + \left[\frac{y}{-3(1+x)}\right]_0^\infty$$

$$= \frac{9}{2(1+y)^2}\left[\frac{1}{2}+\frac{y}{3}\right]$$

$$= \frac{\cancel{9}\,3}{2(1+y)^4}\;\frac{3+2y}{\cancel{6}\,2}$$

$$f(x) = \frac{3}{4}\;\frac{3+2y}{(1+y)^4}$$

$$f(x/y) = \frac{f(x,y)}{f(y)}$$

$$= \frac{9(1+x+y)\,/\,2(1+y)^4)}{3/4 \times \frac{3+2y}{(1+y)^4}}$$

$$= \frac{6\times(1+x+y)}{(1+x)^4\,(3+2y)}$$

$$f(y/x) = \frac{f(x,y)}{f(x)}$$

$$= \frac{9(1+x+y)}{2(1+x)^4 (1+y)^4}$$

$$= \frac{4(1+x)^4}{3(3+2x)}$$

$$= \frac{6(1+x+y)}{(1+x)^4 (3+2x)}$$