Inverse Laplace Transforms

Type 1

If $L(f(t))$ has $= F(s)$ then

$L(e^{-at} f(t)) = F(s+a)$

Hence $L^{-1}(F(s+a)) = e^{-at}(f(t))$.

$= e^{-at}\{L^{-1}(F(s))\}$.

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① Find $L^{-1}\left(\frac{1}{(s+a)^2}\right)$

Soln:

$L^{-1}\left(1/(s+a)^2\right) = e^{-at} L^{-1}(1/s^2)$

$= e^{-at}\, t$

$L(t) = 1/s^2$

$t = L^{-1}(1/s^2)$

Hence the soln.

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② find $L^{-1}\left(\frac{1}{(s+2)^2+16}\right)$

Soln:

$= e^{-2t} L^{-1}\left(\frac{1}{s^2+16}\right)$

$= e^{-2t} L^{-1}\left(\frac{1}{s^2+4^2}\right)$

$= \frac{e^{-2t}}{4} L^{-1}\left(\frac{4}{s^2+4^2}\right)$

$= \frac{e^{-2t}}{4} \sin 4t$

(3) find $L^{-1}\left(\frac{s-3}{(s-3)^2+4}\right)$

$a=3$

soln:

$= e^{3t} L^{-1}\left(\frac{s}{s^2+4}\right)$

$= e^{3t} L^{-1}\left(\frac{s}{s^2+2^2}\right)$

$= e^{3t}\cos 2t.$

Hence the soln.

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(4) find $L^{-1}\left(\frac{s}{s^2+2s+5}\right)$

soln:

$= L^{-1}\left(\frac{s}{(s+1)^2+4}\right)$

$= L^{-1}\left(\frac{s}{(s+1)^2+2^2}\right)$

$= L^{-1}\left(\frac{s+1-1}{(s+1)^2+2^2}\right)$

$= L^{-1}\left(\frac{s+1}{(s+1)^2+2^2}\right) - L^{-1}\left(\frac{1}{(s+1)^2+2^2}\right)$

$= e^{-t} L^{-1}\left(\frac{s}{s^2+2^2}\right) - \frac{e^{-t}}{2} L^{-1}\left(\frac{2}{s^2+2^2}\right)$

$= e^{-t}\cos 2t - \frac{e^{-t}}{2}\sin 2t.$

Hence the soln.

Result:

- If $L(f(t)) = F(s)$ Then $L(t f(t)) = -\frac{d}{ds}F(s)$

$= -F'(s)$

$t f(t) = -L^{-1} F'(s)$

$$\boxed{L^{-1}(F'(s)) = -t f(t) = -t\left[L^{-1}(F(s))\right]}$$

Problems.

① find $L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)$

Here $F'(s) = \frac{s}{(s^2+a^2)^2}$

$F(s) = \int \frac{s}{(s^2+a^2)^2} ds$

Take $s^2+a^2 = t$

$2s\,ds = dt$

$s\,ds = dt/2$

$s^2+4s+5 = t$

$(2s+4)\,ds = dt/2$

$(s+2)\,ds = \frac{dt}{2}$

$= \int \frac{\frac{dt}{2}}{t^2}$

$\Rightarrow \frac{\frac{dt}{2}}{t^2}$

$= \frac{1}{2}t^{-1}$

$= \frac{1}{2}\int t^{-2} dt$

$= \frac{1}{2}\left(\frac{t^{-1}}{-1}\right)$

$= -\frac{1}{2}\left(\frac{1}{t}\right)$

$= -\frac{1}{2}\left(\frac{1}{s^2+a^2}\right)$

$$F(s) = \frac{-1}{2(s^2+a^2)}$$

By using the formula,

$$L^{-1}(F'(s)) = -t L^{-1}(F(s)).$$

$$L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = -t L^{-1}\left(\frac{-1}{2(s^2+a^2)}\right)$$

$$= \frac{t}{2} L^{-1}\left(1/s^2+a^2\right).$$

$$= \frac{t}{2} \cdot 1/a \; L^{-1}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{t}{2a} \sin at,$$

Hence,

$$L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = t/2a \sin at$$

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(2) find $L^{-1}\left(\frac{s}{(s^2-1)^2}\right)$

$$F'(s) = \frac{s}{(s^2-1^2)^2}.$$

$$F(s) = \int \frac{s}{(s^2-1^2)^2} ds.$$

Take $s^2-1^2 = t$.

$$2s\,ds = dt$$

$$s\,ds = dt/2$$

③

$$= \int \frac{dt/2}{t^2}$$

$$= \frac{1}{2} \int t^{-2}\, dt$$

$$= \frac{1}{2} \left(\frac{t^{-1}}{-1} \right)$$

$$= -\frac{1}{2} \left(\frac{1}{t} \right)$$

$$= -\frac{1}{2} \left(\frac{1}{s^2-1^2} \right)$$

$$F(s) = \frac{-1}{2(s^2-1^2)}$$

By using the formula.

$$L^{-1}(F'(s)) = -t L^{-1}(F(s)).$$

$$L^{-1}\left(\frac{s}{(s^2-1)^2} \right) = -t L^{-1} \left(\frac{-1}{2(s^2-1^2)} \right)$$

$$= \frac{t}{2} L^{-1} \left(\frac{1}{s^2-1^2} \right)$$

$$= \frac{t}{2} \sinh t$$

Hence,

$$L^{-1}\left(\frac{s}{(s^2-1)^2} \right) = t/2 \sinh t.$$

Type

If $L(f(t)) = F(s)$ then $L(tf(t)) = -F'(s)$

① find $L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$

Soln:

Here $L(f(t)) = \log\left(\frac{s+1}{s-1}\right)$

$$L(tf(t)) = -\frac{d}{ds}\left(\log\left(\frac{s+1}{s-1}\right)\right)$$

$$= -\frac{d}{ds}\left(\log(s+1) - \log(s-1)\right)$$

$$= -\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$$

$$= -\frac{1}{s+1} + \frac{1}{s-1}$$

$$L(tf(t)) = \frac{1}{s-1} - \frac{1}{s+1}$$

$$t(f(t)) = L^{-1}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$$

$$= L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s+1}\right)$$

$$tf(t) = e^{t} - e^{-t} = 2\sinh t$$

Hence, $f(t) = \frac{2}{t}\sinh t$

Roughwork

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$2\sinh x = e^{x} - e^{-x}.$$

$$2\sinh t = e^{t} - e^{-t}$$

Result:

$$\mathcal{L}^{-1}(sF(s)) = F(t)$$
$$= \frac{d}{dt}\phi(t)$$
$$\boxed{= \frac{d}{dt}\mathcal{L}^{-1}(F(s)).}$$

① find $\mathcal{L}^{-1}\left(\frac{s}{s^2+k^2}\right)$

Soln:

$$\mathcal{L}^{-1}(sF(s)) = d/dt\left(\mathcal{L}^{-1}(F(s))\right).$$
$$= d/dt\ \mathcal{L}^{-1}\left(1/(s^2+k^2)\right)$$
$$= d/dt\left(\frac{\sin kt}{k}\right)$$
$$= 1/k\ d/dt(\sin kt)$$
$$= 1/\cancel{k}\left(\cos kt \cdot \cancel{k}\right)$$
$$= \cos kt$$

Hence the solution.

HW:

① Find $\mathcal{L}^{-1}\left(\frac{s}{(s+2)^2}\right)$

$$\mathcal{L}^{-1}(sF(s)) = d/dt\left(\mathcal{L}^{-1}(F(s))\right).$$
$$= d/dt\ \mathcal{L}^{-1}\left(1/(s+2)^2\right)$$
$$= d/dt\ e^{-2t}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = d/dt\ e^{-2k\,t}$$

2. Find $L^{-1}\left(\frac{1}{s(s+a)}\right)$.

$$= \int_0^t L^{-1}\left(\frac{1}{s+a}\right) dt$$

$$= \int_0^t e^{-at}\, dt$$

$$= \left[\frac{e^{-at}}{-a}\right]_0^t$$

$$= -\frac{1}{a}\left[e^{-at}\right]_0^t$$

$$= -\frac{1}{a}\left[e^{-at} - e^{-a(0)}\right] = -\frac{1}{a}\left[e^{-at} - 1\right] = \cancel{\frac{-e^{-at}}{a}}$$

$$= \frac{1}{a}\left[1 - e^{-at}\right]$$

3. Find $L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right)$

$$= L^{-1}\left(\frac{s}{s(s^2+a^2)^2}\right)$$

$$= \int_0^t L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) dt \longrightarrow$$ previous (sum) (Applied) Ans

$$= \int_0^t \frac{t}{2a} \sin at\, dt$$

$$= \frac{1}{2a} \int_0^t t \sin at\, dt$$

$u = t$ | $\int dv = \int \sin at\, dt$

$du = dt$ | $v = -\frac{\cos at}{a}$

$$= \frac{1}{2a}\left[-\frac{t\cos at}{a}\Big|_0^t + \int_0^t \frac{\cos at}{a}\, dt\right]$$

$$= \frac{1}{2a}\left[-\frac{t\cos at}{a} + \frac{\sin at}{a^2}\Big|_0^t\right]$$

$$= \frac{1}{2a}\left[-\frac{t\cos at}{a} + \frac{\sin at}{a^2} \right]$$

$$= -\frac{t\cos at}{2a^2} + \frac{\sin at}{2a^3} \,//.$$

Hence the soln :-

By using Inverse transformation to solve a problem for partial fractions

1. Find $L^{-1}\left\{ \frac{1}{s(s+1)(s+2)} \right\}$.

Given,

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(s+2)}$$

$$1 = A(s+1)(s+2) + B(s(s+2)) + C(s)(s+1)$$

put $s=0$,,

$$1 = A(1)(2)$$

$$\boxed{A = 1/2}.$$

put $s=-1$

$$1 = B(-1)(-1+2) = -B$$

$$\boxed{B = -1}$$

put $s=-2$,

$$1 = C(-2)(-2+1)$$

$$1 = 2C$$

$$C = 1/2$$

$$\frac{1}{s(s+1)(s+2)} = \frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2}$$

$$= \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}$$

$$L^{-1}\left(\frac{1}{s(s+1)(s+2)}\right) = L^{-1}\left(\frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}\right)$$

$$= L^{-1}\left(\tfrac{1}{2s}\right) - L^{-1}\left(\tfrac{1}{s+1}\right) + L^{-1}\left(\frac{1}{2(s+2)}\right)$$

$$= \frac{1}{2} L^{-1}(1/s) - e^{-t} L^{-1}(1/s) + \frac{1}{2}e^{-2t} L^{-1}(1/s)$$

$$= \frac{1}{2}(1) - e^{-t}(1) + \frac{1}{2}e^{-2t}(1)$$

$L(1) = 1/s$

$1 = L^{-1}(1/s)$

$$= \frac{1}{2} - e^{-t} + 1/2\, e^{-2t}$$

$$L^{-1}\left(\frac{1}{s(s+1)(s+2)}\right) = 1/2 - e^{-t} + 1/2\, e^{-2t}.$$

Hence the soln.

2) Find $L^{-1}\left(\frac{1}{(s+1)(s^2+2s+2)}\right)$

<u>Soln:</u>

Given $L^{-1}\left(\frac{1}{(s+1)(s^2+2s+2)}\right)$

$$\frac{1}{(s+1)(s^2+2s+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+2}.$$

$$\frac{1}{(s+1)(s^2+2s+2)} = \frac{A(s^2+2s+2) + (Bs+C)(s+1)}{(s+1)(s^2+2s+2)}$$

$1 = A(s^2+2s+2) + (Bs+C)(s+1)$ — ①

put $\boxed{s=-1}$

$1 = A(1-2+2) + (B+C)(0)$.

$1 = A(1) + 0$

$\boxed{A=1}$

put $\boxed{s=0}$.

$1 = A(0+0+2) + (0+C)(0+1)$

$1 = A(2) + C$

$1 = 1(2) + C$.

$1 = 2 + C$

$1 - 2 = C$

$\boxed{C=-1}$

$A = 1$, $C = -1$, in eq ①

put $\boxed{s=1}$

$1 = A(1+2+2) + (B+C)(2)$

$1 = A + 2A + 2A + 2B + 2C$.

$= 5A + 2B + 2C$.

$= 5(1) + 2B + 2(-1)$

$= 5(1) + 2B + -2$

$1 = 3 + 2B$

$2B = -2$

$B = -\cancel{2}/\cancel{2}$

$\boxed{B=-1}$

$$\therefore \frac{1}{(s+1)(s^2+2s+2)} = \frac{1}{s+1} + \frac{(-1)s+(-1)}{s^2+2s+2}$$

$$= \frac{1}{s+1} - \frac{s+1}{s^2+2s+2}$$

$$= \left(\frac{1}{s+1} - \frac{s+1}{(s+1)^2+1}\right)$$

$$L^{-1}\left(\frac{1}{(s+1)(s^2+2s+2)}\right) =$$

$$L^{-1}\left(\frac{1}{s+1} - \frac{s+1}{(s+1)^2+1}\right)$$

$$= L^{-1}(1/(s+1)) - L^{-1}\left(\frac{(s+1)}{(s+1)^2+1}\right)$$

$$= L^{-1}\left(e^{-t}(1/s)\right) - e^{-t} L^{-1}\left(\frac{s}{s^2+1}\right)$$

$$= e^{-t} L^{-1}(1/s) - e^{-t} L^{-1}\left(\frac{s}{s^2+1}\right)$$

$$= e^{-t}(1) - e^{-t}\cos t$$

$L(1) = 1/s$

$1 = L^{-1}(1/s)$

$$= e^{-t}(1 - \cos t).$$

3. Find $L^{-1}\left(\frac{1+2s}{(s+2)^2(s-1)^2}\right)$

$$\frac{1+2s}{\quad} = \frac{A}{(s+2)} + \frac{B}{(s+2)^2} + \frac{C}{(s-1)} + \frac{D}{(s-1)^2}.$$

11. $\dfrac{s^2}{(s^2+a^2)^2}$

12. $\dfrac{s}{(s^2+4)^2}$ (B.Sc. 19...

13. $\dfrac{1}{(s^2+9)^2}$.

14. $\log\left(\dfrac{1-s^2}{s^2}\right)$

15. $\log\left(\dfrac{1+s}{s}\right)$.

16. $\log\left(\dfrac{s}{s^2+1}\right)$.

17. $\dfrac{1}{(s^2+a^2)(s^2+b^2)}$

18. $\dfrac{3a^2}{s^3+a^3}$.

19. $\dfrac{s^2-a^2}{(s^2+a^2)^2}$

20. $\dfrac{1}{s^2(s^2+1)(s^2+9)}$

21. $\dfrac{1}{s^4-a^4}$.

22. $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$

(B.Sc. 19...

23. $\dfrac{s+1}{s^2+2s}$.

24. $\dfrac{1}{s(s+2)^3}$.

25. $\dfrac{s}{(s^2+4)(s^2+1)}$

26. $\dfrac{s^2-s+2}{s(s-3)(s+2)}$

27. $\dfrac{2s-1}{s^2(s-1)^2}$.

28. $\dfrac{1}{s(s^2-2s+5)}$

29. $\dfrac{s+3}{(s^2+6s+13)^2}$

30. $\dfrac{2(s+1)}{(s^2+2s+2)^2}$.

§ 8. Laplace transformation can be used to solve ordinary differential equations with constant coefficients.

The method is illustrated by means of examples.

Examples.

Ex.1. Solve the equation $\dfrac{d^2y}{dt^2}+2\dfrac{dy}{dt}-3y=\sin t$ given that

Substituting the values of $y(0)$ and $y'(0)$ in the equations

$$s^2\bar{y} + 2s\bar{y} - 3\bar{y} = \frac{1}{s^2+1}, \text{ where } \bar{y} = L(y).$$

$$(s^2 + 2s - 3)\,\bar{y} = \frac{1}{s^2+1}$$

$$\bar{y} = \frac{1}{(s^2+2s-3)(s^2+1)}$$

$$= \frac{1}{(s+3)(s-1)(s^2+1)}$$

$$\therefore\ y = L^{-1}\,\frac{1}{(s-1)(s+3)(s^2+1)}$$

On splitting into partial fractions, we get

$$y = L^{-1}\left(\frac{-\frac{1}{40}}{s+3} + \frac{\frac{1}{8}}{s-1} + \frac{-\frac{1}{10}s - \frac{1}{5}}{s^2+1}\right)$$

$$= -\frac{1}{40}L^{-1}\left(\frac{1}{s+3}\right) + \frac{1}{8}L^{-1}\left(\frac{1}{s-1}\right)$$

$$= -\frac{1}{40}e^{-3t} + \frac{1}{8}e^{t} - \frac{1}{10}\cos t - \frac{1}{5}\sin t.$$

Ex.2. Show the solution of the differential equa[tion]

$\frac{d^2y}{dt^2} + 4y = A\sin kt$ which is such that $y = 0$ and $\frac{dy}{dt}$ [= 0]

when $t = 0$ is

$$y = A\,\frac{\sin kt - \frac{k}{2}\sin 2t}{4 - k^2} \quad \text{if } k \neq 2.$$

If $k = 2$, show that $y = \frac{A(\sin 2t - 2t\cos 2t)}{8}$ (B.Sc. 198[illegible])

$$y'' + 4y = A\sin kt$$

$$L(y'') + 4L(y) = A\,L(\sin kt)$$

$$s^2\bar{y} - sy(0) - y'(0) + 4\bar{y} = A\,\frac{k}{s^2 + k^2}, \text{ where } L(y) = \bar{y}$$

Since $y(0) = 0$, $y'(0) = 0$, we have

$$(s^2 + 4)\,\bar{y} = A\,.\,\frac{k}{s^2 + k^2}.$$

$$\therefore\ \bar{y} = A\,.\,\frac{k}{(s^2 + 4)(s^2 + k^2)}$$

$$\therefore\ y = A\,k\,L^{-1}\,\frac{1}{(s^2 + 4)(s^2 + k^2)}$$

Case i. If $k \neq 2$,

$$y = A\,k\,L^{-1}\left(\frac{\frac{1}{s^2 + 4} - \frac{1}{s^2 + k^2}}{(k^2 - 4)}\right)$$

$$= \frac{A\,k}{k^2 - 4}\left\{L^{-1}\,\frac{1}{s^2 + 4} - L^{-1}\,.\,\frac{1}{s^2 + k^2}\right\}$$

$$= \frac{A k}{k^2 - 4} \left\{ \frac{\sin 2t}{2} - \frac{\sin kt}{k} \right\}$$

$$= \frac{A}{4 - k^2} \left(\sin kt - \frac{k}{2} \sin 2t \right)$$

Case ii. $k = 2$. Then

$$\bar{y} = 2 A L^{-1} \left\{ \frac{1}{(s^2 + 4)(s^2 + 4)} \right\}$$

$$= 2 A L^{-1} \left\{ \frac{1}{(s^2 + 2^2)^2} \right\}$$

$$= 2 A \cdot \frac{1}{2 (2)^3} (\sin 2t - 2t \cos 2t)$$

$$= \frac{A}{8} (\sin 2t - 2t \cos 2t).$$

Note:- The special advantage of this method in solving ...tial equations is that the initial conditions are satisfied auto... it is unnecessary to find the general solution and dete... constants using the initial conditions.

§ 9. The Laplace transform can also be used to solve s... differential equations.

Ex.1. Solve the simultaneous equations.