Fourier Series.

Definiton

A function $f(x)$ which has only a finite number of discontinousities can be expressed as a trigonometric series in given range $(c + c\ell)$ of x in the form.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}$$

where

$$a_0 = \frac{1}{\ell} \int_c^{c+2\ell} f(x)\, dx.$$

$$a_n = \frac{1}{\ell} \int_c^{c+2\ell} f(x) \cos \frac{n\pi x}{\ell}\, dx$$

$$b_n = \frac{1}{\ell} \int_c^{c+2\ell} f(x) \sin \frac{n\pi x}{\ell}\, dx.$$

Dirichlet's conditions :-

Fouries has shown that the expansions of $f(x)$ in the above from is possible. Only if it satisfies certain conditions. These conditions called Dirichlett's Conditions

Let $f(x)$ be defined in the interval $c < x < c + 2l$ with period $2l$ and satisfy the following conditions.

(i) $f(x)$ is single valued.

(ii) It has a finite number of discontinuities in a period 2π.

(iii) It has a finite number of maxima and minima in a given period.

① Obtain a fourier expansion for the function $f(x) = \frac{1}{2}(\pi - x)$, $0 < x < \pi$.

Soln :-

$2l = 2\pi$

$\Rightarrow l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{2}(\pi - x)\, dx$$

$$= \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi^2 - \frac{4\pi^2}{2} - 0\right] = 0$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx.$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x) \cos nx \, dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos 2n\pi}{n^2} + \frac{1}{n^2}\right]$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\sin nx\,dx.$$

$$= \frac{1}{2\pi}\left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-(-\pi)\frac{\cos 2n\pi}{n} + (\pi)\frac{\cos 0}{n}\right]$$

$$= \frac{1}{2\pi}\left[\frac{\pi}{n} + \frac{\pi}{n}\right] = \frac{1}{2\pi}\left(\frac{2\pi}{n}\right) = \frac{1}{n}$$

$$\boxed{b_n = 1/n}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= 0 + 0 + \sum_{n=1}^{\infty} \frac{1}{n}\sin nx.$$

$$= \frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots \infty$$

(2) If $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in $(0, 2\pi)$ show that $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$.

Soln:-

$$2l = 2\pi$$

$$\Rightarrow l = \pi$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{l} \int_c^{c+2l} f(x)\, dx.$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\pi^2 + x^2 - 2\pi x}{4}\right) dx$$

$$= \frac{1}{4\pi} \left[\pi^2 x + \frac{x^3}{3} - 2\pi \frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi} \left[2\pi^3 + \frac{8\pi^3}{3} - 4\pi^3 - 0\right]$$

$$= \frac{1}{4\pi} \pi^3 \left(2 + 8/3 - 4\right)$$

$$= \frac{\pi^2}{4} \left(8/3 - 2\right) = \frac{\pi^2}{4} (2/3)$$

$$\boxed{a_0 = \pi^2/6}$$

$$a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 \cos nx\, dx$$

$$= \frac{1}{4\pi} \int_0^{2\pi} (\pi^2 + x^2 - 2\pi x) \cos nx \, dx.$$

$$= \frac{1}{4\pi} \Bigg[(\pi^2 + x^2 - 2\pi x) \left(\cancel{\frac{+\sin nx}{n}} \right) -$$

$$(2x - 2\pi) \left(-\frac{\cos nx}{n^2} \right) + (2) \left(\cancel{\frac{-\sin nx}{n^3}} \right) \Bigg]_0^{2\pi}$$

$$= \frac{1}{4\pi} \left[2\pi + \frac{\cos 2n\pi}{n^2} + 2\pi \frac{\cos 0}{n^2} \right]$$

$$= \frac{1}{4\pi} \left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2} \right] = \frac{1}{4\pi} \left(\frac{4\pi}{n^2} \right)$$

$$\boxed{a_n = \frac{1}{n^2}}$$

$$b_n = \frac{1}{l} \int_c^{c+2l} f(x) \sin \frac{n\pi x}{l} \, dx.$$

$$= \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\pi - x}{2} \right)^2 \sin nx \, dx.$$

$$= \frac{1}{4\pi} \int_0^{2\pi} (\pi^2 + x^2 - 2\pi x) \sin nx \, dx.$$

$$= \frac{1}{4\pi} \Bigg[(\pi^2 + x^2 - 2\pi x) \left(-\frac{\cos nx}{n} \right) -$$

$$(2x - 2\pi) \left(\cancel{\frac{-\sin nx}{n^2}} \right) + (2) \left(\frac{\cos nx}{n^3} \right) \Bigg]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[-(\pi^2+4\pi^2-4\pi^2)\frac{\cos 2n\pi}{n}\right.$$

$$\left.+\pi^2\frac{\cos 0}{n}+2\frac{\cos 2n\pi}{n^3}-\frac{2\cos 0}{n^3}\right]$$

$$= \frac{1}{4\pi}\left[\frac{-\pi^2}{n}+\frac{\pi^2}{n}+\frac{2}{n^3}-2/n^3\right]$$

$$\boxed{b_n = 0.}$$

$$f(x) = \frac{(\pi^2/6)}{2} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx \text{ to}$$

$$= \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx \text{ ,, Proved}$$

(3) Determine fourier Series for $f(x) =$

$$\begin{cases} -x & 0 < x < \pi \\ \pi - x & \pi < x < 2\pi \end{cases}$$ Alo show that

$$\sum_{r=1}^{\infty}\frac{1}{(2r+1)^2} = \frac{\pi^2}{8}$$

Soln:-

Interval length $2l = 2\pi$

$l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

$$a_0 = \frac{1}{\pi} \left[\int_0^{\pi} -(x)\, dx + \int_{\pi}^{2\pi} (\pi - x)\, dx \right]$$

$$= \frac{1}{\pi} \left[-\left(\frac{x^2}{2}\right)_0^{\pi} + \left(\pi x - \frac{x^2}{2}\right)_{\pi}^{2\pi} \right]$$

$$= \frac{1}{\pi} \left[-\left(\frac{\pi^2}{2} - 0\right) + \left(2\pi^2 - \frac{4\pi^2}{2} - \pi^2 + \frac{\pi^2}{2}\right) \right]$$

$$= \frac{1}{\pi}\left(-\pi^2\right) = \frac{-\pi^2}{\pi} = -\pi$$

$$\boxed{a_0 = -\pi}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} f(x) \cos nx\, dx + \int_{\pi}^{2\pi} (\pi - x) \cos nx\, dx$$

$$\int u\, dv = uv - \int v\, du$$

$$\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots$$

$u = x$; $v = \cos nx$

$$a_n = \frac{1}{\pi}\left[-\left\{ x\left(\frac{\cancel{\sin nx}}{n}\right) - 1\left(-\frac{\cos nx}{n^2}\right)\right\}_0^{\pi}\right.$$

$$\left. + \left\{(\pi - x)\left(\frac{\cancel{\sin nx}}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[-\cancel{\frac{\cos n\pi}{n^2}} + \frac{\cos 0}{n^2} - \frac{\cos 2n\pi}{n^2} + \cancel{\frac{\cos n\pi}{n^2}}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{\cos 2n\pi}{n^2}\right]$$

Note:

$\sin 0 = 0$

$\sin nx = 0 \quad \forall n$ integer

$\cos 0 = 1 \;:\; \cos n\pi = (-1)^n \quad \forall n$ integer

$\cos \pi = -1$

$\cos 2\pi = 1$

$\cos 3\pi = -1$

$\cos \frac{n\pi}{2} = 0$ (If n is odd.

$$a_n = \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{1}{n^2}\right] = 0$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi}(-x)\sin nx\,dx + \int_{\pi}^{2\pi}(\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\left\{x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right\}_0^{\pi}\right.$$

$$\left. + \left\{(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi\frac{\cos n\pi}{n} - 0 - (-\pi)\frac{\cos 2n\pi}{n}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\cos n\pi + \frac{\pi}{n}\cos 2n\pi\right]$$

$$= \frac{1}{\pi}\times\frac{\pi}{n}\left[\cos n\pi + \cos 2n\pi\right]$$

$$= \frac{1}{n}\left[(-1)^n + 1\right]$$

$$\boxed{b_n = \frac{1}{n}\left[1 + (-1)^n\right]} = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 2/n & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \left(-\pi/2\right) + 0 \sum_{n=2,4,6\ldots}^{\infty} \frac{2}{n}\sin nx$$

$$f(x) = \frac{-\pi}{2} + 2\sum_{n=2,4,6\ldots}^{\infty} \frac{1}{n}\sin nx,$$

Deduce that

$$\sum_{r=1}^{\infty} \frac{1}{(2r+1)^2} = \pi^2/8$$

Taking $x = \pi/4$

LHS $= f(\pi/4) = -\pi/4$.

$$RHS = -\pi/2 + 2 \sum_{n=2,4,6}^{\infty} \frac{1}{n} \sin \frac{n\pi}{4}$$

$$= -\frac{\pi}{2} + 2\left[\frac{1}{2}\sin \pi/2 + \frac{1}{4}\cancel{\sin \pi} + \frac{1}{6}\sin\frac{3\pi}{2} + \cancel{\frac{1}{8}\sin 2\pi} + \frac{1}{10}\sin\frac{5\pi}{2} + \dots\right]$$

$$RHS = -\frac{\pi}{2} + 2\left[\frac{1}{2}\sin\frac{\pi}{2} + \frac{1}{6}\sin\frac{3\pi}{2} + \frac{1}{10}\sin\frac{5\pi}{2} + \dots\right]$$

$$= -\frac{\pi}{2} + 2\left[1/2 - 1/6 + 1/10 - 1/14 + \dots\right]$$

$$-\pi/4 = -\pi/2 + \left[1 - 1/3 + 1/5 - 1/7 + \dots\right]$$

$$1 - 1/3 - 1/5 - 1/7 + \dots \infty = -\pi/4 + \pi/2$$

$$= -\pi + 2\pi/4\pi$$

$$1 - 1/3 + 1/5 - 1/7 + \dots \infty = \pi/4$$

$$1 - 1/3 + 1/5 - 1/7 + \dots \infty = \frac{\pi}{4}.$$

Note :

even term $\longrightarrow$ $n = 2m$

Odd term $\longrightarrow$ $n = 2m+1$ for $m = 0$ to ∞

(or) $n = 2m-1$ for $m = 1$ to ∞

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)} = \pi/4$$

(4) If the Fourier Series of the function $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the interval $(0, 2\pi)$ is $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \infty = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \infty = \frac{\pi^2}{8}$.

Soln :-

$$f(x) = \frac{\pi^2}{12} \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

For $x = 0$ we have $\cos 0 = 1$

$$f(x=0) = \frac{f(0^-) + f(0^+)}{2}$$

$$= \left(-\pi/2\right)^2 + \left(\frac{-\pi}{2}\right)^2 \Big/ 2$$

$$= \frac{\frac{\pi^2}{4} + \frac{\pi^2}{4}}{2} = \frac{\frac{2\pi^2}{4}}{2} = \frac{\pi^2}{4}$$

$$LHS = \frac{\pi^2}{4}$$

$$RHS = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{3\pi^2 - \pi^2}{12} = \frac{2\pi^2}{16}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \infty = \frac{\pi^2}{6} \longrightarrow (1)$$

For $x = \pi$ continous point

$$LHS = F(\pi) = \left(\frac{\pi - x}{2}\right)^2 = 0$$

$$RHS = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}$$

$$= \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\therefore 0 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\frac{-\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\frac{-\pi^2}{12} = \left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots \infty\right]$$

$$\frac{\pi^2}{12} = \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - \cdots \infty\right]$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \cdots\cdots = \frac{\pi^2}{12} \longrightarrow (2).$$

Adding (1) & (2).

$$2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right] = \frac{\pi^2}{6} + \frac{\pi^2}{12}.$$

$$= \frac{2\pi^2 + \pi^2}{12} = \frac{3\pi^2}{12}$$

$$= \pi^2/4.$$

$$= \frac{\pi^2}{4} \times 2$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$$

(5) Obtain Fourier series for the function $f(x) = x \sin x, \quad 0 < x < 2\pi$

Soln:-

Let $2l = 2\pi$

$l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\, dx.$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \, dx$$

$$= \frac{1}{\pi}\Big[x(-\cos x) - 1(-\sin x)\Big]_0^{2\pi}$$

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 -$$

$$= \frac{1}{\pi}\left[-2\pi \cos 2\pi + 0\right]$$

$$= \frac{-2\pi}{\pi} = -2$$

$$\boxed{a_0 = -2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin x \cos nx \, dx$$

∵ Note!

$$\cos A \sin B = \frac{\sin(A+B) - \sin(A-B)}{2}$$

$$\Rightarrow \cos nx \sin x = \frac{1}{2}\left[\sin(nx+x) - \sin(nx-x)\right]$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x \frac{1}{2}\left[\sin(n+1)x - \sin(n-1)x\right] dx$$

$$a_n = \frac{1}{2\pi}\left[\int_0^{2\pi} x \sin(n+1)x \, dx - \int_0^{2\pi} x \sin(n-1)x \, dx\right]$$

$$= \frac{1}{2\pi}\left[x\left\{\left(\frac{-\cos(n+1)x}{n+1}\right) - 1\left(\cancel{\frac{\sin(n+1)x}{(n+1)^2}}\right)\right\}_0^{2\pi}\right.$$

$$\left. - \left\{ x\left(\frac{-\cos(n-1)x}{n-1}\right) - 1\left(\cancel{\frac{\sin(n-1)x}{(n-1)^2}}\right)\right\}_0^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[-2\pi\,\frac{\cos(n+1)2\pi}{n+1} - 0 + 2\pi\,\frac{\cos(n-1)2\pi}{n-1} - 0\right]$$

$$= \frac{2\pi}{2\pi}\left[-\frac{\cos(n+1)2\pi}{n+1} - \frac{\cos(n-1)2\pi}{n-1}\right]$$

$$= \left[\frac{-1}{n+1} + \frac{1}{n-1}\right] \quad \text{Since } (-1)^{\text{even}} = 1.$$

$$= \frac{-(n-1)+(n+1)}{(n+1)(n-1)}$$

$\cos n\pi = (-1)^n$

$\cos(n+1)2\pi = (-1)^{2(n+1)} = 1$

$\cos(n-1)2\pi = (-1)^{2(n-1)} = 1$

$$a_n = \frac{-n+1+n+1}{n^2-1}$$

$$\boxed{a_n = \frac{2}{n^2-1}} \; ; \text{ for } n \neq 1$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx.$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{\sin 2x}{2}\,dx \quad \left[\begin{array}{l}\text{Since} \\ \sin 2\theta = 2\sin\theta \\ \cos\theta\end{array}\right]$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \, dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-x\cos 2x}{2} + \frac{\cancel{\sin 2x}}{4}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-2\pi\cos 4\pi}{2} + 0\right]$$

$$a_1 = \frac{-\cos 4\pi}{2} = -\frac{1}{2}.$$

$$\boxed{a_1 = -\frac{1}{2}}$$

$$b_n = \frac{1}{n}\int_0^{2\pi} x \sin x \sin nx \, dx.$$

Formula

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\sin nx \sin x = \frac{1}{2}\left[\cos(n-1)x - \cos(n+1)x\right]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x \, \frac{1}{2}\left[\cos(n-1)x - \cos(n+1)x\right]dx$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\cos(n-1)x\,dx - \int_0^{2\pi} x\cos(n+1)x\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left\{x\left(\cancel{\frac{\sin(n-1)x}{n-1}}\right) - 1\left(\frac{-\cos(n-1)x}{(n-1)^2}\right)\right\}_0^{2\pi}\right.$$

$$\left. - \left\{x\left(\cancel{\frac{\sin(n+1)x}{n+1}}\right) - 1\left(\frac{-\cos(n+1)x}{(n+1)^2}\right)\right\}_0^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{\cos(n-1)2\pi}{(n-1)^2} - \frac{\cos 0}{(n-1)^2} - \frac{\cos(n+1)2\pi}{(n+1)^2} + \frac{\cos 0}{(n+1)^2}\right]$$

$$= \frac{1}{2\pi}\left[\cancel{\frac{1}{(n-1)^2}} - \cancel{\frac{1}{(n-1)^2}} - \cancel{\frac{1}{(n+1)^2}} + \cancel{\frac{1}{(n+1)^2}}\right]$$

(for $n \neq 1$)

$$\boxed{b_n = 0}$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x \sin x\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,dx$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\,dx - \int_0^{2\pi} x\cos 2x\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{x^2}{2}\right)_0^{2\pi} - \left[x\left(\frac{\sin 2x}{2}\right) - 1\left(\frac{-\cos 2x}{4}\right)\right]_0\right]$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 1 - 2\sin^2\theta$

$= 2\cos^2\theta - 1$

$1 - 2\sin^2\theta = \cos 2\theta$

$1 - \cos 2\theta = 2\sin^2$

$\frac{1-\cos 2\theta}{2} = \sin^2\theta$

$$b_n = \frac{1}{2\pi}\left[\frac{4\pi^2}{2} - 0 - \frac{\cos 4\pi}{4} + \frac{\cos 0}{4}\right]$$

$$= \frac{1}{2\pi}\left[2\pi^2 - \frac{1}{4} + \frac{1}{4}\right]$$

$$\boxed{b_1 = \pi}$$

$$\therefore f(x) = \frac{(-2)}{2} + \left(-\frac{1}{2}\right)\cos x + \sum_{n=2}^{\infty} \frac{2}{(n^2-1)}$$

$$\cos nx + \pi \sin x + \sum_{n=2}^{\infty} 0 \sin nx$$

$$f(x) = -1 - \frac{\cos x}{2} + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{(n^2-1)}$$

$\cos nx$

$$x \sin x = -1 - \frac{\cos x}{2} + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{(n^2-1)} \cos nx.$$

(6) Obtain fourier series for the function

$$f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \infty = \frac{\pi^2}{8}$

Soln:-

$$a_0 = \frac{1}{\pi} \int_0^{c+2l} f(x)\, dx$$

$2l = 2\pi$

$l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi} \int^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} (-\pi)\, dx + \int_{0}^{\pi} x\, dx \right]$$

$$a_0 = \frac{1}{\pi}\left[-\pi\left[x\right]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\pi(0+\pi) + \left(\frac{\pi^2}{2} - 0\right)\right]$$

$$= \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = \frac{1}{\pi}\left(\frac{-\pi^2}{2}\right)$$

$$\boxed{a_0 = \frac{-\pi}{2}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\, dx + \int_{0}^{\pi} x\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left(\frac{\sin nx}{n}\right) - 1\left(-\frac{\cos nx}{n}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

$$\boxed{a_n = \begin{cases} \dfrac{-2}{\pi n^2} & \text{If } n \text{ is odd} \\ 0 & \text{If } n \text{ is even} \end{cases}}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} (-\pi) \sin nx \, dx + \int_{0}^{\pi} x \sin nx \, dx \right]$$

$$= \frac{1}{\pi} \left[(-\pi) \left(\frac{-\cos nx}{n} \right)_{-\pi}^{0} + \left(x \left(\frac{-\cos nx}{n} \right) - 1 \left(\frac{-\sin nx}{n^2} \right) \right)_0^{\pi} \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi \cdot \cos 0}{n} - \frac{\pi \cos(-n\pi)}{n} - \frac{\pi \cos n\pi}{n} - 0 \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi}{n} - \frac{\pi \cos n\pi}{n} - \frac{\pi \cos n\pi}{n} \right]$$

$$= \frac{1}{\pi} \cdot \frac{\pi}{n} \left[1 - 2 \cos n\pi \right]$$

$$b_n = \frac{1}{n} \left(1 - 2 \cos n\pi \right)$$

$$= \frac{1}{n} \left(1 - 2 (-1)^n \right)$$

$$\boxed{b_n = \begin{cases} 3/n & \text{If } n \text{ is odd} \\ -1/n & \text{If } n \text{ is even} \end{cases}}$$

$$\therefore f(x) = \left(-\frac{\pi}{2} \right) + \sum_{n=1,3,5}^{\infty} \frac{-2}{\pi n^2} \cos nx + \sum_{n=1,3,5}^{\infty} \frac{3}{n} \sin nx + \sum_{n=2,4,6}^{\infty} \frac{1}{n} \sin nx.$$

$$f(x) = \frac{-\pi}{4} - \frac{2}{\pi} \sum_{n=1,3,5}^{\infty} \frac{\cos nx}{n^2} + 3 \sum_{n=1,3,5}^{\infty} \frac{\sin nx}{n}$$

$$- \sum_{n=2,4,6}^{\infty} \frac{\sin nx}{n}$$

If $x=0$ (discontinous)

$$f(0) = \frac{f(0^-) + f(0^+)}{2}$$

$$= \frac{-\pi + 0}{2} = -\pi/2$$

$$LHS = -\pi/2.$$

$$RHS = \frac{-\pi}{4} - \frac{2}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} + 0 + 0$$

$$\frac{-\pi}{2} = \frac{-\pi}{4} - \frac{2}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2}$$

$$\therefore \frac{\pi}{4} - \pi/2 = \frac{-2}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2}$$

$$\cancel{+}\frac{-\pi}{4} = \cancel{+} \frac{2}{\pi} \sum_{n=1,3,5}^{\infty} \frac{-1}{n^2}$$

$$\frac{\pi^2}{8} = \sum_{n=1,3,5}^{\infty} \frac{1}{n^2}$$

$$\frac{1}{1^2} + 1/3^2 + 1/5^2 \dots = \pi^2/8$$

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