UNIT : V

Partial Differential Equations

A partial differential equation is an equation which contains one or more partial derivatives.

The order of the partial differential equation is that of the derivative of highest order in the equation.

Let $z = f(x,y)$, where x and y are independent variables, and z is the depend variable on x and y.

Then $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are the first order partial derivatives;

$\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$ are the second order partial derivatives.

Ex.1: $x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = z^2$ is called the first order partial differential equation.

Ex.2: $x^2 \frac{\partial^2 z}{\partial x^2} + y^2 \frac{\partial^2 z}{\partial y^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} = 0$ is called the second order partial differential equation.

Notations

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q; \quad \frac{\partial^2 z}{\partial x^2} = r$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = s \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = t$$

Partial differential may be formed by
(i) Eliminating arbitary constants
(ii) Eliminating arbittary functions

Formation of PDE by eliminating arbitrary constants

Let the relationship between z, x and y be defined by $f(x, y, z, a, b) = 0 \longrightarrow (1)$

Here there are arbitrary constants a and b. To eliminate 'a' and 'b' we need three equations

Differentiating (1) partially with respect to x and y we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0 \longrightarrow (2)$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0 \longrightarrow (3)$$

We can eliminate the arbitrary constants from (1), (2) and (3) and the resulting equation will be of the form $\phi(x, y, z, p, q) = 0$

Note: 1. If the number of constants to be eliminated is equal to the number of independent variables the eliminant of arbitrary constants will result in a first order partial differential equation.

2. If the number of arbitrary constants is more than the number of independent variables a second order partial differential equation will be formed by eliminating arbitrary constants.

Example

1) Form the partial differential equation by eliminating arb. constants from $z = (x^2+a)(y^2+b)$

Solution

$$z = (x^2+a)(y^2+b) \longrightarrow (1)$$

Diff. partially with respect to x and y, we get

$$p = 2x(y^2+b)$$

$$q = 2y(x^2+a)$$

$$pq = 4xy(x^2+a)(y^2+b)$$

$$pq = 4xyz$$

This is the required PDE.

2). Find the PDE by eliminating 'a' and 'b' from $\log(az-1) = x+ay+b \longrightarrow (1)$

Diff. (1) partially with respect to x and y, we get

$$\frac{1}{az-1}\,ap = 1 \longrightarrow (2)$$

$$\frac{1}{az-1}\,aq = a \longrightarrow (3)$$

$$\frac{(3)}{(2)} \Rightarrow \frac{q}{p} = a \Rightarrow ap = q$$

From (1). $ap = az-1$

$$a(z-p) = 1$$

$$\frac{q}{p}(z-p) = 1$$

$$q(z-p) = p$$

This is the required PDE.

3) Form the partial diff. equation of the family of sphere of radius r with the centre at (a, b, c)

Solution

The equation of the sphere is

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2 \longrightarrow (1)$$

Diff. with respect to x, y we get

$$2(x-a)+2(z-c)\frac{\partial z}{\partial x}=0$$

$$x-a=-(z-c)p \longrightarrow (2)$$

$$2(y-b)+2(z-c)\frac{\partial z}{\partial y}=0$$

$$y-b=-(z-c)q \longrightarrow (3)$$

Sub. (2) and (3) in (1)

$$[-(z-c)p]^2+[-(z-c)q]^2+(z-c)^2=r^2$$

$$(z-c)^2(p^2+q^2+1)=r^2 \longrightarrow (4)$$

Diff. (2) with respect to x

$$1=-(z-c)\frac{\partial^2 z}{\partial x^2}-\frac{\partial z}{\partial x}\frac{\partial z}{\partial x}$$

$$1=-(z-c)r-p^2, \text{ where } r=\frac{\partial^2 z}{\partial x^2}$$

$$(z-c)=\frac{-1-p^2}{r} \longrightarrow (5)$$

Sub. (5) in (4)

$$\left[\frac{-1-p^2}{r}\right]^2(p^2+q^2+1)=r^2$$

$$(1+p^2)^2(1+p^2+q^2)=r^2\frac{\partial^2 z}{\partial x^2}$$

———— x ————

4) Form the PDE by eliminating a and b from
$z = ax+by+a^2+b^2$

Solution

$$z = ax+by+a^2+b^2 \longrightarrow (1)$$

Diff. with respect to x and y we get

$$\frac{\partial z}{\partial x} = p = a$$

$$\frac{\partial z}{\partial y} = q = b$$

Substituting these in (1) we get

$$z = px+qy+p^2+q^2$$

5) The equation of any sphere of radius 'r' having its centre in the xoy plane is
$(x-a)^2+(y-b)^2+z^2 = r^2$
Form a PDE by eliminating the constants a and b.

Solution

$$(x-a)^2+(y-b)^2+z^2 = r^2 \longrightarrow (1)$$

Diff. partially with respect to x and y we get

$$2(x-a)+2zp = 0$$

$$\Rightarrow (x-a) = -zp \longrightarrow (2)$$

$$2(y-b)+2zq = 0$$

$$\Rightarrow (y-b) = -zq \longrightarrow (3)$$

Sub. (2) & (3) in (1)

$$(-zp)^2+(-zq)^2+z^2 = r^2$$

$$z^2(p^2+q^2+1) = r^2$$

This is the required PDE

(6) Form the equation of all spheres with centres on the z axis.

Solution

Let the ~~centre~~ equation of the sphere be

$$x^2+y^2+(z-c)^2=r^2 \longrightarrow (1)$$

Diff. partially with respect to x and y we get

$$2x+2(z-c)p=0 \longrightarrow (2)$$

$$2y+2(z-c)q=0 \longrightarrow (3)$$

$$(2) \Rightarrow x=-(z-c)p$$

$$(3) \Rightarrow y=-(z-c)q$$

$$\Rightarrow \frac{x}{y}=\frac{-\cancel{(z-c)}p}{-\cancel{(z-c)}q}$$

$$\Rightarrow py=qx$$

This is the required PDE

Elimination of Arbitrary Functions

Examples

1. Form the PDE by eliminating the arbitrary function 'f' from $z=f(x^2+y^2+z^2)$.

Solution $z=f(x^2+y^2+z^2) \longrightarrow (1)$

Diff. (1) partially with respect to x and y we get

$$p=f'(x^2+y^2+z^2)(2x+2zp) \longrightarrow (2)$$

$$q=f'(x^2+y^2+z^2)(2y+2zq) \longrightarrow (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p}{q}=\frac{\cancel{f'(x^2+y^2+z^2)\,2}(x+zp)}{\cancel{f'(x^2+y^2+z^2)\,2}(y+zq)}$$

$$\frac{p}{q} = \frac{x+zp}{y+zq}$$

$$p(y+zq) = q(x+zp)$$

$$py + \cancel{zpq} = qx + \cancel{zpq}$$

$$\Rightarrow py = qx$$

This is the required PDE

2. Form the PDE by eliminating arbitrary function ϕ from $xyz = \phi(x^2+y^2-z^2)$

Solution $xyz = \phi(x^2+y^2-z^2) \longrightarrow (1)$

Diff. partially with respect to x, y

$$xyp + yz = \phi'(x^2+y^2-z^2)\,(2x-2zp) \longrightarrow (2)$$

$$xyq + xz = \phi'(x^2+y^2+z^2)\,(2y-2zq) \longrightarrow (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{xyp+yz}{xyq+xz} = \frac{x-zp}{y-zq}$$

$$(xyp+yz)(y-zq) = (xyq+xz)(x-zp)$$

$$xy^2p + y^2z - \cancel{xyz}pq - yz^2q = x^2yq + x^2z - \cancel{xyz}pq - xz^2p$$

$$xy^2p + xz^2p - qyz^2 - qyx^2 = zx^2 - zy^2$$

$$px(y^2+z^2) - qy(x^2+z^2) = z(x^2-y^2)$$

This is the required PDE

Methods to solve the first order PDE

The general solution of a first order PDE is $f(x, y, z, p, q) = 0$, where $p = \frac{\partial z}{\partial x}$ & $q = \frac{\partial z}{\partial y}$.

A solution of a PDE which contains the maximum possible no. of arbitrary constants is called a complete integral.

A solution of a PDE which contains the maximum possible no. of arbitrary functions is called a general Integral.

The general integral of PDE $F(x, y, z, p, q) = 0 \longrightarrow (1)$

Let the complete integral of (1) is

$$\phi(x, y, z, a, b) = 0 \longrightarrow (2)$$

where a & b are arb. constants.

Suppose that in (2) one of the constants is a function of the others, say, $b = f(a)$.

Then (2) becomes

$$\phi(x, y, z, a\ f(a)) = 0 \longrightarrow (3)$$

Diff. (3) w.r.to 'a' we get

$$\frac{\partial \phi}{\partial a} = \frac{\partial \phi}{\partial a} f'(a) = 0 \longrightarrow (4)$$

The elimination of 'a' between (3) and (4) if it exists is called the general integral of (1).

Complete Integral and singular Integral

Let $F(x,y,z,p,q)=0 \longrightarrow (1)$

Let the complete integral be $\phi(x,y,z,a,b)=0 \longrightarrow (2)$

Diff. (2) partially w.r.to 'a' and 'b', we get

$$\frac{\partial \phi}{\partial a}=0 \longrightarrow (3)$$

and

$$\frac{\partial \phi}{\partial b}=0 \longrightarrow (4)$$

the elimination of 'a' and 'b' from the equations (2), (3) and (4) if it exists is called the singular integral.

Type I $F(p,q)=0$.

Suppose that $z=ax+by+c$ is a trivial solution of $F(p,q)=0$

Then $p=\frac{\partial z}{\partial x}=a$, $q=\frac{\partial z}{\partial y}=b$ we get

$F(a,b)=0$

Hence the complete solution of the given equation is $z=ax+by+c$ where $F(a,b)=0$

Solving for b from $F(a,b)=0$

we get $b=\phi(a)$ then

$z=ax+\phi(a)y+c \longrightarrow (1)$

is the complete integral of the given equation.

Diff. p.w.r.to a

$0=x+\phi'(a)y$

Diff. partially with respect to c, we get

$0 = 1$ which is absurd

There is no singular integral for the given PDE.

To find the GI

The general integral is, putting $c = f(a)$ in (1)

$$z = ax + \phi(a)y + f(a) \longrightarrow (2)$$

& Diff. p. w. r. to a

$$0 = x + y\phi'(a) + f'(a) \longrightarrow (3)$$

Eliminating 'a' between (2) & (3), we get the general solution

1) Solve $\sqrt{p} + \sqrt{q} = 1$

Solution

Given $\sqrt{p} + \sqrt{q} = 1$

The complete integral is $z = ax + by + c$

where $\sqrt{a} + \sqrt{b} = 1$

$$\Rightarrow \sqrt{b} = 1 - \sqrt{a}$$

Squaring on both sides

$$b = (1 - \sqrt{a})^2$$

$\therefore$ The complete solution is

$$z = ax + (1 - \sqrt{a})^2 y + c \longrightarrow (1)$$

Diff. partially with respect to 'c', we get

$0 = 1$ (absurd)

There is no SI

Taking $c = f(a)$ when f is arbitrary

$$z = ax + (1-\sqrt{a})^2 y + f(a) \longrightarrow (2)$$

Diff. partially w.r.to 'a',

$$0 = x + 2(1-\sqrt{a})y\left[-\tfrac{1}{2}a^{-1/2}\right] + f'(a) \longrightarrow (3)$$

Eliminating 'a' between (2) and (3) we get the general solution.

Type II : Clairaut's form $z = px + qy + f(p,q)$

1) Solve $z = px + qy + p^2q^2$

Solution

Given $z = px + qy + p^2q^2$

The complete solution is

$$z = ax + by + a^2b^2 \longrightarrow (1)$$

Diff. (1) partially w.r.to 'a' & 'b', we get

$$0 = x + 2ab^2 \Rightarrow x = -2ab^2 \longrightarrow (2)$$

$$0 = y + 2a^2b \Rightarrow y = -2a^2b \longrightarrow (3)$$

$$\frac{x}{b} = \frac{y}{a} = \frac{-2ab^{\cancel{2}}}{\cancel{b}} = \frac{-2a^{\cancel{2}}b}{\cancel{a}} = -2ab = \frac{1}{k}$$

$$\frac{x}{b} = \frac{y}{a} = \frac{1}{k} \text{ (say)}$$

$$\Rightarrow a = ky, \quad b = kx$$

Sub. 'a' and 'b' in (2)

$$x = -2(ky)(kx)^2 = -2k^3x^2y$$

$$\Rightarrow k^3 = \frac{-x}{2x^2y} = -\frac{1}{2xy} \longrightarrow (4)$$

Sub. 'a' and 'b' in (1),

$$z = kxy + kxy + k^4 x^2 y^2$$

$$= 2kxy + kx^2y^2(k^3)$$

$$= 2kxy + kx^2y^2\left(\frac{-1}{2xy}\right)$$

$$z = 2kxy - \frac{k}{2}xy = \frac{3}{2}kxy$$

$$z^3 = \frac{27}{8}k^3x^3y^3$$

$$= \frac{27}{8}x^3y^3\left(\frac{-1}{2xy}\right)$$

$$z^3 = \frac{-27}{16}x^2y^2$$

$16z^3 + 27x^2y^2 = 0$ is the singular solution.

put $b = \phi(a)$ in (1)

$$z = ax + \phi(a)y + a^2[\phi(a)]^2 \longrightarrow (5)$$

Diff. partially w.r.to 'a' we get

$$0 = x + \phi'(a) + 2a[\phi(a)]^2 + a^2\,2\phi(a)\phi'(a) \longrightarrow (6)$$

Eliminate 'a' between (5) and (6) we get the general solution.

<u>Type III</u> <u>Equation of the type $f(x,p,q)=0$</u>

1) Solve $p = 2qx$

Let $q = a$

then $p = 2ax$

$dz = p\,dx + q\,dy$

$= 2ax\,dx + a\,dy$

Int. on bothsides we get

$z = \cancel{2}a\frac{x^2}{\cancel{2}} + ay + c$

$z = ax^2 + ay + c \longrightarrow (1)$

equation (1) is the complete integral of the given equation.

Diff. (1) partially w.r.to c,

$1 = 0$

Hence there is no singular integral

———— x ————

Type IV : Equation of the type $f(y, p, q) = 0$.

Assume $p = a$,

$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

$dz = p\,dx + q\,dy$, Int. we get the CI.

1) Solve $q = py + p^2 \longrightarrow (1)$

Solution Assume $p = a$

$\therefore (1) \Rightarrow q = ay + a^2$

$dz = p\,dx + q\,dy$

$= a\,dx + (ay + a^2)\,dy$

Int. on bothsides

$z = ax + \frac{ay^2}{2} + a^2y + b \longrightarrow (2)$

Diff. partially w.r.to 'b' $0 = 1$ (absurd)

There is no singular integral

Let $b = \phi(a)$

$$z = ax + \frac{ay^2}{2} + a^2y + \phi(a) \longrightarrow (3)$$

Diff. (3) partially w.r. to 'a' we get

$$0 = x + \frac{y^2}{2} + 2ay + \phi'(a) \longrightarrow (4)$$

Eliminate 'a' between (3) and (4) we get general solution

—— x ——

Type V : Equation of the type $f(z, p, q) = 0$

Let z be a function of u where

$$u = x + ay$$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = a$$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = \frac{dz}{du}(a)$$

$\therefore f(z, p, q) = 0$ becomes $f\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0$

which is an ode of ~~ode~~ first order.

solving for $\frac{dz}{du}$, we obtain $\frac{dz}{du} = \phi(z, a)$

$$\Rightarrow \frac{dz}{\phi(z, a)} = du$$

Int. $$\int \frac{dz}{\phi(z, a)} = u + c$$

$$f(z, a) = (x + ay) + c$$

This is the CI, singular and general integral are found out as usual

Solve $z^2 = 1+p^2+q^2$.

Solution

Given $z^2 = 1+p^2+q^2 \longrightarrow (1)$

Let $u = x+ay$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = a$$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}, \quad q = a\frac{dz}{du}$$

sub. in (1) we get

$$z^2 = 1+\left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2$$

$$z^2-1 = \left(\frac{dz}{du}\right)^2(1+a^2)$$

$$\left(\frac{dz}{du}\right)^2 = \frac{z^2-1}{a^2+1}$$

$$\frac{dz}{du} = \frac{1}{\sqrt{1+a^2}}\sqrt{z^2-1}$$

$$\Rightarrow \frac{dz}{\sqrt{z^2-1}} = \frac{du}{\sqrt{1+a^2}}$$

Int. on bothsides

$$\cosh^{-1}(z) = \frac{1}{\sqrt{1+a^2}}u+b$$

$$= \frac{1}{\sqrt{1+a^2}}(x+ay)+b$$

Which is the complete integral

Type VI : $f(x, p) = f(y, q)$

Put $f(x, p) = f(y, q) = a$ (say)

Solving for p and q, we get

$$p = f_1(x, a) \text{ and } q = \phi_1(y, a)$$

But $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

$$dz = p\,dx + q\,dy$$

$$= f_1(x, a)\,dx + \phi_1(y, a)\,dy$$

Int., on bothsides

$$z = \int f_1(x, a)\,dx + \int \phi_1(y, a)\,dy$$

this is the C.I.

The singular and general integral are found out as usual.

— x —

1) Solve $p - x^2 = q + y^2$

Solution

Given $p - x^2 = q + y^2$

Let $p - x^2 = q + y^2 = a$

$$\Rightarrow p = a + x^2 \text{ \& } q = a - y^2$$

$$dz = p\,dx + q\,dy$$

$$= (a + x^2)\,dx + (a - y^2)\,dy$$

Int. $z = ax + \frac{x^3}{3} + ay - y^3/3 + b \longrightarrow (1)$

is the complete integral

There is no singular integral

put $b = \phi(a)$

$$z = ax + \frac{x^3}{3} + ay - \frac{y^3}{3} + \phi(a) \longrightarrow (2)$$

Diff. partially w. r to 'a', we get

$$0 = x + y + \phi'(a)$$

Eliminate 'a' between (1) and (2) to get the general solution.

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2) Find the complete solution of

$$p + q = \sin x + \sin y$$

Solution

Given $p + q = \sin x + \sin y$

$$p - \sin x = \sin y - q = a$$

$$\Rightarrow p = a + \sin x, \quad q = \sin y - a$$

$$dz = p\,dx + q\,dy$$

$$= (a + \sin x)\,dx + (\sin y - a)\,dy$$

Int.

$$z = ax - \cos x - \cos y - ay + b$$

is the complete integral
