UNIT IV

Differential Equations

Differential Equation

Equations in which an unknown function, and its derivatives or differentials occur are called differential equations.

for example,

(i) $x + y\frac{dy}{dx} = 3y$, (ii) $\frac{dy}{dx} + \frac{x-y}{x+y} = 0$, (iii) $\frac{d^2y}{dx^2} + y = sinx$, and (iv)$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

are all differential equations

Ordinary differential equation

If, in a differential equation, the unknown function is a function of one independent variable, then it is known as an ordinary differential equation.

Partial differential equation

If the unknown function is a function of two or more independent variables and the equation of involves partial derivatives of the unknown function, then it is known as a partial differential equation

In the above examples, equations from (i) to (iii) are ordinary differential equations while the fourth is a partial differential equation..

Order of a differential equation

The order of a differential equation is the order of the highest differential coefficient which occurs in it. In the examples given above, (i) and (ii) are of first order while (iii) and (iv) are of order 2.

Degree of a differential equation

The degree of a differential equation is the degree of the highest order differential coefficient which occurs in it, after the equation has been cleared of radicals and fractions. The above listed equations are all of degree 1. To decide, for example, the degree of the differential equation

$$\rho = \frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2}$$, we rewrite it as

$$\rho^2 \left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$$ and observe that it is of degree 2.

Linear differential equation

If in a differential equation, the derivatives and the dependent variables appear in first power and there are no products of these, and also the coefficients of the various terms are either constants or functions of the independent variable, the equation is said to be a linear differential equation.

SOLUTION OF A DIFFERENTIAL EQUATION

A relation between the dependent and independent variables, which, when substituted in the equation, satisfies it, is known as a solution or a primitive of the equation.
Note that, in the solution, the derivatives of the dependent variable should not be present.

The solution, in which the number of arbitrary constants occurring is equal to the order of the equation, is known as the general solution or the complete integral. By giving particular values to the arbitrary constant appearing in the general solution we obtain particular solutions of the equation.
For example, $y = Ae^{2x} + Be^{-2x}, y = 3e^{2x} + 2e^{-2x}$ are respectively the general solution and the particular solution of the equation $\frac{d^2y}{dx^2} - 4y = 0$
Solutions of equations which do not contain any arbitrary constants and which are not derivable from the general solution by giving particular values to one or more of the arbitrary constants, are called singular solutions.

11. $(2x^3y^2 + 4x^2y + 2xy^2 + xy^4 + 2y)dx + (2y^3 + 2x^2y + 2x)dy = 0$
[**Ans :** $(2x^2y^2 + 4xy + y^4) = Ce^{-x^2}$]

12. $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$ [**Ans :** $3x^2y^4 + 6xy^2 + 2y^6 = k$]

13. $(y^2 + 2x^2y)dx + (3x^3 - xy)dy = 0$ [**Ans :** $12\sqrt{xy} - 2(y/x)^{3/2} = k$]

14. $(4xy + 2x^2)dx + (3y^4 + 5xy^3)dy = 0$ [**Ans :** $x^1y^1(x + y^3) = C$]

15. $y(3ydx - 2xdy) + x^2(10ydx - 6xdy) = 0$ [**Ans :** $x^3y + 2x^5 = cy^3$]

7.5 DIFFERENTIAL EQUATIONS OF FIRST ORDER AND HIGHER DEGREE

Consider the equation

$$p^n + A_1p^{n-1} + A_2p^{n-2} + \ldots.A_n = 0 \qquad \ldots(7.6)$$

where $p = \dfrac{dy}{dx}$ and $A_1, A_2, \ldots\ldots, A_n$ are all functions of x and y. The solutions of thös equation of first order and higher degree under various categories are considered below.

7.5.1 Type (i): Equations Solvable for p

The L.H.S. of the equations (7.6) can be resolved into a number of linear factors $(p-f_1), (p-f_2), \ldots, (p-f_n)$ so that it takes the form $(p-f_1)(p-f_2)\ldots(p-f_n) = 0$.

Equating each of these factors to zero, we get n differential equations of first order and first degree. Solving these equations individually and combining them we get the primitive of the given equation.

Example 7.25

Solve $p^2 - 7p + 12 = 0$

Solution

$$(p-4)(p-3) = 0 p = 3, \ p = 4$$

$$\frac{dy}{dx} = 3, \ \frac{dy}{dx} = 4$$

Solving them, we have $y = 3x + c$, $y = 4x + c$.

Combining these solutions, the solution of the given equation can be put as $(y - 3x - c)(y - 4x - c) = 0$

Note : Since the primitive of the first order equation can have only one arbitrary constant, we take the constants of integration in both cases as c itself.

Example 7.26

Solve $xyp^2 - p(x^2 + y^2) + xy = 0$

Solution The given equation is

$$px(py - x) - y(py - x) = 0 \text{ or}$$

$$(py - x)(px - y) = 0$$

$$\therefore py = x, \; px - y = 0$$

If $py = x$, then $y\dfrac{dy}{dx} = x$

$$\int y\,dy = \int x\,dx \text{ or } \frac{y^2}{2} = \frac{x^2}{2} + c \text{ or } y^2 - x^2 = k \quad ...(1)$$

If $px = y$, then $x\dfrac{dy}{dx} = y$,

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\log y = \log x + \log k \text{ or } y = kx \quad ...(2)$$

$\therefore$ The solution is $(y - kx)(y^2 - x^2 - k) = 0$

Example 7.27

Solve $xp^2 + (y - x)p - y = 0$

Solution Solving for p, $\quad p = \dfrac{-(y-x) \pm \sqrt{(y-x)^2 + 4xy}}{2x}$

$$= \frac{-(y-x) \pm \sqrt{(y+x)^2}}{2x}$$

$$= \frac{(x-y) \pm (x+y)}{2x}$$

$$\therefore p = 1, \quad p = -\frac{y}{x}$$

$$\frac{dy}{dx} = 1, \quad \frac{dy}{dx} = -\frac{y}{x}$$

Integrating the first, we have $y = x + c$ ...(1)

Integrating the second, we have

$$\int \frac{dy}{y} = -\int \frac{dx}{x} \text{ or } \log y = -\log x + \log c = \log \frac{c}{x}$$

$$y = \frac{c}{x} \text{ or } xy = c \quad ...(2)$$

$\therefore$ The solution of the given equation is $(y - x - c)(xy - c) = 0$

Example 7.28

Solve $p^2 + 2py\cot x = y^2$

Solution Solving for p,

$$p = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$$

$$= \frac{-2y\cot x \pm 2y\operatorname{cosec} x}{2} = y(-\cot x \pm \operatorname{cosec} x)$$

Integrating $p = y(-\cot x + \operatorname{cosec} x)$

$$\int \frac{dy}{y} = \int(-\cot x + \operatorname{cosec} x)dx$$

$$\log y = -\log\sin x - \log(\operatorname{cosec} x + \cot x) + \log c$$

$$\log(y\sin x) = \log\frac{c}{\operatorname{cosec} x + \cot x}$$

$$\therefore y\sin x = \frac{c\sin x}{1+\cos x} \text{ or } y(1+\cos x) = c \qquad \text{...(1)}$$

When $p = y(-\cot x - \operatorname{cosec} x)$

$$\int\frac{dy}{y} = -\int\cot x\,dx - \int\operatorname{cosec} x\,dx$$

$$\log y = -\log\sin x + \log(\operatorname{cosec} x + \cot x) + \log c$$

$$\log(y\sin x) = \log[c(\operatorname{cosec} x + \cot x)]$$

$$y\sin x = c(\operatorname{cosec} x + \cot x)$$

$$y\sin^2 x = c(1+\cos x)$$

$$y(1+\cos x)(1-\cos x) = c(1+\cos x)$$

$$\therefore y(1-\cos x) = c \qquad \text{...(2)}$$

The solution is $[y(1-\cos x) - c][y(1+\cos x) - c] = 0$

Example 7.29

Solve $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$

Solution

$$p^2(p+2x) - y^2p(p+2x) = 0$$

$$(p+2x)(p^2 - py^2) = 0$$

$$p(p-y^2)(p+2x) = 0$$

$$p = 0, \quad p = y^2, \quad p = -2x.$$

$$p = 0 \quad \text{i.e.,} \ \frac{dy}{dx} = 0 \text{ gives } y = c \qquad \text{...(1)}$$

$$p = y^2 \quad \text{i.e.,} \ \frac{dy}{y^2} = dx \text{ gives}$$

$$-\frac{1}{y} = x + c \text{ or } xy + cy = -1 \qquad \text{...(2)}$$

and $p + 2x = 0$ i.e., $dy = -2x\,dx$ gives

$$y = -x^2 + c \qquad \text{...(3)}$$

combining (1), (2) and (3) we get the solution as

$$(y - c)(xy + cy + 1)(y + x^2 - c) = 0$$

EXERCISES

1) $yp^2 + (x - y)p - x = 0$ **[Ans :** $(xy - c)(x^2y - c) = 0$]

2) $p^2y - 2px + x^2 = 0$ **[Ans :** $(3x^2 - 6y + 4 - c)^2 = 16(1 - y)^3$]

3) $2p^2 - (x + 2y^2)p + xy^2 = 0$ **[Ans :** $\left(x + \frac{1}{y} + c\right)\left(\frac{x^2}{4} - y + c\right) = 0$]

4) $p(p - y) = x(x + y)$ **[Ans :** $(y + x + 1 - ce^x)(2y + x^2 - c) = 0$]

5) $x^2p^2 + xy\,p - 6y^2 = 0$ **[Ans :** $(y - cx^2)(yx^3 - c) = 0$]

6) $p^2 - 2py = 3y^2$ *(Bharathiar '90)*

[Ans : $(y - ce^{3x}$ $(y - ce^{-x} = 0$]

7) $p^2 + 3p + 2 = 0$ *(GCT/96)*

[Ans : $(y + 2x + C_1)(y + x + C_2 = 0$]

7.5.2 Type (ii): Equations Solvable for y

Suppose the differential equation can be solved for y giving a relation

$$y = f(x, p) \qquad \text{...(7.7)}$$

Differentiating w.r. to x we get

$$p = g\left(x, p, \frac{dp}{dx}\right) \qquad \text{...(7.8)}$$

Let the solution of this first order linear equation in p be

$$\Phi(x, p, c) = 0 \qquad \text{...(7.9)}$$

Eliminating p between (7.7) and (7.9) we get a relation between x and y which is the solution. If the elimination is not possible we express the solution in the form

$$x = f_1(p, c),\ y = f_2(p, c), \qquad p \text{ being a parameter.}$$

Example 7.30

Solve $y = 2px + p^2$

Solution Differentiating w.r. to x, we get

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

$$p = 2p + 2\frac{dp}{dx}(x + p)$$

$$-p = 2\frac{dp}{dx}(x + p) \text{ or } \frac{dx}{dp} + \frac{2x}{p} = -2$$

This is a linear equation in x.

$$P = \frac{2}{p} \quad Q = -2$$

$$\int P dp = 2\log p \text{ and } e^{\int P dp} = p^2$$

$$\therefore xp^2 = -2\int p^2 dp,\ xp^2 = -2\frac{p^3}{3} + c.$$

Thus the solution is $x = -2/3p + cp^{-2}$ ...(1)

Using this in the given equation we get

$$y = -\frac{1}{3}p^2 + 2cp^{-1} \qquad \text{...(2)}$$

(1) and (2) together gives the solution of the given equation.

Example 7.31

Solve $y = -px + p^2x^4$

Solution

$$y = -px + p^2x^4$$

$$\frac{dy}{dx} = -p - x\frac{dp}{dx} + x^4 2p\frac{dp}{dx} + p^2 \cdot 4x^3$$

$$p = -p + 4p^2x^3 + \frac{dp}{dx}(2px^4 - x)$$

$$2p + x\frac{dp}{dx} - 2px^4\frac{dp}{dx} - 4p^2x^3 = 0$$

$$\left(2p + x\frac{dp}{dx}\right) - 2px^3\left(x\frac{dp}{dx} + 2p\right) = 0$$

$$\left(2p + x\frac{dp}{dx}\right)(1 - 2px^3) = 0$$

$$x\frac{dp}{dx}+2p = 0 \text{ or } 1-2px^3 = 0$$

$$x\frac{dp}{dx}+2p = 0 \text{ or } \frac{dp}{p} = -2\frac{dx}{x}$$

$$\log p = -2\log x + \log c \Rightarrow p = \frac{c}{x^2}$$

Using this relation in the given equation (1) we get $xy = c^2x - c$ which is the solution of the given equation.

Note : $1-2px^3 = 0$ is not considered since it does not contain the derivative term $\frac{dp}{dx}$. Substituting $p = \frac{1}{2x^3}$ in the given equation we get $4x^2y+1 = 0$. This relation $4x^2y+1 = 0$ cannot be derived from the general solution $xy = c^2x - c$ obtained above by giving any particular value for c. Such a solution is called a singular solution.

It is easily seen that $4x^2y+1 = 0$ satisfies the given differential equation (1) and hence it is a solution of the equation.

Example 7.32

Solve $y = x + p^2 - 2p$

Solution

$$\frac{dy}{dx} = 1+2p\frac{dp}{dx} - 2\frac{dp}{dx}$$

$$p-1 = 2\frac{dp}{dx}(p-1) \text{ or } (p-1)\left(2\frac{dp}{dx}-1\right) = 0$$

$$\frac{dp}{dx} = \frac{1}{2}, \quad p = 1$$

$$2\int dp = \int dx$$

$$2p = x + c \text{ or } p = \frac{x+c}{2}$$

Using this in the given equation we get

$$(y-x) = \left(\frac{x+c}{2}\right)\left(\frac{x+c-4}{2}\right)$$

$$4(y-x) = (x+c)(x+c-4)$$

$$4(y-x) = (x+c)^2 - 4(x+c)$$

$$4(y+x) = (x+c)^2 \text{ is the solution.}$$

Note : $p = 1$, used in the given equation, gives $y = x - 1$. Verify that this is indeed, a singular solution.

EXERCISES

Solve the following equations:

1) $p^2 + py = x^2 + xy$ [Ans : $[c + x^2 - 2y][c - (y + x - 1)e^x] = 0$]

2) $y = (1 + p^2)^{-1/2} + b$ [Ans : $(x + c)^2 + (y - b)^2 = 1$]

3) $xp^2 + x = 2yp$ [Ans : $2y = cx^2 + \frac{1}{x}$]

4) $y = 2px - p^2$ [Ans : $3y = p^2 + \frac{6c}{p}; 3x = 2p + \frac{c}{p^2}$]

5) $p^2x - 2yp + ax = 0$ [Ans : $2yc = c^2x^2 + a$]

7.5.3 Type (iii): Equations Solvable for x

In this case the equation can be put in the form

$$x = f(y, p) \qquad \text{...(7.10)}$$

Differentiating w.r. to y we get

$$\frac{dx}{dy} = g\left(y, p, \frac{dp}{dy}\right) \qquad \text{...(7.11)}$$

If the solution of (2) be obtained as

$$\Phi(y, p, c) = 0 \qquad \text{...(7.12)}$$

then, by eliminating p between (7.10) and (7.12), we get the solution. If the elimination is not possible, we express x and y in terms of p and p is regarded as a parameter.

Example 7.33

Solve $y = 3x + \log p$

Solution Differentiating w.r. to y we get,

$$1 = 3\frac{dx}{dy} + \frac{1}{p}\frac{dp}{dy}$$

$$1 = \frac{3}{p} + \frac{1}{p}\frac{dp}{dy} \Rightarrow p - 3 = \frac{dp}{dy}$$

$$\int \frac{dp}{p-3} = \int dy; \log(p-3) = y + c$$

$$(p - 3) = e^{y+c} \text{ or } p = 3 + ke^{y}$$

Using this in the given equation we get

$$y = 3x + \log(3 + ke^y) \quad \text{or} \quad e^{-3x} = 3e^{-3} + k$$

Example 7.34

Solve $y = px + \sin^{-1} p$

Solution Differentiating w.r. to y we get,

$$1 = p\frac{dx}{dy} + x\frac{dp}{dy} + \frac{1}{\sqrt{1-p^2}}\frac{dp}{dy}$$

$$1 = p\frac{1}{p} + \frac{dp}{dy}\left(x + \frac{1}{\sqrt{1-p^2}}\right)$$

$$\therefore \frac{dp}{dy} = 0 \quad \text{or} \quad p = c \quad \text{and } x = -\frac{1}{\sqrt{1-p^2}} \qquad ...(1)$$

Using $p = c$ in the given equation we get $y = cx + \sin^{-1} c$ as the solution.

Note : From (1) we can get $\frac{dy}{dx} = c$ or $y = cx + d$. But since this contains 2 arbitrary constants this cannot be a primitive of the given equation which is of first order.

$x = -\frac{1}{\sqrt{1-p^2}}$ leads to the singular solution.

Example 7.35

Solve $4yp^2 + 2xp = y$

Solution Rewriting we get,

$$2x = \frac{y}{p} - 4yp$$

Differentiating w.r. to y we get

$$2\frac{dx}{dy} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 4p - 4y\frac{dp}{dy}$$

$$\left(\frac{1+4p^2}{p}\right) = -y\frac{dp}{dy}\left(\frac{1+4p^2}{p^2}\right)$$

$$\therefore \left(\frac{1+4p^2}{p^2}\right)\left(p + y\frac{dp}{dy}\right) = 0$$

$$\therefore p + y\frac{dp}{dy} = 0 \quad \text{or} \quad \frac{dp}{p} = -\frac{dy}{y} \quad \log p + \log y = \log c$$

$$p = \frac{c}{y}$$

Using this in the given equation we get the solution as $y^2 = 2cx + 4c^2$.

Note : $4y^2 + x^2 = 0$ is the singular solution.

EXERCISES

1. $yp^2 - xp + 3y = 0$ [Ans : $y = cp^{3/2}(2+p^2)^{-5/4}$; $x = cp^{1/2}(p^2+3)(p^2+2)^{-5/4}$]

2. $y = 2px + y^2p^3$ [Ans : $y = 2cx + c^3$]

3. $ayp^2 + (2x-b)p - y = 0$ [Ans : $ac + (2x-b)c - y^2 = 0$]

4. $x + \dfrac{p}{\sqrt{1+p^2}} = a$ [Ans : $(y+c)^2 + (x-a)^2 = 1$]

5. $y = 3px + 4p^3$ [Ans : $x = \frac{c}{3}p^{-3/2} - \frac{12}{7}p^2$; $y = cp^{-1/2} - \frac{8}{7}p^3$]

6. $xp^2 + x = 2yp$ [Ans : $y = \frac{1}{2c} + \frac{c}{2}x^2$]

7. $y^2 \log y = xpy + p^2$ (GCT 96) [Ans : $cx = \log y - c^2$]

7.5.4 Type (iv): Clairaut's Equation

An equation of the form

$$y = px + f(p) \tag{7.13}$$

is known as Clairaut's equation.

Differentiating w.r.t. to x we get

$$p = p.1 + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\text{or} \quad [x + f'(p)]\frac{dp}{dx} = 0$$

$$\frac{dp}{dx} = 0 \quad \text{gives} \quad p = c$$

Using this in the given equation we get $y = cx + f(c)$ as the solution.

Note :

1. The general solution of Clairaut's equation $y = cx + f(c)$ can be interpreted geometrically as family of straight lines c being a parameter.
2. Singular solution of the Clairaut's equation (7.13) $y = px + f(p)$ is the eliminant of p between the equation (7.13) and the relation

$$0 = x + f'(p) \quad \text{...(7.14)}$$

It can be observed that (7.14) is obtained by partially differentiating (7.13) with respect to p, treating p as a parameter. So singular solution is the eliminant of p

between (7.13) and (7.14), p being regarded as a parameter or it is the eliminant of c between

$$y = cx + f(c) \quad \text{and } 0 = x + \frac{\partial f}{\partial c}.$$

Clearly, the eliminant is the envelope of the family of straight lines $y = cx + f(c)$ represented by the general solution.

Example 7.36

Solve $(y - px)(p - 1) = p$

Solution $y - px = \dfrac{p}{p-1} \qquad \therefore y = px + \dfrac{p}{p-1}$

Since this is a Clairaut's equation, we get the general solution by replacing p by c.

$\therefore$ The general solution is $y = cx + \dfrac{c}{c-1}$

Example 7.37

Solve $y = 2px + yp^2$

Solution Multiplying by y we get,

$$y^2 = 2pxy + y^2p^2 \qquad (1)$$

put $y^2 = Y$

$$2y\frac{dy}{dx} = \frac{dY}{dx} = P \text{ (say) i.e., } 2yp = P$$

Equation (1) takes the form $Y = Px + \dfrac{P^2}{4}$

This being a Clairaut's equation has the solution $Y = cx + \dfrac{c^2}{4}$

$\therefore$ The general solution of the given equation is $y^2 = cx + \dfrac{c^2}{4}$

Example 7.38

Solve $\sin px \cos y = \cos px \sin y + p$

Solution

$$\sin px \cos y - \cos px \sin y = p$$

$$\sin(px - y) = p, \ \therefore y = px - \sin^{-1} p$$

Since this is a Clairaut's equation, the solution is $y = cx - \sin^{-1} c$.

EXERCISES

Solve the following :

1. $y = x\dfrac{dy}{dx} + a\dfrac{dx}{dy}$ [Ans : $y = cx + \dfrac{a}{c}$]

2. $y = x\frac{dy}{dx} + e^{dy/dx}$

[Ans : $y = cx + e^c$]

3. $y = (x-a)\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$

[Ans : $y = cx - ac - c^2$]

4. $xy(y - px) = x + py$

[Hint : Put $x^2 = X, y^2 = Y$]

[Ans : $y^2 = cx^2 + 1 + c$]

5. $y = 3px + 6y^2p^2$

[Hint : Multiply by y^2 and put $y^3 = Y$]

[Ans : $y^3 = 3cx + 6c^2$]

6. $p^2x(x-2) + p(2y - 2xy - x + 2) + y^2 + y = 0$

[Ans : $(y - cx + 2c)(y - cx + 1) = 0$]

7. $e^{3x}(p-1) + p^3e^{2y} = 0$

[Hint : Put $e^x = X$ and $e^y = Y$]

[Ans : $e^y = ce^x + c^3$]

7.6 LINEAR DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER WITH CONSTANT COEFFICIENTS

The general form of a linear differential equation of the nth order with constant coefficients is

$$a_0\frac{d^n y}{dx^n} + a_1\frac{d^{n-1} y}{dx^{n-1}} + \ldots.. + a_n y = X \quad \ldots(7.15)$$

where $a_0 (\neq 0), a_1, a_2, \ldots\ldots, a_n$ are constants and X is a function of x.

If $X = 0$,(7.15) becomes

$$a_0\frac{d^n y}{dx^n} + a_1\frac{d^{n-1} y}{dx^{n-1}} + \ldots.. + a_n y = 0 \quad \ldots(7.16)$$

The equation (7.16) is called the homogeneous linear equation corresponding to equation (7.15).

It can be seen that

(i) if $y = f_1(x), y = f_2(x), y = f_n(x)$ are n linearly independent solutions of the equation (2) then

$$y = C_1 f_1(x) + C_2 f_2(x) + \ldots. + C_n f_n(x)$$

where $C_1, C_2, \ldots.., C_n$ are arbitrary constants is also its solution. This solution containing n arbitrary constants is known as the general solution of equation (7.16).

(ii) if $y = \varphi(x)$ be a solution of (7.15) not containing any arbitrary constants, then

$$y = C_1 f_1(x) + C_2 f_2(x) + \ldots + C_n f_n(x) + \varphi(x) \quad \ldots(7.17)$$