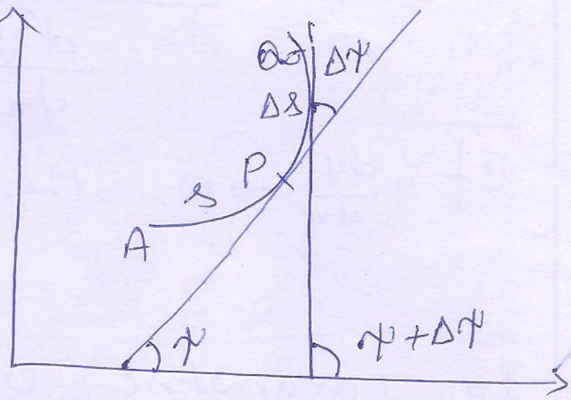
UNIT : 2

Differential calculus

Curvature and Radius of curvature



Let P and Q be any two closed points on a plane curve. Let the arcual distances of P and Q measured from a fixed point A on the given curve be s and $s+\Delta s$, so that

$\widehat{PQ} = \Delta s$

Let the tangents at P and Q to the curve make angles ψ and $\psi+\Delta\psi$ with a fixed line in the plane of the curve, say, the x axis. then the angle between the tangents at P and Q $= \Delta\psi$.

Hence $\dfrac{\Delta\psi}{\Delta s}$ is the average rate of bending of the curve (or average rate of change of direction of the tangent to the curve in the arcual interval $\widehat{PQ}$) or average curvature of the arc PQ.

$\underset{\Delta s\to 0}{Lt}\dfrac{\Delta\psi}{\Delta s} = \dfrac{d\psi}{ds}$ is the curvature of the curve at the point P and is denoted by k (kappa).

Radius of curvature of a curve at any point on it is defined as the reciprocal of the curvature of the curve at that point and denoted by ρ. Thus $\rho = \dfrac{1}{k} = \dfrac{ds}{d\psi}$.

Formula for Radius of curvature in cartesian co-ordinates

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

If $\frac{dy}{dx} \to \infty$, then $\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}$

In parametric coordinates

$$\rho = \frac{\left(\dot{x}^2+\dot{y}^2\right)^{3/2}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}$$

where $\dot{x} = \frac{dx}{dt}$ or $\frac{dx}{d\theta}$ & $\dot{y} = \frac{dy}{dt}$ or $\frac{dy}{d\theta}$.

WORKED EXAMPLES

1) Find the radius of curvature of the curve $x^4+y^4=2$ at the point $(1,1)$

Solution

Given $x^4+y^4=2$

$\Rightarrow y^4 = 2-x^4$

Differentiating with respect to x, we get

$\cancel{4}y^3\frac{dy}{dx} = -\cancel{4}x^3$

$\Rightarrow \frac{dy}{dx} = -\frac{x^3}{y^3}$

Once again diff. w.r.to x, we get

$$\frac{d^2y}{dx^2} = -\left[\frac{y^3 3x^2 - x^3 3y^2\frac{dy}{dx}}{(y^3)^2}\right]$$

$$\left(\frac{dy}{dx}\right)_{(1,1)} = \frac{-1}{1} = -1$$

$$\left(\frac{d^2y}{dx^2}\right)_{(1,1)} = -\left[\frac{1(3)-3(-1)}{1}\right] = -6$$

The radius of curvature $\rho = \frac{(1+y'^2)^{3/2}}{y''}$

$$(\rho)_{(1,1)} = \frac{[1+(-1)^2]^{3/2}}{-6} = \frac{2^{3/2}}{-2\times 3} = \frac{-2^{3/2-1}}{3}$$

$$= \frac{-\sqrt{2}}{3}$$

$|\rho| = \frac{\sqrt{2}}{3}$ [$\because$ radius is taken in +ve sign]

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2) Find the radius of curvature at $(a,0)$ on the curve $xy^2 = a^3 - x^3$.

Solution: Given $xy^2 = a^3 - x^3$

Diff. w. r. to x, we get

$$x \cdot 2y\frac{dy}{dx} + 1 \cdot y^2 = -3x^2$$

$$2xy\frac{dy}{dx} = -(y^2+3x^2)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(y^2+3x^2)}{2xy}$$

$$\left(\frac{dy}{dx}\right)_{(a,0)} = \infty$$

$$\Rightarrow \frac{dx}{dy} = \frac{-2xy}{(y^3+3x^2)}$$

$$\left(\frac{dx}{dy}\right)_{(a,0)} = 0$$

$$\frac{d^2x}{dy^2} = \frac{(-3x^2-y^2)\left(2x+2y\frac{dx}{dx}\right)-2xy\left(-6x\frac{dx}{dy}-2y\right)}{(-3x^2-y^2)^2}$$

$$\left(\frac{d^2x}{dy^2}\right)_{(a,0)} = \frac{(-3a^2-0)(2a+0)-0}{(-3a^2)^2}$$

$$= \frac{-\cancel{6}^{2}a^3}{+\cancel{9}_{3}a^4} = \frac{-2}{3a}$$

$\therefore$ The radius of curvature

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}$$

$$= \frac{1}{\left(\frac{-2}{3a}\right)} = \frac{-3a}{2}$$

$$|\rho| = \frac{3a}{2}$$

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3). Find the radius of curvature on $y=e^x$ at the point where the curve cuts the y axis.

Solution : Given $y=e^x$

At y axis $x=0$.

$\therefore\ y=e^0=1$

$\therefore$ (0,1) is the pt of intersection of the curve $y=e^x$ cut the y axis.

$\therefore$ The radius of curvature for $y=e^x$ at (0,1) is required.

$$y = e^x$$

$$\left(\frac{dy}{dx}\right) = e^x$$

$$\left(\frac{dy}{dx}\right)_{(0,1)} = 1$$

$$\frac{d^2y}{dx^2} = e^x$$

$$\left(\frac{d^2y}{dx^2}\right)_{(0,1)} = 1$$

$$\rho = \frac{(1+y'^2)^{3/2}}{y''} = (1+1)^{3/2} = 2^{3/2}$$

$$\rho = 2\sqrt{2}$$

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4) Find the radius of curvature at any point (x, y) on the curve $y = c\log\sec(x/c)$

Solution

$$y = c\log\sec(x/c)$$

$$\frac{dy}{dx} = \cancel{c}\,\frac{1}{\cancel{\sec(x/c)}}\,\cancel{\sec\left(\frac{x}{c}\right)}\tan(x/c)\,\frac{1}{\cancel{c}}$$

$$= \tan(x/c)$$

$$\frac{d^2y}{dx^2} = \sec^2(x/c)\left(\frac{1}{c}\right)$$

$$\rho = \frac{(1+y'^2)^{3/2}}{y''} = \frac{(1+\tan^2(x/c))^{3/2}}{\frac{1}{c}\sec^2(x/c)} = \frac{[\sec^4(x/c)]^{3/4}}{\frac{1}{c}\sec^2(x/c)}$$

$$= c\,\frac{\sec^3(x/c)}{\sec^2(x/c)} = c\sec(x/c), \quad \rho = c\sec(x/c)$$

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5) Find the radius of curvature for $\sqrt{x}+\sqrt{y}=1$ at $\left(\frac{1}{4}, \frac{1}{4}\right)$

solution

Gn: $\sqrt{x}+\sqrt{y}=1$

$\Rightarrow \sqrt{y} = 1-\sqrt{x}$

ie, $y^{1/2} = 1 - x^{1/2}$

Diff. w. r.to x,

$$\cancel{\tfrac{1}{2}}\, y^{-1/2} \frac{dy}{dx} = -\cancel{\tfrac{1}{2}}\, x^{-1/2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x^{-1/2}}{y^{-1/2}} = -\sqrt{\frac{y}{x}}$$

$$\left(\frac{dy}{dx}\right)_{(1/4,\,1/4)} = -1$$

$$\frac{d^2y}{dx^2} = \frac{-\left[\sqrt{x}\,\frac{1}{2}y^{-1/2}\frac{dy}{dx} - \sqrt{y}\,\frac{1}{2}\,x^{-1/2}\right]}{(\sqrt{x})^2}$$

$$\left(\frac{d^2y}{dx^2}\right)_{(1/4,\,1/4)} = \left[\frac{+\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)^{-1/2}(-1) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)^{-1/2}}{\left(\sqrt{1/4}\right)^2}\right]$$

$$= -\left[-\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)^{-1/2} - \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)^{-1/2}\right] \Big/ \frac{1}{4}$$

$$= \frac{\left[\left(\frac{1}{4}\right)^{1-1/2} + \left(\frac{1}{4}\right)^{1-1/2}\right]}{\frac{1}{4}} = \frac{\left(\frac{1}{4}\right)^{1/2} + \left(\frac{1}{4}\right)^{1/2}}{\left(\frac{1}{4}\right)}$$

$$= \frac{\left(\frac{1}{2}+\frac{1}{2}\right)}{1/4} = \frac{1}{\left(1/4\right)} = 4$$

$\therefore$ the radius of curvature is

$$\rho = \frac{(1+y'^2)^{3/2}}{y''}$$

$$= \frac{[1+(-1)^2]^{3/2}}{4} = \frac{\cancel{2}\sqrt{2}}{\cancel{4}_2} = 1/\sqrt{2}$$

$$\rho = \frac{1}{\sqrt{2}}$$

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6) If ρ is the radius of curvature at any point (x,y) on the curve $y = \frac{ax}{a+x}$ show that $\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$

Solution : Given $y = \frac{ax}{a+x}$

Diff. w. r to x, we get

$$\frac{dy}{dx} = \frac{(a+x)a - ax(1)}{(a+x)^2} = \frac{a^2}{(a+x)^2}$$

$$\frac{d^2y}{dx^2} = a^2\left[-2(a+x)^{-3}\right] = \frac{-2a^2}{(a+x)^3}$$

$$\rho = \frac{\left\{1+\left[\frac{a^2}{(a+x)^2}\right]^2\right\}^{3/2}}{\left|\frac{-2a^2}{(a+x)^3}\right|} = \left[1+\frac{a^4}{(a+x)^4}\right]^{3/2} \times \frac{(a+x)^3}{+2a^2}$$

$$\rho = \left[\frac{(a+x)^4+a^4}{(a+x)^4}\right]^{3/2} \times \frac{(a+x)^3}{+2a^2} = \frac{[(a+x)^4+a^4]^{3/2}}{[(a+x)^4]^{3/2}} \times \frac{(a+x)^3}{+2a^2}$$

$$\rho = \left[\frac{\{(a+x)^4+a^4\}^{3/2}}{(a+x)^{\cancel{6}\,3}}\right]\left[\frac{(a+x)^3}{+2a^2}\right]$$

$$\rho = +\frac{1}{2a^{\cancel{3}}(a+x)^3}\left[(a+x)^4+a^4\right]^{3/2}$$

$\times \frac{2}{a}$

$$\Rightarrow \frac{2\rho}{a} = \frac{\cancel{2}}{\cancel{2}\,a^3(a+x)^3}\left[(a+x)^4+a^4\right]^{3/2}$$

Taking the power $2/3$, on bothsides

$$\left(\frac{2\rho}{a}\right)^{2/3} = \frac{1}{\left[a^{\cancel{3}}(a+x)^{\cancel{3}}\right]^{2/\cancel{3}}}\left\{\left[(a+x)^4+a^4\right]^{\cancel{3}/\cancel{2}}\right\}^{\cancel{2}/\cancel{3}}$$

$$= \frac{(a+x)^4+a^4}{a^2(a+x)^2}$$

$$= \frac{(a+x)^{\cancel{4}\,2}}{a^2\cancel{(a+x)^2}} + \frac{a^{\cancel{4}\,2}}{\cancel{a^2}(a+x)^2}$$

$$= \left(\frac{a+x}{a}\right)^2 + \left(\frac{a}{a+x}\right)^2$$

Given $y = \frac{ax}{a+x}$

$$\Rightarrow \frac{y}{x} = \frac{a}{a+x} \text{ \& } \frac{x}{y} = \frac{a+x}{a}$$

$$\therefore \left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{x}{y}\right)^2 + (y/x)^2$$

Hence proved.

7) S.T. the measure of curvature of the curve $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$ at any point (x, y) is $\frac{ab}{2(ax+by)^{3/2}}$

<u>solution</u>

Gn. $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$

Differentiate w. r. to x

$$\frac{1}{\sqrt{a}}\,\frac{1}{2}\,x^{-1/2} + \frac{1}{\sqrt{b}}\,\frac{1}{2}\,y^{-1/2}\frac{dy}{dx} = 0$$

$$\frac{1}{\sqrt{a}}\frac{1}{\cancel{2}\sqrt{x}} = -\frac{1}{\sqrt{b}}\frac{1}{\cancel{2}\sqrt{y}}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sqrt{by}}{\sqrt{ax}} = -\frac{\sqrt{b}}{\sqrt{a}}\frac{\sqrt{y}}{\sqrt{x}}$$

Diff. w. r to x,

$$\frac{d^2y}{dx^2} = \frac{-\sqrt{b}}{\sqrt{a}}\left[\frac{\sqrt{x}\frac{1}{2\sqrt{y}}\left(\frac{dy}{dx}\right) - \sqrt{y}\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}\right]$$

$$= -\frac{\sqrt{b}}{\sqrt{a}}\frac{1}{x}\left[\frac{\cancel{\sqrt{x}}}{2\cancel{\sqrt{y}}}\left(\frac{-\sqrt{b}}{\sqrt{a}}\frac{\cancel{\sqrt{y}}}{\cancel{\sqrt{x}}}\right) - \frac{1}{2}\frac{\sqrt{y}}{\sqrt{x}}\right]$$

$$= \frac{\sqrt{b}}{2\sqrt{a}}\frac{1}{x}\left[\frac{\sqrt{b}}{\sqrt{a}} + \frac{\sqrt{y}}{\sqrt{x}}\right]$$

$$= \frac{\sqrt{b}}{2x\sqrt{a}}\left[\frac{\sqrt{bx}+\sqrt{ay}}{\sqrt{ax}}\right]$$

But $\frac{\sqrt{x}}{\sqrt{a}} + \frac{\sqrt{y}}{\sqrt{b}} = 1 \Rightarrow \frac{\sqrt{bx}+\sqrt{ay}}{\sqrt{ab}} = 1$

$$\Rightarrow \sqrt{bx}+\sqrt{ay} = \sqrt{ab}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\sqrt{b}}{2x\sqrt{a}}\left[\frac{\sqrt{\cancel{a}b}}{\sqrt{\cancel{a}x}}\right]$$

$$\frac{d^2y}{dx^2} = \frac{b}{2\sqrt{a}\,x^{3/2}}$$

$\therefore$ Radius of curvature $\rho = \frac{(1+y'^2)^{3/2}}{y''}$

$$\rho = \left[1+\left(\frac{-\sqrt{by}}{\sqrt{ax}}\right)^2\right]^{3/2} \times \left(\frac{2\sqrt{a}\,x^{3/2}}{b}\right)$$

$$= \left(1+\frac{by}{ax}\right)^{3/2} \times \frac{2\sqrt{a}\,x^{3/2}}{b}$$

$$= \left(\frac{ax+by}{ax}\right)^{3/2} \times \frac{2\sqrt{a}\,x^{3/2}}{b}$$

$$\rho = \frac{(ax+by)^{3/2}}{(ax)^{3/2}} \times \frac{2\sqrt{a}\, x^{3/2}}{b}$$

$$\rho = \frac{2(ax+by)^{3/2}}{ab}$$

$\therefore$ curvature $\kappa = \frac{1}{\rho} = \frac{ab}{2(ax+by)^{3/2}}$

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8) Prove that the radius of curvature at any point of a cycloid $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ is $4a\cos(\theta/2)$

Solution Given $x = a(\theta + \sin\theta)$

$y = a(1-\cos\theta)$

$\dot{x} = a(1+\cos\theta)$, $\dot{y} = a(\sin\theta)$

$\ddot{x} = a(-\sin\theta)$, $\ddot{y} = a\cos\theta$

the radius of curvature at any point θ is

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

$$\rho = \frac{\left[a^2(1+\cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}}{a(1+\cos\theta)a\cos\theta - a\sin\theta(-a\sin\theta)}$$

$$= \frac{\left[a^2(1+\cos^2\theta + 2\cos\theta) + a^2\sin^2\theta\right]^{3/2}}{a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta}$$

$$= \frac{\left[a^2 + 2a^2\cos\theta + a^2(\cos^2\theta + \sin^2\theta)\right]^{3/2}}{a^2\cos\theta + a^2(\cos^2\theta + \sin^2\theta)}$$

$$\rho = \frac{[2a^2 + 2a^2\cos\theta]^{3/2}}{a^2 + a^2\cos\theta} = \frac{[2a^2(1+\cos\theta)]^{3/2}}{a^2(1+\cos\theta)}$$

$$= 2^{3/2} a^3 (1+\cos\theta)^{3/2 - 1} a^{-2}$$

$$= 2^{3/2} a \sqrt{1+\cos\theta}$$

$$= 2^{3/2} a \sqrt{2\cos^2(\theta/2)}$$

$$= 2^2 a \cos(\theta/2)$$

$$\rho = 4a\cos(\theta/2)$$

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9) Find the radius of curvature at 't' for this curve $x = a(\cos t + t\sin t)$
$y = a(\sin t - t\cos t)$

<u>solution</u>

$x = a(\cos t + t\sin t)$

$\frac{dx}{dt} = a(-\sin t + \sin t + t\cos t)$
$= at\cos t$

$y = a(\sin t - t\cos t)$

$\frac{dy}{dt} = a(\cancel{\cos t} - \cancel{\cos t} + t\sin t)$

$\frac{dy}{dt} = at\sin t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \cancel{at}\sin t \times \frac{1}{\cancel{at}\cos t} = \tan t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{d}{dt}(\tan t)\frac{dt}{dx} = \sec^2 t \, \frac{1}{at\cos t} = \frac{\sec^3 t}{at}$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left[1 + (\tan t)^2\right]^{3/2}}{\frac{\sec^3 t}{at}}$$

$$= \frac{at}{\sec^3 t}\left(\sec^{\cancel{2}} t\right)^{3/\cancel{2}}$$

$$= \frac{at}{\sec^3 t}\sec^3 t$$

$$\rho = at$$

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Leibnitz theorem

A rule for finding the n^{th} derivative of the product of two functions.

It is expressed as a series of $(n+1)$ terms involving successive differentials of separate functions and binomial co-efficients, as follows

$$D^n(uv) = (D^n u)v + n(D^{n-1}u)(Dv) + \frac{n(n-1)}{1\cdot 2}(D^{n-2}u)(D^2 v)$$
$$+ \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}(D^{n-3}u)(D^3 v) + \dots + u(D^n v)$$

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1) Find the 4^{th} derivative of the function

$y = e^x \sin x$

Solution: Let $u = e^x$, $v = \sin x$

$$D^4(e^x \sin x) = D^4(e^x)\sin x + 4D^3(e^x)D(\sin x)$$
$$+ \frac{4\cdot 3}{1\cdot 2} D^2(e^x)D^2(\sin x) + \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3} D(e^x)D^3(\sin x)$$
$$+ \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4} e^x D^4(\sin x) \longrightarrow (1)$$

$$D(e^x) = \frac{d}{dx}(e^x) = e^x$$

$$D^2(e^x) = e^x, \quad D^3(e^x) = e^x, \quad D^4(e^x) = e^x$$

$$D(\sin x) = \cos x, \quad D^2(\sin x) = D[D(\sin x)]$$
$$= D(\cos x) = -\sin x$$

$$D^3(\sin x) = D[D^2(\sin x)] = D[-\sin x] = -\cos x$$

$$D^4(\sin x) = D[D^3(\sin x)] = D(-\cos x) = \sin x$$

$\therefore$ (1) becomes

$$D^4(e^x \sin x) = e^x \sin x + 4e^x (\cancel{\cos x}) + 6e^x(-\sin x) + 4e^x(\cancel{-\cos x}) + e^x(\sin x)$$

$$= e^x \sin x (2-6)$$

$$\boxed{D^4(e^x \sin x) = -4e^x \sin x}$$

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2) find the third derivative of the function $y = e^{2x} \ln x$

solution

Given $y = e^{2x} \ln x$

Let $u = e^{2x}$, $v = \ln x$, $n = 3$

$$D^3(uv) = (D^3u)v + 3(D^2u)(Dv) + \frac{3 \cdot \cancel{2}}{1 \cdot \cancel{2}}(Du)(D^2v) + \frac{\cancel{3 \cdot 2 \cdot 1}}{\cancel{1 \cdot 2 \cdot 3}} u D^3(v)$$

$$D^3(uv) = (D^3u)v + 3(D^2u)(Dv) + 3(Du)(D^2v) + uD^3(v) \quad \longrightarrow (1)$$

$u = e^{2x}$	$v = \ln x$
$Du = 2e^{2x}$	$Dv = \frac{1}{x}$
$D^2u = 4e^{2x}$	$D^2v = -\frac{1}{x^2}$
$D^3u = 8e^{2x}$	$D^3v = \frac{2}{x^3}$

$\therefore$ (1) becomes

$$D^3(e^{2x}\ln x) = 8e^{2x}\ln x + 3(4e^{2x})\left(\frac{1}{x}\right) + 3(2e^{2x})(-1/x^2) + e^{2x}\left(\frac{2}{x^3}\right)$$

$$D^3(e^{2x}\ln x) = e^{2x}\left[8\ln x + \frac{12}{x} - \frac{6}{x^2} + \frac{2}{x^3}\right] \quad //$$

Find the nth order derivative of the function $y = x^2 \cos x$

Solution

Assume that $u = \cos x$, $v = x^2$

$$D^n(uv) = (D^n u)v + n(D^{n-1}u)(Dv) + \frac{n(n-1)}{2!}(D^{n-2}u)(D^2 v) + \frac{n(n-1)(n-2)}{3!}(D^{n-3}u)(D^3 v) + \cdots + u(D^n v) \longrightarrow (1)$$

$v = x^2$

$Dv = 2x$

$D^2 v = 2$

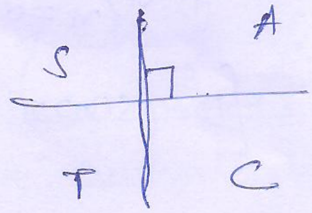
$\therefore$ (1) becomes

$$D^n(\cos x\, x^2) = (D^n \cos x)\, x^2 + n(D^{n-1}\cos x)(2x) + \frac{n(n-1)}{\cancel{2!}} D^{n-2}(\cos x)\,(\cancel{2}) \longrightarrow (2)$$

$D(\cos x) = -\sin x = \cos\left(\frac{\pi}{2} + x\right) = \cos\left(x + \frac{\pi}{2}\right)$

$D^2(\cos x) = -\cos x = \cos\left(\frac{2\pi}{2} + x\right) = \cos\left(x + \frac{2\pi}{2}\right)$

$D^3(\cos x) = \sin x = \cos\left(x + \frac{3\pi}{2}\right)$



$\vdots$

$D^{n-2}(\cos x) = \cos\left(x + \frac{(n-2)\pi}{2}\right)$

$D^{n-1}(\cos x) = \cos\left(x + \frac{(n-1)\pi}{2}\right)$

$D^n(\cos x) = \cos\left(x + \frac{n\pi}{2}\right)$

$$\therefore D^n(\cos x\, x^2) = \cos\left(x + \frac{n\pi}{2}\right)x^2 + n\cos\left(x + \frac{(n-1)\pi}{2}\right)(2x) + \frac{n(n-1)(\cancel{2})}{\cancel{2!}}\cos\left(x + \frac{(n-2)\pi}{2}\right)$$

$$D^n(\cos x\, x^2) = \cos\left(x + \frac{n\pi}{2}\right)x^2 + 2xn\cos\left(x + \frac{(n-1)\pi}{2}\right) + n(n-1)\cos\left(x + \frac{(n-2)\pi}{2}\right)$$

x

4) Find the 5th derivative of the function
$y = x^2 \sin 2x$ at $x=0$

Solution

Let $u = \sin 2x$, $v = x^2$

$D^5(\sin 2x\, x^2) = D^5(uv) = (D^5u)(v) + 5(D^4u)(Dv)$
$+ 10(D^3u)(D^2v) + 10(D^2u)(D^3v)$
$+ 5(Du)(D^4v) + uD^5(v)$

	$v = x^2$
$Du = \cos 2x(2) = 2\cos 2x$	$Dv = 2x$
$D^2u = -4\sin 2x$	$D^2v = 2$
$D^3u = -8\cos 2x$	$D^3v = 0$
$D^4u = 16\sin 2x$	
$D^5u = 32\cos 2x$	$(v)_{x=0} = 0$
$(Du)_{x=0} = 2$	$(Dv)_{x=0} = 0$
$(D^2u)_{x=0} = -4$	$(D^2v)_{x=0} = 2$
$(D^3u)_{x=0} = -8$	

$\left[D^5(\sin 2x\, x^2)\right]_{x=0} = 10(-8)(2) = -160$ //

x