Allied Mathematics I

UNIT I – MATRIX THEORY

Matrix: A rectangular array of numbers arranged in rows and columns is called a matrix and is denoted by capital letters.

Square Matrix: If the number of rows of a matrix is equal to the number of columns of a matrix then the matrix is called a square matrix.

Column matrix: A matrix contains only one column then the matrix is called a column matrix or a column vector.

Eigen values and Eigen Vectors

Definition: Let $A = [a_{ij}]$ be a square matrix of order n. If there exists a non-zero column vector X and a scalar λ such that

$$AX = \lambda X \longrightarrow (1)$$

then λ is called an eigenvalue of the matrix A and X is called the eigen vector corresponding to the eigenvalue λ.

The equation $|A - \lambda I| = 0$ is called the characteristic equation of A. The roots of the characteristic equation are called the characteristic roots or eigenvalues of A.

Corresponding to each value of λ, the equation (1) possess a non-zero solution X, X is called the eigenvector of A corresponding to the eigenvalue λ.

Properties of Eigenvalues

1. A square matrix A and its transpose A^T have the same eigenvalues.

2. The sum of the eigenvalues of a matrix A is equal to the sum of the principal diagonal elements of A.

3. The product of the eigenvalues of A is equal to $|A|$.

4. If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigenvalues of A, then
   (i) $k\lambda_1, k\lambda_2, \ldots k\lambda_n$ " " " " kA
   (ii) $\lambda_1^n, \lambda_2^n, \ldots \lambda_n^n$ " " " " A^n
   (iii) $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots \frac{1}{\lambda_n}$ " " " " A^{-1} if $|A| \neq 0$

5. A matrix A is singular $(|A| = 0)$ iff zero is a characteristic roots of A

6. The eigenvalues of a diagonal matrix are the diagonal elements.

7. The eigenvalues of an upper or lower triangular matrix are diagonal elements

Properties of Eigenvectors

1. If the eigen values of a matrix A are distinct, then the corresponding eigenvectors are linearly independent.

2. If the eigen values are equal, then the linearly independent eigenvectors may be linearly independent or linearly dependent.

WORKED EXAMPLES

1.) Find the sum and product of the eigen values of $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Solution

Sum of the eigenvalue of the given matrix
$= $ sum of the leading diagonal elements
$= 2+2+2 = 6$

And, Product of the eigenvalues $= |A|$

$$= \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= 2(4-0) - 0 + 1(0-2) = 8-2 = 6$$

$\therefore$ Sum of the eigenvalues $= 6$
& the product of the eigenvalues $= 6$

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2) It is given that two of the eigenvalues of the matrix $\begin{bmatrix} 2 & -2 & 2 \\ -2 & -1 & -1 \\ 2 & -1 & -1 \end{bmatrix}$ are equal. Using the properties of the eigenvalues, find the remaining eigenvalues of this matrix.

Solution

Given two of the eigenvalues are equal,
ie, $\lambda_1 = \lambda_2$.

Sum of the eigenvalues $=$ sum of the leading diagonal elements

$$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 2-1-1 = 0$$

but $\lambda_1 = \lambda_2$

$$\Rightarrow 2\lambda_2 + \lambda_3 = 0 \Rightarrow \boxed{\lambda_3 = -2\lambda_2} \longrightarrow (1)$$

Product of the eigenvalues $= |A|$

ie, $\lambda_1 \lambda_2 \lambda_3 = 2(1-1) + 2(2+2) + 2(2+2)$

$\lambda_2 \lambda_2 \lambda_3 = 2(4) + 2(4) \quad [\because \lambda_1 = \lambda_2]$

$\lambda_2^2 \lambda_3 = 16 \longrightarrow (2)$

Sub. (1) in (2) $\quad \lambda_2^2(-\cancel{2}\lambda_2) = \cancel{16}\ 8$

$\lambda_2^3 = -8 = (-2)^3$.

$\Rightarrow \lambda_2 = -2$

sub. $\lambda_2 = -2$ in (1), $\lambda_3 = -2(-2) = 4$.

$\therefore$ The eigenvalues of the given matrix are $-2, -2, 4$.

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3) Find the eigenvalues of A^3 where $A = \begin{bmatrix} 3 & 1 & 5 \\ 0 & 4 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

Solution: Since A is ~~an~~ upper triangular matrix, its eigenvalues are its leading diagonals.

$\therefore$ The eigenvalues of A are $3, 4, -1$.

$\therefore$ The " " A^3 are $3^3, 4^3, (-1)^3$

ie, $27, 64, -1$.

$\therefore$ The eigenvalues of A^3 are $27, 64, -1$.

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4) Find the eigenvalues of A^{-1} if the matrix A is $\begin{bmatrix} 2 & 5 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{bmatrix}$

Solution: Since A is an upper triangular matrix, its eigenvalues are its leading diagonals ie, $2, 3, 4$

& $|A| = 2(12) - 5(0) - 1(0) = 24$.

$\therefore$ The eigenvalues of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.

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5) If $-1, 2, 3$ are the eigenvalues of A, write down the eigenvalues of $4A$, A^2, $A+2I$, $A-3I$

Solution: Given: Eigenvalues of A are $-1, 2, 3$

$\therefore$ " " $4A$ are $4(-1), 4(2), 4(3)$

ie, $-4, 8, 12$

Eigenvalues of A^2 are $(-1)^2, 2^2, 3^2$

ie, $1, 4, 9$

Eigenvalues of $A+2I$ are $-1+2,\ 2+2,\ 3+2$

ie, $1, 4, 5$

Eigenvalues of $A-3I$ are $-1-3,\ 2-3,\ 3-3$

ie, $-4, -1, 0$.

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b) Determine the eigenvalues and eigenvectors of

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$$

Solution : Let $A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}$

To find the eigenvalues

the characteristic equation is $|A-\lambda I|=0$

ie, $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0 \longrightarrow (1)$

where D_1 = sum of the leading diagonals

D_2 = " " minors of leading diagonals

$D_3 = |A|$

$D_1 = 2+1-3 = 0$

$$D_2 = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ -7 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix}$$

$= (-3-2) + (-6+0) + (2-4) = -6-5-2 = -13$

$D_3 = |A| = 2(-3-2) - 2(-6+7) = 2(-5) - 2(1)$

$= -10-2 = -12$

Sub. D_1, D_2 & D_3 in (1), the characteristic equation is $\lambda^3 - 0\lambda^2 + (-13)\lambda - (-12) = 0$

$\lambda^3 - 13\lambda + 12 = 0 \longrightarrow (2)$

When $\lambda = 1$, $(2) \Rightarrow 1 - 13 + 12 = 0$. Divide equation (2) by 1.

By synthetic division,

$$\begin{array}{c|cccc} 1 & 1 & 0 & -13 & 12 \\ & & 1 & 1 & -12 \\ \hline & 1 & 1 & -12 & \boxed{0} \end{array}$$

$\lambda^2 + \lambda - 12 = 0$

$\Rightarrow \lambda = 3, -4$

$\therefore$ The eigenvalues of A are $1, 3, -4$

<u>To find the eigenvectors</u>:

<u>case 1</u> When $\lambda = 1$, $(A - \lambda I)X_1 = 0$ becomes

$$\left[\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix} - 1\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

ie, $$\begin{pmatrix} 2-1 & 2 & 0 \\ 2 & 1-1 & 1 \\ -7 & 2 & -3-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ -7 & 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$X_1 =$ The co-factor of $(A - \lambda I)$

$$= \begin{bmatrix} \begin{vmatrix} 0 & 1 \\ 2 & -4 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ -7 & -4 \end{vmatrix} \\ \begin{vmatrix} 2 & 0 \\ -7 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 0-2 \\ -(-8+7) \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

ie, $X_1 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$

Case 2 : When $\lambda = 3$, $(A-\lambda I)X_2 = 0$ becomes

$$\left[\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2-3 & 2 & 0 \\ 2 & 1-3 & 1 \\ -7 & 2 & -3-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} -1 & 2 & 0 \\ 2 & -2 & 1 \\ -7 & 2 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$X_2 =$ the co-factor of $(A-\lambda I)$

$$= \begin{bmatrix} \begin{vmatrix} -2 & 1 \\ 2 & -6 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ -7 & -6 \end{vmatrix} \\ \begin{vmatrix} 2 & -2 \\ -7 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 12-2 \\ -(-12+7) \\ 4-14 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ -10 \end{bmatrix}$$

i.e. $X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

Case 3 : When $\lambda = -4$, $(A-\lambda I)X_3 = 0$ becomes

$$\left(\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix} - (-4)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 6 & 2 & 0 \\ 2 & 5 & 1 \\ -7 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$X_3 =$ the co-factor of $(A-\lambda I) = \begin{bmatrix} \begin{vmatrix} 5 & 1 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ -7 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 5 \\ -7 & 2 \end{vmatrix} \end{bmatrix}$

$$= \begin{bmatrix} 3 \\ -(2+7) \\ 4+35 \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ 39 \end{bmatrix}$$

ie, $X_3 = \begin{bmatrix} 1 \\ -3 \\ 13 \end{bmatrix}$.

Hence the eigenvalues of A are 1, 3, −4 and the corresponding eigen vectors are $\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 13 \end{pmatrix}$

x

7. Find the eigenvalues and eigenvectors of $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Solution

Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

ie, $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0 \longrightarrow (1)$

where D_1 = sum of the leading diagonals

$= 2+3+2 = 7$

D_2 = sum of the minors of leading diagonals

$D_2 = \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = (6-2)+(4-1)+(6-2)$

$= 4+3+4 = 11$

$D_3 = |A| = 2(6-2) - 2(2-1) + 1(2-3)$

$= 2(4) - 2(1) - 1 = 8-2-1 = 5$

sub. D_1, D_2 & D_3 in (1), $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0 \rightarrow (2)$

when $\lambda = 1$, (2) becomes $1-7+11-5 = 0$

$\therefore$ 1 is a root of (2)

Dividing eqn. (2) by 1, using synthetic division we obtain

$$\begin{array}{c|cccc} 1 & 1 & -7 & 11 & -5 \\ & & 1 & -6 & 5 \\ \hline & 1 & -6 & 5 & \boxed{0} \end{array}$$

$\lambda^2 - 6\lambda + 5 = 0$

$\Rightarrow \lambda = 1, 5$ $\therefore$ The eigenvalues of A are 5, 1, 1

Case 1 : When $\lambda = 5$, $(A - \lambda I)X_1 = 0$ becomes

$$\left[\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

X_1 = Co-factor of I row of $(A - \lambda I)$

$$= \begin{bmatrix} \begin{vmatrix} -2 & 1 \\ 2 & -3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} \\ \begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 6-2 \\ -(-3-1) \\ 2+2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case 2 : When $\lambda = 1$, $(A - \lambda I)X_2 = 0$ becomes

$$\left[\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} - 1\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Rightarrow x_1 + 2x_2 + x_3 = 0$$

when $x_1 = 0$, $x_3 = -2x_2$, putting $x_2 = 1$, $x_3 = -2$

when $x_2 = 0$, $x_3 = -x_1$, putting $x_1 = 1$, $x_3 = -1$

$$\therefore X_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\therefore$ The eigenvalues of A are $5, 1, 1$ and the corresponding eigenvectors are $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

8) Find the eigenvalues and eigenvectors of $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Solution : Let $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

ie, $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0 \longrightarrow (1)$

where D_1 = Sum of the leading diagonals

$= 6+3+3 = 12$

D_2 = Sum of the minors of leading diagonals

ie, $D_2 = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$

$= (9-1) + (18-4) + (18-4) = 8 + 14 + 14 = 36$

$D_3 = |A| = 6(9-1) + 2(-6+2) + 2(2-6)$

$= 6(8) + 2(-4) + 2(-4) = 48 - 8 - 8 = 48 - 16 = 32$

Sub. D_1, D_2 & D_3 in (1), we obtain the characteristic equation.

$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0. \longrightarrow (2)$

when $\lambda = 2$, equation (2) becomes $2^3 - 12(2^2) + 36(2) - 32 = 0$

By synthetic division,

$$\begin{array}{r|rrrr} 2 & 1 & -12 & 36 & -32 \\ & & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & 0 \end{array}$$

$\lambda^2 - 10\lambda + 16 = 0$

$\lambda = 2, 8$

$\therefore$ The eigenvalues of A are $8, 2, 2$

To find the eigenvectors

Case 1 : When $\lambda = 8$, $(A - \lambda I)X_1 = 0$ becomes

$$\left(\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - 8 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

X_1 = co-factor of I row of $(A-\lambda I)$

$$= \begin{bmatrix} \begin{vmatrix} -5 & -1 \\ -1 & -5 \end{vmatrix} \\ -\begin{vmatrix} -2 & -1 \\ 2 & -5 \end{vmatrix} \\ \begin{vmatrix} -2 & -5 \\ 2 & -1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 25-1 \\ -(10+2) \\ 2+10 \end{bmatrix} = \begin{bmatrix} 24 \\ -12 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Case 2 : When $\lambda=2$, $(A-\lambda I)X_2=0$ becomes

$$\left(\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Rightarrow 2x_1 - x_2 + x_3 = 0$$

When $x_1=0$, $x_2=x_3$, putting $x_3=1$, $x_2=1$

when $x_2=0$, $x_3=-2x_1$, putting $x_1=1$, $x_3=-2$

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

$\therefore$ The eigenvalues of A are 8, 2, 2 and the corresponding eigenvectors are $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$.

Cayley-Hamilton Theorem

Every square matrix satisfies its own characteristic equation.

9. verify Cayley – Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ and also use it to find A^{-1}.

Solution

The characteristic equation is $|A - \lambda I| = 0$

ie, $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0 \longrightarrow (1)$

where $D_1 = 1+2+1 = 4$

$$D_2 = \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 7 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix}$$

$$= (2-6) + (1-7) + (2-12)$$

$$= -4 - 6 - 10 = -20$$

$$D_3 = |A| = 1(2-6) - 3(4-3) + 7(8-2)$$

$$= -4 - 3 + 7(6) = -7 + 42 = 35$$

Sub. D_1, D_2, D_3 in (1), $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$

Cayley – Hamilton theorem states that

$$A^3 - 4A^2 - 20A - 35I = 0 \longrightarrow (2)$$

$$A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

Substituting A, A^2, A^3 in (2)

$$\begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus Cayley – Hamilton theorem is verified.

Multiplying eqn. (2) by A^{-1}, $A^2 - 4A - 20I - 35A^{-1} = 0$

$$A^{-1} = \frac{1}{35}(A^2 - 4A - 20I)$$

$$= \frac{1}{35}\left[\begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix} - 4\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} - 20\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$A^{-1} = \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$$

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12. Use Cayley-Hamilton theorem to find the value of the matrix given by $(A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I)$, if the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Solution

The characteristic equation is $|A - \lambda I| = 0$

ie, $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$

where $D_1 = 2+1+2 = 5$

$$D_2 = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2 + (4-1) + (2-0)$$
$$= 2+3+2 = 7$$

$$D_3 = |A| = 2(2) - 1(0) + 1(0-1) = 3$$

$\therefore \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$ is the characteristic equation.

$A^3 - 5A^2 + 7A - 3I = 0$ (1) by Cayley-Hamilton Theorem.

$$\begin{array}{r|l} & A^5 + A \\ \hline A^3 - 5A^2 + 7A - 3I & A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I \\ & A^8 - 5A^7 + 7A^6 - 3A^5 \\ \hline & A^4 - 5A^3 + 8A^2 - 2A \\ & A^4 - 5A^3 + 7A^2 - 3A \\ & (-)\quad (-)\quad (-)\quad (-) \\ \hline & A^2 + A + I \end{array}$$

$\therefore A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$

$=(A^3-5A^2+7A-3I)(A^5+A)+A^2+A+I$

Using (1)

$LHS = 0(A^5+A)+A^2+A+I$

$$A^2=\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}=\begin{bmatrix}5&4&4\\0&1&0\\4&4&5\end{bmatrix}$$

$\therefore$ The given polynomial

$$=\begin{bmatrix}5&4&4\\0&1&0\\4&4&5\end{bmatrix}+\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}8&5&5\\0&3&0\\5&5&8\end{bmatrix}$$

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13. Find the eigenvalues of A and hence find A^n (n is a positive integer) given that $A=\begin{bmatrix}1&2\\4&3\end{bmatrix}$ and also find A^3.

Solution The characteristic equation of A is

$$|A-\lambda I|=0$$

ie $\lambda^2-D_1\lambda+D_2=0 \longrightarrow (1)$

where D_1 = sum of the main diagonals $=1+3=4$

$D_2=|A|=3-8=-5$

substituting D_1 & D_2 in (1) $\lambda^2-4\lambda-5=0$ is the characteristic equation.

$\therefore$ The eigenvalues of A are $-1, 5$

When λ^n divided by $(\lambda^2-4\lambda-5)$, let the quotient be $Q(\lambda)$ and remainder be $(a\lambda+b)$

then $\lambda^n=(\lambda^2-4\lambda-5)Q(\lambda)+(a\lambda+b) \longrightarrow (2)$

When $\lambda = -1$ in (1),

$$(-1)^n = a(-1)+b$$

ie, $-a+b = (-1)^n \longrightarrow (3)$

When $\lambda = 5$ in (1)

$$5^n = 5a+b \longrightarrow (4)$$

Solving (3) and (4), we get

$$-a+b = (-1)^n$$
$$\underline{\overset{(-)}{}\;5a+b = 5^n}$$
$$-6a = (-1)^n - 5^n$$

$$6a = 5^n - (-1)^n \Rightarrow a = \frac{5^n-(-1)^n}{6}$$

sub. a in (3),

$$b = (-1)^n + a = (-1)^n + \left[\frac{5^n-(-1)^n}{6}\right]$$

$$= \frac{6(-1)^n + 5^n - (-1)^n}{6}$$

$$b = \frac{5^n + 5(-1)^n}{6}$$

Replacing λ by A in (2),

$$A^n = (A^2-4A-5I)\,Q(A) + aA + bI$$

$$= 0 + \left[\frac{5^n-(-1)^n}{6}\right]\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix} + \left[\frac{5^n+5(-1)^n}{6}\right]\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

ie, $A^n = \left[\frac{5^n-(-1)^n}{6}\right]\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix} + \left[\frac{5^n+5(-1)^n}{6}\right]\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

To find A^3

$$A^3 = \left[\frac{5^3-(-1)^3}{6}\right]\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix} + \left[\frac{5^3+5(-1)^3}{6}\right]\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}21 & 42\\ 84 & 63\end{bmatrix} + \begin{bmatrix}20 & 0\\ 0 & 20\end{bmatrix} = \begin{bmatrix}41 & 42\\ 84 & 83\end{bmatrix}$$

———— × ————

12. Using Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ find A^4.

Solution

The characteristic equation is $|A-\lambda I| = 0$

ie, $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0 \longrightarrow (1)$

where $D_1 = 2+2+2 = 6$

$D_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$

$= (4-1) + (4-2) + (4-1) = 3+2+3 = 8$

$D_3 = |A| = 2(4-1) + 1(-2+1) + 2(1-2)$

$= 2(3) - 1 - 2 = 3$

Substituting $D_1, D_2 \& D_3$ in (1), then the characteristic equation is

$\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0 \longrightarrow (2)$

According to Cayley-Hamilton theorem, A satisfies equation (2), $A^3 - 6A^2 + 8A - 3I = 0 \rightarrow (3)$

$(3) \times A \Rightarrow A^4 - 6A^3 + 8A^2 - 3A = 0$

$\Rightarrow A^4 = 6A^3 - 8A^2 + 3A \longrightarrow (4)$

$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$

$A^3 = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$

$\therefore A^4 = 6\begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - 8\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} + 3\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

(9)

$$A^4 = \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

——— x ———