24

Non-linear Programming

"Each venture is a new beginning to explore something hidden"

24 : 1. INTRODUCTION

Like linear programming, *Non-linear Programming* is a mathematical technique for determining the optimal solutions to many business problems. In a *non-linear programming problem*, either the objective function is non-linear, or one or more constraints have non-linear relationship or both.

24 : 2. FORMULATING A NON-LINEAR PROGRAMMING PROBLEM (NLPP)

We consider some real-life problems, that we shall formulate as NLPPs.

SAMPLE PROBLEMS

2401. ***A company faces a responsive price-volume relationship for its products, the lower a product's price—the greater is the sales quantity, even in face of resultant price decreases by competitors. If the sales-revenue does not vary proportionately with price, reflect this phenomenon in the non-linear objective function of the price.***

Mathematical Formulation of the Problem. Let $x(p)$ represent the sales quantity as a function of the price p, say in the product-mix problem. Clearly, the associated sales revenue is $px(p)$. Now if the sales quantity function be given by the demand equation $x(p) = \alpha - \beta p$ for α, β constants, over the range of p, then the sales révenue component in the objective function is quadratic, $z = px(p) = \alpha p - \beta p^2$; in the decision variables p. If each unit costs c to produce (where p and c are in the same units) then total profit P is given by

$$P = z - cx(p) = \alpha p - \beta p - c\alpha + c\beta p = (\alpha + c\beta)p - c\alpha - \beta p^2.$$

2402. (Production Allocation Problem) *A manufacturing company produces two products : Radios and TV sets. Sales-price relationships for these two products are given below :*

Product	*Quantity demanded*	*Unit price*
Radios	1,500 − 5 p_1	p_1
TV sets	3,800 − 10 p_2	p_2

The total cost functions for these two products are given by $200x_1 + 0.1x_1^2$ *and* $300x_2 + 0.1x_2^2$ ***respectively. The production takes place on two assembly lines. Radio sets are assembled on Assembly line I and TV sets are assembled on Assembly line II. Because of the limitations of the assembly-line capacities, the daily production is limited to no more than 80 radio sets and 60 TV sets. The production of both types of products requires electronic components. The production of each of these sets requires five units and six units of electronic equipment components respectively. The electronic components are supplied by another manufacturer, and the supply is limited to 600 units per day. The company has 160 employees, i.e., the labour supply amounts to 160 man-days. The production of one unit of radio set requires 1 man-day of labour, whereas 2 man-days of labour are required for a TV set. How many units of radio and TV sets should the company produce in order to maximize the total profit? Formulate the problem as a non-linear programming problem.***

Mathematical Formulation of the Problem. Let us assume that whatever is produced is sold in the market. Let x_1 and x_2 stand for the quantities of radio sets and TV sets respectively, manufactured by the firm. Then we are given that

$$x_1 = 1{,}500 - 5p_1 \quad \text{or} \quad p_1 = 300 - 0.2\,x_1$$
$$x_2 = 3{,}800 - 10p_2 \qquad\quad p_2 = 380 - 0.1\,x_2$$

Further, if C_1, C_2 stand for the total cost of production of these amounts of radio sets and TV sets respectively, then we are also given that

$$C_1 = 200x_1 + 0.1x_1^2 \quad \text{and} \quad C_2 = 300x_2 + 0.1x_2^2$$

Now, the revenue on radio sets is $p_1 x_1$ and on TV sets is $p_2 x_2$. Thus the total revenue R is measured by

$$R = p_1x_1 + p_2x_2$$

which can be written as

$$R = (300 - 0.2x_1)\,x_1 + (380 - 0.1x_2)\,x_2$$
$$= 300x_1 - 0.2x_1^2 + 380x_2 - 0.1x_2^2.$$

The total profit z is measured by the difference between the total revenue R and the total cost $C = C_1 + C_2$. Thus

$$z = R - C_1 - C_2 = 100x_1 - 0.3x_1^2 + 80x_2 - 0.2x_2^2.$$

The objective function thus obtained is a non-linear function.

In the present case, production is influenced by the available resources. The two assembly lines have limited capacity to produce radio and TV sets. Since no more than 80 radio sets can be assembled on assembly line I and 60 TV sets on assembly line II per day, we have the restrictions : $x_1 \le 80$ and $x_2 \le 60$.

There is another side constraint in the daily requirement of the electronic components, so that $5x_1 + 6x_2 \le 600$. The number of available employees is limited to 160 man-days. Thus $x_1 + 2x_2 \le 160$. Also obviously, since the manufacturer cannot produce negative number of units, we must have $x_1 \ge 0$ and $x_2 \ge 0$.

Hence the given problem can be put in the following mathematical format :

> Determine two real numbers, x_1 and x_2 so as to maximize
>
> $$z = 100x_1 - 0.3x_1^2 + 80x_2 - 0.2x_2^2$$
>
> subject to the contraints :
>
> $$0 \le x_1 \le 80, \quad 0 \le x_2 \le 60, \quad 5x_1 + 6x_2 \le 600,$$
> $$x_1 + 2x_2 \le 160, \quad x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.$$

This problem is a non-linear programming problem, since the objective function is non-linear in x_1 and x_2.

Remarks. In a non-linear programming problem, the objective function z may be linear in x_1 and x_2, whereas the constraints are non-linear in x_1 and x_2, or both z and the constraints may be non-linear in x_1 and x_2. For example, the decision-making problem

Maximize $f(x_1, x_2) = 3x_1 + 5x_2$ subject to the constraints :

$$x_1 + x_2 \le 3, \quad x_1^2 + x_2^2 \le 10, \quad \text{and} \quad x_1, x_2 \ge 0$$

is a non-linear programming problem.

PROBLEMS

2403. (*One-Potato, Two-Potato Problem*) A frozen-food company processes potatoes into packages of French fries, hash browns and flakes (for meshed potatoes). At the beginning of the manufacturing process, the raw potatoes are sorted by length and quality, and then allocated to the separate product lines.

The company can purchase its potatoes from two sources, which differ in their yields of various sizes and quality. Each source yields different fractions of the products French fries, hash browns and flakes. Suppose that it is possible, at different costs, to alter these yields somewhat. Let f_1, f_2 and f_3 be the fractional yield per unit of weight of source *1* potatoes made into the three products, similarly, let

g_1, g_2 and g_3 be the yields for source 2. Suppose that each f_i and g_i can vary within $\pm 10\%$ of the yields shown below :

Product	Source I	Source 2	Purchase limitations
French fries	0.2	0.3	1.8
Hash browns	0.2	0.1	1.2
Flakes	0.3	0.3	2.4
Relative Profit	5	6	

Let $C_1(f_1, f_2, f_3)$ and $C_2(g_1, g_2, g_3)$ be the expense associated with obtaining these yields.

The problem is to determine how many potatoes should the company purchase from each source? Formulate the problem as a non-linear programming problem.

2404. A manufacturing concern operates its two available machines to polish its metal products. The two machines are equally efficient, although their maintenance costs are different. The daily maintenance and operation cost of the machines is given in rupees as the non-linear function :

$$f(x_1, x_2) = 100 - 1.2x_1 - 1.5x_2 + 0.3x_1^2 + 0.5x_2^2$$

where x_1 and x_2 are the number of hours of operation of machine I and machine II respectively.

The past records of the firm indicate that the combined operating hours of two machines should be at least 35 hours a day in order to perform a satisfactory job. However, the production manager wishes to operate machine I at least 6 hours more than machine II because of the higher repair cost of the latter. Find the optimal hours of operating the two machines and the minimum daily cost. Formulate the problem as a non-linear programming problem.

2405. A company manufactures two products A and B. It takes 30 minutes to process one unit of product A and 15 minutes for each unit of B and the maximum machine time available is 35 hours per week. Products A and B require 2 kgs. and 3 kgs. of raw material per unit respectively. The available quantity of raw material is envisaged to be 180 kgs. per week.

The products A and B which have unlimited market potential sell for Rs. 200 and Rs. 500 per unit respectively. If the manufacturing costs for products A and B are $2x^2$ and $3y^2$ respectively, find how much of each product should be produced per week, where

x = Quantity of Product A to be produced, and

y = Quantity of Product B to be produced.

2406. (***Portfolio Selection Problem***) An individual investor has an opportunity to invest a fixed amount of money in n different bonds and stocks. Let x_j be the proportion of his assets invested in the jth security. Then the vector $\mathbf{x} = (x_1, x_2, ..., x_n)$ is called a portfolio and the return R corresponding to a given portfolio $\mathbf{x}$ is a random variable. The investor is risk-averse and is therefore interested in determining a portfolio $\mathbf{x}$ that will minimise the variance of R subject to the restriction that his expected return is not less than some specified amount c (per unit invested). Formulate this portfolio selection problem as an NLPP.

24 : 3. GENERAL NON-LINEAR PROGRAMMING PROBLEM

Definition 1 **(General Non-linear Programming Problem).** *Let z be a real valued function of n variables defined by*

(a) $$z = f(x_1, x_2, ..., x_n).$$

Let $\{b_1, b_2, ..., b_m\}$ be a set of constants such that

(b) $$\begin{cases} g^1(x_1, x_2, ..., x_n) \ \{\le, \ge \text{ or } =\} \ b_1 \\ g^2(x_1, x_2, ..., x_n) \ \{\le, \ge \text{ or } =\} \ b_2 \\ \vdots \\ g^m(x_1, x_2, ..., x_n) \ \{\le, \ge \text{ or } =\} \ b_m \end{cases}$$

where g^i's are real valued functions of n variables, $x_1, ..., x_n$. Finally, let

(c) $$x_j \ge 0, \qquad j = 1, 2, ..., n.$$

If either $f(x_1, ..., x_n)$ or some $g^i(x_1, ..., x_n)$, $i = 1, 2, ..., m$; or both are non-linear, then the problem of determining the n-type $(x_1, x_2, ... x_n)$ which makes z a maximum or minimum and satisfies (b) and (c), is called a general non-linear programming problem (GNLPP).

In matrix notations the GNLPP may be written as :

Determine $\mathbf{x}^T \in R^n$ so as to maximize or minimize the objective function $z = f(x)$, subject to the constraints :

$$g^i(\mathbf{x}) \{\leq, \geq \text{ or } =\} b_i, \quad \mathbf{x} \geq 0 \qquad i = 1, 2, \ldots, m,$$

where either $f(\mathbf{x})$ or some $g^i(\mathbf{x})$ or both are non-linear in $\mathbf{x}$.

Sometimes it is convenient to write the constraints $g^i(x) \{\leq, \geq \text{ or } =\} b_i$ as $h^i(\mathbf{x}) \{\leq, \geq \text{ or } =\} 0$ for $h^i(x) = g^i(\mathbf{x}) - b_i$.

There is one simplex-like solution procedure for the solution of the general non-linear programming problem. However, numerous solution methods have been developed since the appearance of the fundamental theoretical paper by Kuhn and Tucker. A few primary types of available solution techniques will be discussed in this and the next chapter.

24 : 4. CONSTRAINED OPTIMIZATION WITH EQUALITY CONSTRAINTS

If the non-linear programming problem is composed of some differentiable objective function and equality side constraints, the optimization may be achieved by the use of *Lagrange multipliers** as illustrated below :

Consider the problem of maximizing or minimizing $z = f(x_1, x_2)$ subject to the constraints :

$$g(x_1, x_2) = c \quad \text{and} \quad x_1, x_2 \geq 0,$$

where c is a constant.

We assume that $f(x_1, x_2)$ and $g(x_1, x_2)$ are differentiable w.r.t. x_1 and x_2. Let us introduce differentiable function $h(x_1, x_2)$ differentiable w.r.t. x_1 and x_2 and defined by $h(x_1, x_2) \equiv g(x_1, x_2) - c$. Then the problem can be restated as

$$\text{Maximize } z = f(x_1, x_2) \text{ subject to the constraints :}$$

$$h(x_1, x_2) = 0 \quad \text{and} \quad x_1, x_2 \geq 0.$$

To find the necessary conditions for a maximum (or minimum) value of z, a new function is formed by introducing a Lagrange multiplier λ, as

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda h(x_1, x_2).$$

The number λ is an unknown constant, and the function $L(x_1, x_2, \lambda)$ is called the *Lagrangian function with Lagrange multiplier* λ. The necessary conditions for a maximum or minimum (stationary value) of $f(x_1, x_2)$ subject to $h(x_1, x_2) = 0$ are thus given by

$$\frac{\partial L(x_1, x_2, \lambda)}{\partial x_1} = 0, \quad \frac{\partial L(x_1, x_2, \lambda)}{\partial x_2} = 0 \text{ and } \frac{\partial L(x_1, x_2, \lambda)}{\partial \lambda} = 0.$$

Now, these partial derivatives are given by

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1},$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2}, \text{ and}$$

$$\frac{\partial L}{\partial \lambda} = -h,$$

where L, f and h stand for the functions $L(x_1, x_2, \lambda)$, $f(x_1, x_2)$, and $h(x_1, x_2)$ respectively, or simply by

$$L_1 = f_1 - \lambda h_1, \quad L_2 = f_2 - \lambda h_2 \quad \text{and} \quad L_\lambda = -h.$$

The *necessary conditions for maximum or minimum of* $f(x_1, x_2)$ are thus given by

$$f_1 = \lambda h_1, \quad f_2 = \lambda h_2 \quad \text{and} \quad -h(x_1, x_2) = 0$$

Note. These *necessary conditions* become sufficient conditions for a maximum (minimum) if the objective function is concave (convex) and the side constraints are in the form of equalities.

---

*The method of *Lagrange multipliers* is a systematic way of generating the necessary conditions for a stationary point.

SAMPLE PROBLEM

2407. **(*Input-Allocation Problem*)** *A manufacturing concern produces a product consisting of two raw materials, say A_1 and A_2. The production function is estimated as*

$$z = f(x_1, x_2) = 3.6x_1 - 0.4x_1^2 + 1.6x_2 - 0.2x_2^2$$

where z represents the quantity (in tons) of the product produced and x_1 and x_2 designate the input amounts of raw materials A_1 and A_2. The company has Rs. 50,000 to spend on these two raw materials. The unit price of A_1 is Rs. 10,000 and of A_2 is Rs. 5,000. Determine how much input amounts of A_1 and A_2 be decided so as to maximize the production output.

Solution. Since the company must operate within the available funds, the budgetary constraint is

$$10{,}000x_1 + 5{,}000x_2 \le 50{,}000 \quad \text{or} \quad 2x_1 + x_2 \le 10.$$

We reduce this inequality constraint to an equality one by imposing an additional assumption that the company has to spend every available single paisa on these raw materials. Then the constraint is $2x_1 + x_2 = 10$. Also, obviously $x_1 \ge 0$ and $x_2 \ge 0$. The problem of the company can thus be written as the following NLPP :

$$\text{Maximize } z = f(x_1, x_2) = 3.6x_1 - 0.4x_1^2 + 16x_2 - 0.2x_2^2 \text{ subject to the constraints :}$$

$$2x_1 + x_2 = 10 \quad \text{and} \quad x_1, x_2 \ge 0$$

or $\quad$ Maximize $z = f(x_1, x_2)$ subject to the constraints :

$$h(x_1, x_2) = 0 \quad \text{and} \quad x_1, x_2 \ge 0$$

where $h(x_1, x_2) = 2x_1 + x_2 - 10$. Observe that $f(x_1, x_2)$ and $h(x_1, x_2)$ are both differentiable w.r.t. x_1 and x_2. Also we observe that the objective function $z = f(x_1, x_2)$ is a concave function and the said constraint is an equality constraint. Therefore, the necessary and sufficient conditions for a maximum are

$$f_1 = \lambda h_1, \quad f_2 = \lambda h_2 \quad \text{and} \quad -h(x_1, x_2) = 0$$

That is,

$$3.6 - 0.8x_1 = 2\lambda, \quad 1.6 - 0.4x_2 = \lambda, \quad \text{and} \quad 2x_1 + x_2 = 10.$$

The first two of these yield

$$\lambda = 1.8 - 0.4x_1 = 1.6 - 0.4x_2$$

and so the elimination of λ gives $0.4x_1 - 0.4x_2 - 0.2 = 0$.

Now since $x_2 = 10 - 2x_1$, the last equation gives

$$0.4x_1 - 0.4(10 - 2x_1) - 0.2 = 0$$

or $$1.2x_1 - 4.2 = 0, \quad \text{or} \quad x_1 = 3.5$$

Thus $$x_2 = 10 - 2x_1 = 3.$$

The maximum value of the objective function is thus given by

$$z = f(3.5, 3) = 3.6(3.5) - 0.4(3.5)^2 + 1.6(3) - 0.2(3)^2$$
$$= 10.7 \text{ (tonnes)}.$$

Thus, in order to have a maximum production of 10.7 tonnes, the company must input 3.5 units of raw material A and 3 units of raw material B.

2408. *Obtain the necessary and sufficient conditions for the optimum solution of the following NLPP :*

$$\textit{Minimize } z = f(x_1, x_2) = 3e^{2x_1+1} + 2e^{x_2+5} \textit{ subject to the constraints :}$$

$$x_1 + x_2 = 7, \quad x_1, x_2 \ge 0.$$ *[Kerala M.Sc. (Math.) 2001]*

Solution. Let us introduce a new differentiable Lagrangian function $L(x_1, x_2, \lambda)$ defined by

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda(x_1 + x_2 - 7)$$
$$= 3e^{2x_1+1} + 2e^{x_2+5} - \lambda(x_1 + x_2 - 7)$$

where λ is the Lagrangian multiplier.

Since the objective function $z = f(x_1, x_2)$ is convex and the side constraint an equality one, the necessary and sufficient conditions for the minimum of $f(x_1, x_2)$ are given by

$$\frac{\partial L}{\partial x_1} = 6e^{2x_1+1} - \lambda = 0 \quad \text{or} \quad \lambda = 6e^{2x_1+1}$$

$$\frac{\partial L}{\partial x_2} = 2e^{x_2+5} - \lambda = 0 \quad \text{or} \quad \lambda = 2e^{x_2+5}$$

$$\frac{\partial L}{\partial \lambda} = -(x_1 + x_2 - 7) = 0 \quad \text{or} \quad x_1 + x_2 = 7$$

These imply $\quad 6e^{2x_1+1} = 2e^{x_2+5} = 3e^{7-x_1+5}$

or $\quad \log 3 + 2x_1 + 1 = 7 - x_1 + 5 \quad$ or $\quad x_1 = \frac{1}{3}(11 - \log 3).$

Thus $\quad x_2 = 7 - \frac{1}{3}(11 - \log 3).$

Necessary Conditions for a General NLPP

Consider the general NLPP :

Maximize (or minimize) $z = f(x_1, x_2, \ldots, x_n)$ subject to the constraints :

$$g^i(x_1, \ldots, x_n) = c_i \quad \text{and} \quad x_j \geq 0; \quad i = 1, 2, \ldots, m \ (< n)$$

The constraints can be reduced to

$$h^i(x_1, \ldots, x_n) = 0 \quad \text{for} \quad i = 1, 2, \ldots, m,$$

by the transformation $h^i(x_1, \ldots, x_n) = g^i(x_1, \ldots, x_n) - c_i$ for all $i = 1, 2, \ldots, m \ (< n)$.

The problem can then be written in the matrix form as

Maximize (or minimize) $z = f(\mathbf{x}), \ \mathbf{x} \in \mathbf{R}^n$ subject to the constraints :

$$h^i(\mathbf{x}) = 0, \ \mathbf{x} \geq 0.$$

To find the necessary conditions for a maximum or minimum of $f(\mathbf{x})$, the Lagrangian function $L(\mathbf{x}, \lambda)$, is formed by introducing m Lagrangian multipliers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$. This function is defined by

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i h^i(\mathbf{x}).$$

Assuming that L, f and h^i are all differentiable partially w.r.t. $x_1, x_2, \ldots, x_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_m$, the necessary conditions for a maximum (minimum) of $f(\mathbf{x})$ are :

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h^i(\mathbf{x})}{\partial x_j} = 0, \qquad j = 1, 2, \ldots, n$$

$$\frac{\partial L}{\partial \lambda_i} = -h^i(\mathbf{x}) = 0, \qquad i = 1, 2, \ldots, m.$$

These $m + n$ necessary conditions can be represented in the following abbreviated form :

$$L_j = f_j - \sum_{i=1}^{m} \lambda_i h^i_j = 0 \quad \text{or} \quad f_j = \sum_{i=1}^{m} \lambda_i h^i_j; \qquad j = 1, 2, \ldots, n$$

and

$$L\lambda_j = -h^i = 0 \quad \text{or} \quad h^i = 0; \qquad i = 1, 2, \ldots, m$$

where $\quad f_j = \dfrac{\partial f(\mathbf{x})}{\partial x_j}, \quad h^i = h^i(\mathbf{x}) \quad$ and $\quad h^i_j = \dfrac{\partial h^i(\mathbf{x})}{\partial x_j}.$

Remark. **These necessary conditions also become sufficient for a maximum (minimum) of the objective function if the objective function is concave (convex) and the side constraints are equality ones.**

SAMPLE PROBLEM

2409. ***Obtain the set of necessary conditions for the non-linear programming problem :***

Maximize $z = x_1^2 + 3x_2^2 + 5x_3^2$ *subject to the constraints :*

$$x_1 + x_2 + 3x_3 = 2, \quad 5x_1 + 2x_2 + x_3 = 5$$

$$x_1, x_2, x_3 \geq 0.$$

Solution. Here we have $\mathbf{x} = (x_1, x_2, x_3)$, $f(\mathbf{x}) = x_1^2 + 3x_2^2 + 5x_3^2$, $g^1(\mathbf{x}) = x_1 + x_2 + 3x_3$, $g^2(\mathbf{x}) = 5x_1 + 2x_2 + x_3$ and $c_1 = 2$, $c_2 = 5$. Defining $h^i(\mathbf{x}) = g^i(\mathbf{x}) - c_i$, $i = 1, 2$, we have the constraints : $h^i(\mathbf{x}) = 0$ for $i = 1, 2$.

The necessary conditions for the stationary point are

$$\frac{\partial L}{\partial x_1} = 4x_2 - 24 - \lambda = 0 \qquad \frac{\partial L}{\partial x_3} = 4x_3 - 8 - \lambda = 0$$

$$\frac{\partial L}{\partial x_3} = 4x_3 - 12 - \lambda = 0 \qquad \frac{\partial L}{\partial \lambda} = -(x_1 + x_2 + x_3 - 11) = 0.$$

The solution of the simultaneous equations yields the stationary point

$$\mathbf{x}_0 = (x_1, x_2, x_3) = (6, 2, 3)\,; \quad \lambda = 0.$$

The sufficient condition for the stationary point to be a minimum is that the minors Δ_3 and Δ_4 be both negative. Now, we have

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 4 \end{vmatrix} = -8 \quad \text{and} \quad \Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{vmatrix} = -48$$

which are both negative. Thus, $\mathbf{x}_0 = (6, 2, 3)$ provides the solution to the *NLLP.*

PROBLEM

2411. (*a*) Examine $z = 6x_1 x_2$ for maxima and minima under the requirement $2x_1 + x_2 = 10.$

(*b*) What happens if the problem becomes that of maximizing $z = 6x_1 x_2 - 10x_3$ under the constraint equation $3x_1 + x_2 + 3x_3 = 10.$

Sufficient Conditions for a General Problem with $m(<n)$ Constraints

Introducing the m Lagrange multipliers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, let the Lagrangian function for a general *NLPP* with more than one constraint be :

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i h^i(\mathbf{x}) \qquad (m < n)$$

The reader may verify that the equations

$$\frac{\partial L}{\partial x_i} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \lambda_i} = 0 \qquad (i = 1, 2, \ldots, m\,;\ j = 1, 2, \ldots, n)$$

yield the necessary conditions for stationary points of $f(\mathbf{x})$. Thus the optimization of $f(\mathbf{x})$ subject to $h(\mathbf{x}) = 0$ is equivalent to the optimization of $L(\mathbf{x}, \lambda)$. We state here the sufficiency conditions for the Lagrange multiplier method of stationary point of $f(\mathbf{x})$ to be a maxima or minima without proof. For this we assume that the function $L(\mathbf{x}, \lambda)$, $f(\mathbf{x})$ and $h(\mathbf{x})$ all possess partial derivatives of order one and two w.r.t. the decision variables.

Let ,
$$V = \left(\frac{\partial^2 L(\mathbf{x}, \lambda)}{\partial x_i\, \partial x_j}\right)_{n \times n}$$

be the matrix of second order partial derivatives of $L(\mathbf{x}, \lambda)$ w.r.t. decision variables

$$U = [\,h^i_j(\mathbf{x})\,]_{m \times n}$$

where $h^i_j(\mathbf{x}) = \dfrac{\partial h^i(\mathbf{x})}{\partial x_j}$, $i = 1, 2, \ldots, m\,;\ j = 1, 2, \ldots, n.$

Define the square matrix

$$\mathbf{H}^B = \left[\begin{array}{c|c} \mathbf{O} & \mathbf{U} \\ \hline \mathbf{U}^T & \mathbf{V} \end{array}\right]_{(m+n) \times (m+n)}$$

where $\mathbf{O}$ is an $m \times m$ null matrix. The matrix $\mathbf{H}^B$ is called the *bordered Hessian matrix*. Then, the sufficient conditions for maximum and minimum stationary points are given below :

Let $(\mathbf{x}_0, \lambda_0)$ for the function $L(\mathbf{x}, \lambda)$ be its stationary point. Let $\mathbf{H}^B_0$ be the corresponding bordered Hessian matrix computed at this stationary point. Then $\mathbf{x}_0$ is a

(*a*) maximum point, if starting with principal minor of order $(2m+1)$, the last $(n-m)$ principal minors of $\mathbf{H}^B_0$ from an alternating sign pattern starting with $(-1)^{m+n}$; and

(*b*) minimum point, if starting with principal minor of order $(2m+1)$, the list $(n-m)$ principal minors of $\mathbf{H}^B_0$ have the sign of $(-1)^m$.

Remark. It may be observed that the above conditions are only sufficient for identifying an extreme point, but not necessary. That is, a stationary point may be an extreme point without satisfying the above conditions.

SAMPLE PROBLEM

2412. *Solve the non-linear programming problem :*

Optimize $z = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$ *subject to the constraints :*

$$x_1 + x_2 + x_3 = 15, \quad 2x_1 - x_2 + 2x_3 = 20.$$

[Delhi B.Sc. (Stat.) 2002]

Solution. Here we have

$$f(\mathbf{x}) = 4x_1^1 + 2x_2^2 + x_3^2 - 4x_1x_2, \quad h^1(\mathbf{x}) = x_1 + x_2 + x_3 - 15$$

$$h^2(\mathbf{x}) = 2x_1 - x_2 + 2x_3 - 20.$$

Construct the Lagrangian function

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda_1 h^1(\mathbf{x}) - \lambda_2 h^2(\mathbf{x})$$

$$= (4x_1^1 + 2x_2^2 + x_3^2 - 4x_1x_2) - \lambda_1(x_1 + x_2 + x_3 - 15) - \lambda_2(2x_1 - x_2 + 2x_3 - 20)$$

The stationary point $(\mathbf{x}_0, \lambda_0)$ has thus given the following necessary conditions :

$$\frac{\partial L}{\partial x_1} = 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 = 0 \qquad \frac{\partial L}{\partial x_2} = 4x_2 - 4x_1 - \lambda_1 + \lambda_2 = 0$$

$$\frac{\partial L}{\partial x_3} = 2x_3 - \lambda_1 - 2\lambda_2 = 0 \qquad \frac{\partial L}{\partial \lambda_1} = -[x_1 + x_2 + x_3 - 15] = 0$$

$$\frac{\partial L}{\partial \lambda_2} = -[2x_1 - x_2 + 2x_3 - 20] = 0.$$

The solution to these simultaneous equations yields

$$\mathbf{x}_0 = (x_1, x_2, x_3) = (33/9, \ 10/3, \ 8) \quad \text{and} \quad \lambda_0 = (\lambda_1, \lambda_2) = (40/9, \ 52/9).$$

The bordered Hessian matrix at this solution $(\mathbf{x}_0, \lambda_0)$ is given by

$$\mathbf{H}^B_0 = \left[\begin{array}{cc:ccc} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ \hdashline 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{array}\right]$$

Here since $n = 3$ and $m = 2$, therefore $n - m = 1$, $(2m + 1 = 5)$. This means that one needs to check the determinant of $\mathbf{H}^B_0$ only it must have the sign of $(-1)^2$.

Now since det $\mathbf{H}^B_0 = 96 > 0$, $\mathbf{x}_0$ is a minimum point

PROBLEMS

Solve the following non-linear programming problems, using the method of Lagrangian multipliers.

2413. Minimize $z = 6x_1^2 + 5x_2^2$ **subject to the constraints :**

$$x_1 + 5x_2 = 3, \quad x_1, x_2 \geq 0.$$

[Kerala M.Sc. (Math.) 2001]

2414. Minimize $f(x_1, x_2) = 3x_1^2 + x_2^2 + 2x_1x_2 + 6x_1 + 2x_2$ **subject to the constraints :**

$$2x_1 + x_2 = 4, \quad x_1, x_2 \geq 0.$$

[Nagarjuna M.Sc. (Stat.) 1989]

Ans : $x_1 = 5$, $x_2 = 6$.

2415. Minimize $z = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$ subject to the constraints :

$$x_1 + x_2 + x_3 = 20, \quad x_1, x_2, x_3 \ge 0.$$

2416. Minimize $z = x_1^2 + x_2^2 + x_3^2$ subject to the constraints :

$$4x_1 + x_2^2 + 2x_3 = 14, \quad x_1, x_2, x_3 \ge 0.$$

2417. Minimize $z = x_1^2 + x_2^2 + x_3^2$ subject to the constraints :

$$x_1 + x_2 + 3x_3 = 2, \quad 5x_1 + 2x_2 + x_3 = 5, \quad x_1, x_2, x_3 \ge 0.$$

[Andhra M.E. (Mech. & Ind.) 1996]

2418. Maximize $z = 6x_1 + 8x_2 - x_1^2 - x_2^2$ subject to the constraints :

$$4x_1 + 3x_2 = 16, \quad 3x_1 + 5x_2 = 15, \quad x_1, x_2 \ge 0.$$

24 : 5. CONSTRAINED OPTIMIZATION WITH INEQUALITY CONSTRAINTS

We shall now derive the *Kuhn-Tucker Conditions* (necessary and sufficient) for the optimal solution of general *NLPP*. Consider the general *NLPP* :

Optimize $z = f(x_1, x_2, \ldots, x_n)$ subject to the constraints :

$$g(x_1, \ldots, x_n) \le C \quad \text{and} \quad x_1, x_2, \ldots, x_n \ge 0$$

where C is a constant.

Introducing the function $h(x_1, \ldots, x_n) = g - C$, the constraint reduces to $h(x_1, \ldots, x_n) \le 0$. The problem thus can be written as

Optimize $z = f(\mathbf{x})$ subject to $h(\mathbf{x}) \le 0$ and $\mathbf{x} \ge \mathbf{0}$, where $\mathbf{x} \in \mathbf{R}^n$.

We now slightly modify the problem by introducing new variable S, defined by $S^2 = -h(\mathbf{x})$, or $h(\mathbf{x}) + S^2 = 0$.

The new variable S is called a *slack variable* and appears as its square in the constraint equation so as to ensure its being non-negative. This avoids an additional constraint $S \ge 0$. Now the problem can be restated as

Optimize $z = f(\mathbf{x})$ $\mathbf{x} \in \mathbf{R}^n$ subject to the constraints :

$$h(\mathbf{x}) + S^2 = 0 \quad \text{and} \quad \mathbf{x} \ge \mathbf{0}.$$

This is a problem of constrained optimization in $n+1$ variables and a single equality constraint and can thus be solved by the Lagrangian multiplier method.

To determine the stationary points, we consider the Lagrangian function defined by

$$L(\mathbf{x}, S, \lambda) = f(\mathbf{x}) - \lambda\,[h(\mathbf{x}) + S^2],$$

where λ is the Lagrange multiplier. The necessary conditions for stationary points are

$$\frac{\partial L}{\partial x_j} \equiv \frac{\partial f}{\partial x_j} - \lambda \frac{\partial h}{\partial x_j} = 0 \quad \text{for } j = 1, 2, \ldots, n. \qquad \ldots(1)$$

$$\frac{\partial L}{\partial \lambda} \equiv -[h(\mathbf{x}) + S^2] = 0. \qquad \ldots(2)$$

$$\frac{\partial L}{\partial S} \equiv -2S\lambda = 0. \qquad \ldots(3)$$

Equation (3) states that $\frac{\partial L}{\partial S} = 0$, which requires either $\lambda = 0$ or $S = 0$. If $S = 0$, (2) implies that $h(\mathbf{x}) = 0$. Thus (2) and (3) together imply $\lambda\, h(\mathbf{x}) = 0$.

The variable S was introduced merely to convert the inequality constraint into an equality one, and therefore may be discarded. Moreover, since $S^2 \ge 0$, (2) gives $h(\mathbf{x}) \le 0$. Whenever $h(\mathbf{x}) < 0$, we get $\lambda = 0$ and whenever $\lambda > 0$, $h(\mathbf{x}) = 0$. However, λ is unrestricted in sign whenever $h(\mathbf{x}) = 0$.

The necessary conditions for the point $\mathbf{x}$ to be a point of maximum are thus restated as (in the abbreviated form) :

$f_j - \lambda h_j = 0$	$(j = 1, 2, \ldots, n)$
$\lambda h = 0$	*Maximize f*
$h \le 0$	*subject to :*
$\lambda \ge 0$*	$h \le 0.$

The set of such necessary conditions is called *Kuhn-Tucker Conditions.*

A similar argument holds for the minimization non-linear programming problem :

Minimize $z = f(\mathbf{x})$ subject to the constraints :

$$g(\mathbf{x}) \ge C \quad \text{and} \quad \mathbf{x} \ge \mathbf{0}.$$

Introduction of $h(\mathbf{x}) = g(\mathbf{x}) - C$ reduces the first constraint to $h(\mathbf{x}) \ge 0$. The new surplus variable S_0 can be introduced in $h(\mathbf{x}) \ge 0$ so that we may have the equality constraint $h(\mathbf{x}) - S_0^2 = 0$. The appropriate Lagrangian function is

$$L(\mathbf{x}, S_0, \lambda) = f(\mathbf{x}) - \lambda\,[h(\mathbf{x}) - S_0^2].$$

The following set of Kuhn-Tucker conditions is obtained :

$f_j - \lambda h_j = 0$	$(j = 1, 2, \ldots, n)$
$\lambda h = 0$	*Minimize f*
$h \ge 0$	*subject to :*
$\lambda \ge 0$	$h \ge 0.$

Theorem 24-1 **(Sufficiency of Kuhn-Tucker Conditions)** *The Kuhn-Tucker conditions for a maximization NLPP of Maximizing $f(\mathbf{x})$ subject to the constraints $h(\mathbf{x}) \le 0$ and $\mathbf{x} \ge \mathbf{0}$, are sufficient conditions for a maximum of $f(\mathbf{x})$, if $f(\mathbf{x})$ is concave and $h(\mathbf{x})$ is convex.*

Proof. The result follows if we are able to show that the Lagrangian function

$$L(\mathbf{x}, S, \lambda) = f(\mathbf{x}) - \lambda[h(\mathbf{x}) + S^2],$$

where S is defined by $h(\mathbf{x}) + S^2 = 0$, is concave in $\mathbf{x}$ under the given conditions.

In that case the stationary point obtained from the Kuhn-Tucker conditions must be the global maximum point.

Now, since $h(\mathbf{x}) + S^2 = 0$, it follows from the necessary conditions that $\lambda S^2 = 0$. Since $h(\mathbf{x})$ is convex and $\lambda \ge 0$, it follows that $\lambda h(\mathbf{x})$ is also convex and $-\lambda h(\mathbf{x})$ is concave. Thus, we conclude that $f(\mathbf{x}) - \lambda h(\mathbf{x})$ and hence $f(\mathbf{x}) - \lambda[h(\mathbf{x}) + S^2] = L(\mathbf{x}, S, \lambda)$ is concave in $\mathbf{x}$.

Remark. **By a similar argument it can be shown that for the minimization *NLPP*, Kuhn-Tucker conditions are also the sufficient conditions for the minimum of the objective function, if the objective function $f(\mathbf{x})$ is convex and the function $h(\mathbf{x})$ is concave.**

SAMPLE PROBLEM

2419. *Maximize* $z = 3.6x_1 - 0.4x_1^2 + 1.6x_2 - 0.2x_2^2$ *subject to the constraints :*

$$2x_1 + x_2 \le 10 \quad \text{and} \quad x_1, x_2 \ge 0.$$

Solution. Here

$$f(x) = 3.6x_1 - 0.4x_1^2 + 1.6x_2 - 0.2x_2^2$$

$$g(x) = 2x_1 + x_2, \quad c = 10$$

$$h(x) = g(x) - c = 2x_1 + x_2 - 10.$$

---

* More precisely since $\lambda = \dfrac{f_j}{h_j}\left(= \dfrac{\partial f/\partial x_j}{\partial h/\partial x_j} = \dfrac{\partial f}{\partial h}\right)$ measures the rate of variation of f w.r.t. h, then as the right-hand side of $h(\mathbf{x}) \le 0$ increases about zero, the solution space becomes less constrained and hence $f(\mathbf{x})$ cannot decrease. This means that $\lambda \ge 0$.

The Kuhn-Tucker conditions are :

$$\frac{\partial f(\mathbf{x})}{\partial x_1} - \lambda \frac{\partial h(\mathbf{x})}{\partial x_1} = 0, \quad \frac{\partial f(\mathbf{x})}{\partial x_2} - \lambda \frac{\partial h(\mathbf{x})}{\partial x_2} = 0 \quad \lambda h(\mathbf{x}) = 0, \quad h(\mathbf{x}) \le 0, \quad \lambda \ge 0$$

where λ is the Lagrangian multiplier.

That is,

$$3.6 - 0.8x_1 = 2\lambda \quad ...(1)$$

$$1.6 - 0.4x_2 = \lambda \quad ...(2)$$

$$\lambda [2x_1 + x_2 - 10] = 0 \quad ...(3)$$

$$2x_1 + x_2 - 10 \le 0 \quad ...(4)$$

$$\lambda \ge 0 \quad ...(5)$$

From equation (3) either $\lambda = 0$ or $2x_1 + x_2 - 10 = 0$.

Let $\lambda = 0$, then (2) and (1) yield $x_1 = 4.5$ and $x_2 = 4$. With these values of x_1 and x_2 however, (4) cannot be satisfied. Thus optimal solution cannot be obtained here for $\lambda = 0$. Let then $\lambda \ne 0$, which implies [from (3)] that $2x_1 + x_2 - 10 = 0$. This together with (1) and (2) yields the stationary value

$$\mathbf{x}_0 = (x_1, x_2) = (3.5, 3)$$

Now it is easy to observe that $h(\mathbf{x})$ is convex in $\mathbf{x}$, and $f(\mathbf{x})$ is concave in $\mathbf{x}$. Thus Kuhn-Tucker conditions are the sufficient conditions for the maximum. Hence $\mathbf{x}_0 = (3.5, 3)$ is the solution to the given NLPP. The maximum value of z (corresponding to $\mathbf{x}_0$) is given by

$$z_0 = 10.7.$$

Kuhn-Tucker Conditions for General NLPP with $m\ (< n)$ Constraints

Introducing $\mathbf{S} = (S_1, S_2, \ldots, S_m)$, let the Lagrangian function for a general *NLPP* with $m\ (< n)$ constraints be

$$L(\mathbf{x}, \mathbf{S}, \lambda) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i [h^i(\mathbf{x}) + S_i^2]$$

where $\lambda = (\lambda_1, \ldots, \lambda_m)$ are the Lagrangian multipliers.

The necessary conditions for $f(\mathbf{x})$ to be a maximum are :

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h^i}{\partial x_j} = 0 \quad \text{for } j = 1, 2, \ldots, n \quad ...(1)$$

$$\frac{\partial L}{\partial \lambda_i} = h^i + S_i^2 = 0 \quad \text{for } i = 1, 2, \ldots, m \quad ...(2)$$

$$\frac{\partial L}{\partial S_i} = -2S_i\lambda_i = 0 \quad \text{for } i = 1, 2, \ldots, m \quad ...(3)$$

where $L = L(\mathbf{x}, \mathbf{S}, \lambda)$, $f = f(\mathbf{x})$ and $h^i = h^i(\mathbf{x})$.

Equation (3) states that either $\lambda_i = 0$ or $S_i = 0$. By an argument parallel to that considered in the case of single inequality constraint; the conditions (3) and (2) together are replaced by the conditions (5), (6) and (7) below :

$$\lambda_i h^i = 0 \quad \text{for } i = 1, 2, \ldots, m \quad ...(5)$$

$$h^i \le 0 \quad \text{for } i = 1, 2, \ldots, m \quad ...(6)$$

$$\lambda_i \ge 0 \quad \text{for } i = 1, 2, \ldots, m \quad ...(7)$$

The Kuhn-Tucker conditions for a maximum may thus be restated as

$f_j = \sum_{i=1}^{m} \lambda_i h^i_j$	$(j = 1, 2, \ldots, n)$	
$\lambda_i h^i = 0$	$(i = 1, 2, \ldots, m)$	*Maximize f*
$h^i \le 0$	$(i = 1, 2, \ldots, m)$	*subject to :* $h^i \le 0$
$\lambda_i \ge 0$		
where $h^i = \frac{\partial h^i}{\partial x_j}$	$(i = 1, 2, \ldots, m)$.	

Theorem 24-2. **(Sufficiency of Kuhn-Tucker Conditions)** *For the NLPP of maximizing* $f(\mathbf{x})$, $\mathbf{x} \in \mathbf{R}^n$, *subject to the inequality constraints* $h^i(\mathbf{x}) \le 0$ $(i = 1, 2, \ldots, m)$, *the Kuhn-Tucker conditions are also the sufficient conditions for a maximum if* $f(\mathbf{x})$ *is concave and all* $h^i(\mathbf{x})$ *are convex functions of* $\mathbf{x}$.

Proof. Exercise for the reader.

The Kuhn-Tucker conditions for a minimization non-linear programming problem may be obtained in a similar manner. These conditions in that case come out to be :

$$f_j = \sum_{i=1}^{m} \lambda_i h_j^i \qquad (j = 1, 2, \ldots, n)$$

$$\lambda_i h^i = 0 \qquad \textit{Minimize } f$$

$$h^i \ge 0 \qquad \textit{subject to :}$$

$$\lambda_i \ge 0 \qquad h^i \ge 0 \qquad (i = 1, 2, \ldots, m).$$

It can be shown that for this minimization problem, Kuhn-Tucker conditions are also sufficient conditions for the minima if $f(\mathbf{x})$ is convex and all $h^i(\mathbf{x})$ are concave in $\mathbf{x}$, that is, $-h^i(\mathbf{x})$ are also all convex.

Note. If $f(\mathbf{x})$ is strictly concave (convex), the Kuhn-Tucker conditions are sufficient conditions for an absolute maximum (minimum).

Remarks 1. We may consider $\mathbf{x} \ge \mathbf{0}$ or $-\mathbf{x} \le \mathbf{0}$, to have been included in the inequality constraint $h^i(\mathbf{x}) \le 0$.

2. In both the maximization and minimization NLPP, the Lagrange multipliers λ_i corresponding to the equality constraints $h^i(\mathbf{x}) = 0$ must be unrestricted in sign.

3. A general NLPP may contain the constraints of the '≥' or '=' or '≤' type. In the case of maximization NLPP, all constraints must be converted into those of '≤' type and in the case of minimization NLPP, into those of '≥' type by suitable multiplication by −1.

SAMPLE PROBLEM

2420. *Determine* x_1, x_2 *and* x_3 *so as to*

$$\textit{Maximize} \quad z = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2 \quad \textit{subject to the constraints :}$$

$$x_1 + x_2 \le 2, \quad 2x_1 + 3x_2 \le 12, \quad x_1, x_2 \ge 0.$$

[IAS 1992]

Solution. Here $\quad f(\mathbf{x}) = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2 \qquad \mathbf{x} \in \mathbf{R}^n$

$$h^1(\mathbf{x}) = x_1 + x_2 - 2, \quad h^2(\mathbf{x}) = 2x_1 + 3x_2 - 12.$$

Clearly, $f(\mathbf{x})$ is concave* and $h^1(\mathbf{x})$, $h(\mathbf{x})$ are convex in $\mathbf{x}$. Thus the Kuhn-Tucker conditions will be the necessary and sufficient conditions for a maximum. These conditions are obtained by the partial differentiation of the Lagrangian function

$$L(\mathbf{x}, \mathbf{S}, \lambda) = f(\mathbf{x}) - \lambda_1 [h^1(\mathbf{x}) + S_1^2] - \lambda_2 [h^2(\mathbf{x}) + S_2^2]$$

where $\mathbf{S} = (S_1, S_2)$, $\lambda = (\lambda_1, \lambda_2)$, S_1, S_2 being slack variables and λ_1, λ_2 the Lagrange multipliers.

The Kuhn-Tucker conditions are given by

(1) $$f_j = \sum_{i=1}^{m} \lambda_i h_j^i \qquad (j = 1, 2, 3)$$

(2) $$\lambda_i h^i = 0 \qquad (i = 1, 2)$$

(3) $$h^i \le 0 \qquad (i = 1, 2)$$

(4) $$\lambda_i \ge 0 \qquad (i = 1, 2)$$

* The objective function is concave if the principal minors of bordered Hessian matrix, alternate in sign, beginning with the negative sign. If the principal minors are positive, the objective function is convex. In the present case

$$\mathbf{H}^B = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad n = 3, \; m = 2, \; |\mathbf{H}^B| < 0. \;\; \text{Thus } f(\mathbf{x}) \text{ is concave.}$$

Thus, in this problem, these are

(1) (i) $-2x_1+4=\lambda_1+2\lambda_2$ (ii) $-2x_2+6=\lambda_1+3\lambda_2$ (iii) $-2x_3=0$.

(2) (i) $\lambda_1(x_1+x_2-2)=0$ (ii) $\lambda_2(2x_1+3x_2-12)=0$

(3) (i) $x_1+x_2-2\le 0$ (ii) $2x_1+3x_2-12\le 0$

(4) $\lambda_1\ge 0,\ \lambda_2\ge 0$.

Now, there arise four cases :

Case 1. $\lambda_1=0$ *and* $\lambda_2=0$. (*i*), (*ii*) and (*iii*) yield $x_1=2,\ x_2=3,\ x_3=0$.

However, this solution violates (3) [(*i*) and (*ii*) both], and it must therefore be discarded.

Case 2. $\lambda_1=0$ *and* $\lambda_2\ne 0$. (2) yield $2x_1+3x_2=12$ and (1) (*i*) and (*ii*) yield $-2x_1+4=2\lambda_2$, $-2x_2+6=3\lambda_2$. The solution of these simultaneous equations yields $x_1=24/13$, $x_2=36/13$, $\lambda_2=2/13>0$; also (1) (*iii*) gives $x_3=0$. However, this solution violates (3) (*i*). This solution is also thus discarded.

Case 3. $\lambda_1\ne 0$ *and* $\lambda_2\ne 0$. (2) (*i*) and (*ii*) yield $x_1+x_2=2$ and $2x_1+3x_2=12$. These together yield $x_1=-6$ and $x_2=8$. Thus (1) (*i*), (*ii*) and (*iii*) give $x_3=0,\ \lambda_1=68,\ \lambda_2=-26$. However, this solution is to be discarded since $\lambda_2=-26$ violates (4).

Case 4. $\lambda_1\ne 0$ *and* $\lambda_2=0$. (2) (*i*) yield $x_1+x_2=0$. This together with (1) (*i*) and (*ii*) gives $x_1=1/2$ and $x_2=3/2$, $\lambda_1=3>0$. Further from (1) (*iii*) $x_3=0$. We observe that this solution does not violate any of the Kuhn-Tucker conditions.

Hence the optimum (maximum) solution to the given *NLPP* is

$$x_1=1/2,\ x_2=3/2,\ x_3=0 \text{ with } \lambda_1=3,\ \lambda_2=0,$$

the maximum value of the objective function being $z_0=17/2$.

PROBLEMS

Use the Kuhn-Tucker conditions to solve the following non-linear programming problems :

2421. Minimize $z=2x_1^2+12x_1x_2-7x_2^2$ subject to the constraints :

$$2x_1+5x_2\le 98,\ x_1,x_2\ge 0.$$

2422. Maximize $z=8x_1+10x_2-x_1^2-x_2^2$ subject to the constraints :

$$3x_1+2x_2\le 6,\ x_1\ge 0,\ x_2\ge 0$$ [*IAS 1991*]

2423. Minimize $z=x_1^2+x_2^2+x_3^2$ subject to the constraints :

$$2x_1+x_2\le 5,\ x_1+x_2\le 2,\ x_1\ge 1,\ x_2\ge 2,\ x_3\ge 0.$$ [*IAS 1993*]

2424. Minimize $z=0.3x_1^2-2x_1+0.4x_2^2-2.4x_2+0.6x_1x_2+100$ subject to the constraints :

$$2x_1+x_2\ge 4,\ x_1,x_2\ge 0.$$

2425. Minimize $z=-\log x_1-\log x_2$ subject to the constraints :

$$x_1+x_2\le 2 \text{ and } x_1\ge 0,\ x_2\ge 0.$$

2426. Minimize $z=2x_1+3x_2-x_1^2-2x_2^2$ subject to the conditions :

$$x+3x_1\le 6,\ 5x_1+2x_2\le 10, \text{ and } x_1\ge 0,\ x_2\ge 0.$$

Ans: $x_1=1$, $x_2=3/4$, Min $z=17/8$ [*Madurai B.E. (Electronics) 1990*]

2427. Maximize $z=2x_1-x_1^2+x_2$ subject to the constraints :

$$2x_1+3x_2\le 6,\ 2x_1+x_2\le 4 \text{ and } x_1,x_2\ge 0.$$ [*Dibrugarh M.Sc. (Stat.) 1994*]

2428. Maximize $z=3x_1+x_2$ subject to the constraints :

$$x_1^2+x_2^2\le 5,\ x_1-x_2\le 1 \text{ and } x_1\ge 0,\ x_2\ge 0.$$ [*Madras B.E. (Civil) 1991*]

2429. Maximize $z_1=8x_1^2+2x_2^2$ subject to the constraints :

$$x_1^2+x_2^2\le 9,\ x_1\le 2 \text{ and } x_1,x_2\ge 0.$$

2430. Minimize $f(x_1,x_2)=(x_1-1)^2+(x_2-5)^2$ subject to :

$$-x_1^2+x_2\le 4 \text{ and } -(x_1-2)^2+x_2\le 3.$$ [*Madurai B.E. (Electronics) 1989*]

25

Non-linear Programming Methods

"Let us recognise the fact that every complexity is explainable through echo of simplicity"

25 : 1. INTRODUCTION

As discussed earlier an LPP can easily be solved by simplex method or its variations. The optimum solution lies at one of the extreme points of the convex feasible region. But in a non-linear programming problem (NLPP), the optimum solution can be found anywhere on the boundary of the feasible region and even at some interior point of it. In spite of the substantial advancement in the solution methods of *NLPP* in recent years, an efficient simplex-like technique for a *GNLPP* is yet to be found. Some available techniques for solving some special cases of *GNLPP* shall be treated in the present chapter.

25 : 2. GRAPHICAL SOLUTION

The *graphical method* for the solution of an *NLPP* involving only two variables is best illustrated by the following sample problems :

SAMPLE PROBLEMS

2501. *Minimize the distance of the origin from the convex region bounded by the constraints :*

$$x_1 + x_2 \geq 4, \quad 2x_1 + x_2 \geq 5 \quad and \quad x_1 \geq 0, x_2 \geq 0.$$

Verify that the Kuhn-Tucker necessary conditions hold at the point of minimum distance.

Solution. The problem of minimizing the distance of the origin from the convex region is equivalent to minimizing the radius of a circle with origin as centre, say $r^2 = x_1^2 + x_2^2$ such that it touches (passes though) the convex region bounded by the given constraints. Thus, the problem is formulated as

$$\text{Minimize } z\,(= r^2) = x_1^2 + x_2^2 \text{ subject to the constraints :}$$

$$x_1 + x_2 \geq 4, \quad 2x_1 + x_2 \geq 5, \quad x_1, x_2 \geq 0.$$

Graphical Solution. Consider a set of rectangular cartesian axis OX_1X_2 in the plane. Each point has coordinates of the type (x_1, x_2) and conversely every ordered pair (x_1, x_2) of real numbers determine a point in the plane.

Clearly, any point which satisfies the conditions $x_1 \geq 0$ and $x_2 \geq 0$ lies in the first quadrant and conversely for any point (x_1, x_2) in the first quadrant, $x_1 \geq 0$, and $x_2 \geq 0$. Thus our search for the number pair (x_1, x_2) is restricted to the points in the first quadrant only. Now, since $x_1 + x_2 \geq 4$ and $2x_1 + x_2 \geq 5$, the desired point must be somewhere in the unbounded convex region *ABC* (shown shaded) in Fig. 25.1. Since our search is for such a pair (x_1, x_2) which gives a minimum value of $x_1^2 + x_2^2$ and lies in the convex

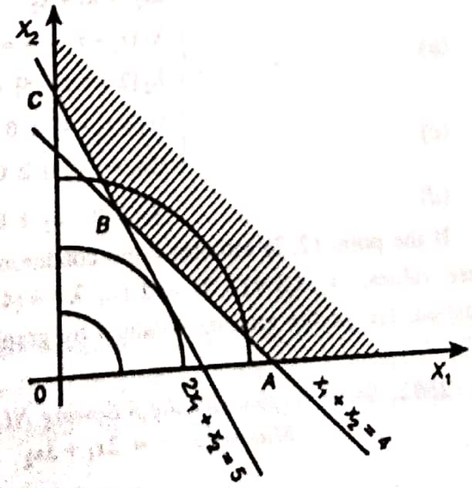


Fig. 25.1

region, the desired point will be that point of the region at which a side of the convex region is tangent to the circle. Then we proceed as follows :

Differentiating the equation of the circle, we have $2x_1dx_1 + 2x_2dx_2 = 0$, yielding

$$\frac{dx_2}{dx_1} = -\frac{x_1}{x_2} \qquad ...(1)$$

Considering the equalities $2x_1 + x_2 = 5$ and $x_1 + x_2 = 4$, we have on differentiation

$$2dx_1 + dx_2 = 0 \quad \text{and} \quad dx_1 + dx_2 = 0.$$

These yield

$$\frac{dx_2}{dx_1} = -2 \quad \text{and} \quad \frac{dx_2}{dx_1} = -1 \text{ respectively.} \qquad ...(2)$$

Thus from (1) and (2), we obtain

$$\frac{-x_1}{x_2} = -1 \text{ or } x_1 = x_2 \quad \text{and} \quad \frac{-x_1}{x_2} = -2 \text{ or } x_1 = 2x_2.$$

This shows that the circle $r^2 = x_1^2 + x_2^2$ has as a tangent to it

(i) the line $x_1 + x_2 = 4$ at the point (2, 2)

(ii) the line $2x_1 + x_2 = 5$ at the point (2, 1).

The point (2, 1) does not lie in the convex region and hence is to be discarded. Thus, the minimum distance from the origin to the convex region bounded by the constraints is

$$z_0 = 2^2 + 2^2 = 8 \text{ and is given by the point (2, 2).}$$

Verification of Kuhn-Tucker Conditions. We now verify that the above minima satisfy the Kuhn-Tucker conditions also. Here we have

$$f(\mathbf{x}) = x_1^2 + x_2^2, \quad h^1(\mathbf{x}) = x_1 + x_2 - 4, \quad h^2(\mathbf{x}) = 2x_1 + x_2 - 5$$

and the problem is that of minimizing $f(\mathbf{x})$ subject to the constraints $h^1(\mathbf{x}) \geq 0$, $h^2(\mathbf{x}) \geq 0$ and $\mathbf{x} \geq 0$. The Kuhn-Tucker conditions for this minimization *NLPP* are :

$$f_j(\mathbf{x}) = \lambda_1 h^1_j(\mathbf{x}) + \lambda_2 h^2_j(\mathbf{x}), \qquad j = 1, 2$$

$$\lambda_i h^i(\mathbf{x}) = 0 \qquad i = 1, 2$$

$$h^i(\mathbf{x}) \geq 0 \qquad i = 1, 2$$

$$\lambda_i \geq 0 \qquad i = 1, 2$$

where $f_j(\mathbf{x}) = \dfrac{\partial f(\mathbf{x})}{\partial x_j} = h^i_j(\mathbf{x}) = \dfrac{\partial h^i(\mathbf{x})}{\partial x_j}$, $(j = 1, 2)$, and λ_1, λ_2 are Lagrangian multipliers.

These conditions thus are as given below :

(a) $\begin{cases} 2x_1 = \lambda_1 + 2\lambda_2 \\ 2x_2 = \lambda_1 + \lambda_2 \end{cases}$

(b) $\begin{cases} \lambda_1 [x_1 + x_2 - 4] = 0 \\ \lambda_2 [2x_1 + x_2 - 5] = 0 \end{cases}$

(c) $\begin{cases} (x_1 + x_2 - 4) \geq 0 \\ (2x_1 + x_2 - 5) \geq 0 \end{cases}$

(d) $\lambda_1 \geq 0, \ \lambda_2 \geq 0.$

If the point (2, 2) satisfies these conditions, then we must have from (a), that $\lambda_1 = 4$, $\lambda_2 = 0$. With these values, $(x_1, x_2) = (2, 2)$ and $(\lambda_1, \lambda_2) = (4, 0)$, it is clear that the conditions (b), (c) and (d) are satisfied. Hence the minima obtained by graphical method satisfies the Kuhn-Tucker conditions for a minima.

2502. *Solve graphically the following NLPP :*

Maximize $z = 2x_1 + 3x_2$ *subject to the constraints :*

$$x_1 x_2 \leq 8, \quad x_1^2 + x_2^2 \leq 20, \quad x_1, x_2 \geq 0.$$

Verify that the Kuhn-Tucker conditions hold for the maxima you obtain.

Solution. In this *NLPP*, the objective function is linear whereas the constraints are non-linear. Consider a set of rectangular cartesian axes OX_1X_2 in the first quadrant only.

Now $x_1x_2 = 8$ represents a rectangular hyperbola with coordinate axes as its asymptotes; and $x_1^2 + x_2^2 = 20$ represents a circle of radius $\sqrt{20}$ with origin as its centre. Thus since $x_1x_2 \le 8$ and $x_1^2 + x_2^2 \le 20$, the desired point may be somewhere in the non-convex feasible region *OABCD* (shown shaded) in Fig. 25.2. Since our search is for such a pair (x_1, x_2) which gives a maximum value of $2x_1 + 3x_2$ and lies in the convex region, the desired point is obtained by moving parallel to $2x_1 + 3x_2 = k$, for some constant k, so long as $k = 2x_1 + 3x_2$ touches the extreme boundary point of feasible region. Thus in our problem the boundary point $C = (2, 4)$ corresponds to maximum z. Hence the optimal solution is $z_0 = 16$, $x_1 = 2$, $x_2 = 4$.

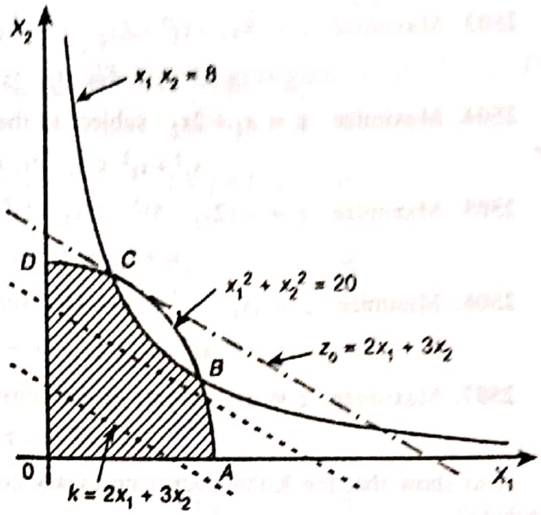


Fig. 25.2

Verification of Kuhn-Tucker Conditions. We now verify that the above maximum solution satisfies the Kuhn-Tucker conditions also.

Here we have

$$f(\mathbf{x}) = x_1 + 3x_2$$
$$h^1(\mathbf{x}) = x_1x_2 - 8$$
$$h^2(\mathbf{x}) = x_1^2 + x_2^2 - 20$$

and the problem is that of maximizing $f(\mathbf{x})$ subject to the constraints $h^1(\mathbf{x}) \le 0$, $h^2(\mathbf{x}) \le 0$ and $\mathbf{x} \ge 0$.

The Kuhn-Tucker conditions for this maximizing *NLPP* are

$$f_j(\mathbf{x}) = \lambda_1 h^1{}_j(\mathbf{x}) + \lambda_2 h^2{}_j(\mathbf{x})\,; \qquad j = 1, 2$$
$$\lambda_i h^i(\mathbf{x}) = 0 \qquad i = 1, 2$$
$$h^i(\mathbf{x}) \le 0 \qquad i = 1, 2$$
$$\lambda_i \ge 0 \qquad i = 1, 2$$

where $f_j(\mathbf{x}) = \dfrac{\partial f(\mathbf{x})}{\partial x_j}$, $h^i{}_j(\mathbf{x}) = \dfrac{\partial h^i(\mathbf{x})}{\partial x_j}$ for $j = 1, 2$, and λ_1, λ_2 are Lagrangian multipliers.

These conditions are thus given as

(*a*) $\begin{cases} 2 = \lambda_1 x_2 + 2\lambda_2 x_1 \\ 3 = \lambda_1 x_1 + 2\lambda_2 x_2 \end{cases}$

(*b*) $\begin{cases} \lambda_1 [x_1 x_2 - 8] = 0 \\ \lambda_2 [x_1^2 + x_2^2 - 20] = 0 \end{cases}$

(*c*) $\begin{cases} x_1 x_2 - 8 \le 0 \\ x_1^2 + x_2^2 - 20 \le 0 \end{cases}$

(*d*) $\lambda_1 \ge 0, \ \lambda_2 \ge 0.$

If the point (2, 4) satisfies these conditions then we must have from (*a*) $\lambda_1 = 1/6$ and $\lambda_2 = 1/3$. With these values, namely $(x_1, x_2) = (2, 4)$ and $(\lambda_1, \lambda_2) = (\frac{1}{6}, \frac{1}{3})$, it is obvious that the conditions (*b*), (*c*) and (*d*) are satisfied. Hence the maxima obtained by graphical method satisfy Kuhn-Tucker conditions for a maxima.

PROBLEMS

Solve the following non-linear programming problems graphically :

2503. Maximize $z = 8x_1 - x_1^2 + 8x_2 - x_2^2$ subject to the constraints :

$$x_1 + x_2 \le 12, \quad x_1 - x_2 \ge 4 \quad \text{and} \quad x_1, x_2 \ge 0.$$

2504. Maximize $z = x_1 + 2x_2$ subject to the constraints :

$$x_1^2 + x_2^2 \le 1, \quad 2x_1 + x_2 \le 2, \quad x_1, x_2 \ge 0.$$

2505. Maximize $z = -(2x_1 - 5)^2 - (2x_2 - 1)^2$ subject to :

$$x_1 + 2x_2 \le 1, \quad x_1 \ge 0 \quad \text{and} \quad x_2 \ge 0.$$

2506. Minimize $z = (x_1 - 2)^2 + (x_2 - 1)^2$ subject to the constraints :

$$-x_1^2 + x_2 \ge 0, \quad -x_1 - x_2 + 2 \ge 0, \quad x_1, x_2 \ge 0.$$

2507. Maximize $z = x_1$ subject to the constraints :

$$(1 - x_1)^3 - x_2 \ge 0, \quad x_1, x_2 \ge 0.$$

Also show that the Kuhn-Tucker necessary conditions for the maxima do not hold. What do you conclude?

2508. Maximize $z = x_1$ subject to the constraints :

$$(3 - x_1)^3 - (x_2 - 2) \ge 0, \quad (3 - x_1)^3 + (x_2 - 2) \ge 0, \quad x_1, x_2 \ge 0.$$

Also show that the Kuhn-Tucker necessary conditions for the maxima do not hold in this case.

25 : 3. KUHN-TUCKER CONDITIONS WITH NON-NEGATIVE CONSTRAINTS

In the preceding chapter, we obtained the *necessary* conditions for a point $\mathbf{x}^\circ \in \mathbf{R}^n$ to be a relative maximum of $f(\mathbf{x})$ subject to the constraints $h^i(\mathbf{x}) \le 0$, $i = 1, 2, \ldots, m$, $\mathbf{x} \ge 0$. These conditions, called Kuhn-Tucker conditions, were found by converting each inequality constraint to an equation through the addition of a squared slack variable, S_i^2, imposing the first-order conditions for maxima, on the first partial derivative of the Lagrangian function, and then simplifying the outcome. The following conditions resulted:

(*a*) $\quad f_j = \sum_{i=1}^{m} \lambda_i h^i_j \qquad j = 1, 2, \ldots, n$

(*b*) $\quad -\lambda_i h^i(\mathbf{x}) = 0 \qquad i = 1, 2, \ldots, m$

(*c*) $\quad h^i(\mathbf{x}) \le 0$ and $\qquad i = 1, 2, \ldots, m$

(*d*) $\quad \lambda_i \ge 0 \qquad i = 1, 2, \ldots, m.$

The reader may have observed that in obtaining these conditions, the non-negativity constraints $\mathbf{x} \ge \mathbf{0}$ were completely ignored. However, we always had in mind to discard all such solutions of (*a*) to (*d*) that violate $\mathbf{x} \ge \mathbf{0}$.

Now we shall consider the non-negativity constraint $\mathbf{x} \ge \mathbf{0}$ as one of the constraints, *viz.*, $h(\mathbf{x}) \ge 0$, where $h(\mathbf{x}) = \mathbf{x}$, and derive the Kuhn-Tucker conditions for the resulting problem.

We restate the problem as

$$\text{Maximize } z = f(\mathbf{x}) \quad \mathbf{x} \in \mathbf{R}^n \text{ subject to the constraints :}$$

$$h^i(\mathbf{x}) \le 0, \quad -\mathbf{x} \le \mathbf{0}; \qquad i = 1, 2, \ldots, m$$

Clearly, there are $m + n$ inequality constraints, and thus we add the squares of $(m+n)$ slack variables $S_1, \ldots, S_m, S_{m+1}, \ldots, S_{m+n}$ in the inequalities so as to convert them into equations ;

$$h^i(\mathbf{x}) + S_i^2 = 0 \qquad \text{for } i = 1, 2, \ldots, m$$

$$-x_j + S_{m+j}^2 = 0 \qquad \text{for } j = 1, 2, \ldots, n.$$

To find the necessary conditions for maximum of $f(\mathbf{x})$, we construct the associated Lagrangian function

$$L(\mathbf{x}, \mathbf{S}, \lambda) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i [h^i(\mathbf{x}) + S_i^2] - \sum_{j=1}^{n} \lambda_{m+j} [-x_j + S_{m+j}^2]$$

where $\mathbf{S} = (S_1, S_2, \ldots, S_{m+n})$; and $\lambda = (\lambda_1, \ldots, \lambda_{m+n})$ are the Lagrangian multipliers. The Kuhn-Tucker conditions are :

$$\frac{\partial L}{\partial x_j} = f_j - \sum_{i=1}^{m} \lambda_i h^i_j + \lambda_{m+j} = 0 \qquad \text{for } j = 1, 2, \ldots, n$$

$$\frac{\partial L}{\partial S_i} = -2\lambda_i S_i = 0 \qquad \text{for } i = 1, 2, \ldots, m$$

$$\frac{\partial L}{\partial S_{m+j}} = -2S_{m+j}\lambda_{m+j} = 0 \qquad \text{for } j = 1, 2, \ldots, n$$

$$\frac{\partial L}{\partial \lambda_i} = -(h^i(\mathbf{x}) + S_i^2) = 0 \qquad \text{for } i = 1, 2, \ldots, m$$

and

$$\frac{\partial L}{\partial \lambda_{m+j}} = -[-x_j + S_{m+j}^2] = 0 \qquad \text{for } j = 1, 2, \ldots, n.$$

The Kuhn-Tucker conditions, as obtained from these, upon simplification are

(*a*)	$f_j = \sum_{i=1}^{m} \lambda_i h^i_j - \lambda_{m+j}$	$(j = 1, 2, \ldots, n)$	
(*b*)	$\lambda_i [h^i(\mathbf{x})] = 0$	$(i = 1, 2, \ldots, m)$	*Maximize* f
(*c*)	$-\lambda_{m+j} x_j = 0$	$(j = 1, 2, \ldots, n)$	*subject to :*
(*d*)	$h^i(\mathbf{x}) \le 0$	$(i = 1, 2, \ldots, m)$	$h^i(\mathbf{x}) \le 0$
(*e*)	$\lambda_i, \lambda_{m+j}, x_j \ge 0$	$(i = 1, 2, \ldots, m ;$ $j = 1, 2, \ldots, n)$	$\mathbf{x} \ge \mathbf{0}$

Remarks. As before these conditions are sufficient also if $f(\mathbf{x})$ is concave and all $h^i(\mathbf{x})$ are convex in $\mathbf{x}$. Similarly, the Kuhn-Tucker conditions for *GNLPP* minimization case are sufficient also if $f(\mathbf{x})$ is convex and all $h^i(\mathbf{x})$ are concave in $\mathbf{x}$.

25 : 4. QUADRATIC PROGRAMMING

Quadratic programming is concerned with the *NLPP* of maximizing (or minimizing) the quadratic objective function, subject to a set of linear inequality constraints.

Definition. **(Quadratic programming problem)** *Let* $\mathbf{x}^T$ *and* $\mathbf{c} \in \mathbf{R}^n$. *Let* $\mathbf{Q}$ *be a symmetric* $n \times n$ *real matrix. Then, the problem of maximizing (determining* $\mathbf{x}$ *so as to maximize)*

$$f(\mathbf{x}) = \mathbf{cx} + \tfrac{1}{2}\mathbf{x}^T \mathbf{Q} \mathbf{x} \quad \textit{subject to the constraints}$$

$$\mathbf{Ax} \le \mathbf{b} \quad \textit{and} \quad \mathbf{x} \ge \mathbf{0}$$

where $\mathbf{b}^T \in \mathbf{R}^m$ *and* $\mathbf{A}$ *is an* $m \times n$ *real matrix, is called a* ***general quadratic programming problem.***

Remarks. $\mathbf{x}^T\mathbf{Q}\,\mathbf{x}$ represents a quadratic form. The reader may recall that a quadratic form $\mathbf{x}^T\mathbf{Q}\,\mathbf{x}$ is said to be *positive-definite (negative-definite)* if $\mathbf{x}^T\mathbf{Q}\,\mathbf{x} > 0\ (< 0)$ for $\mathbf{x} \ne \mathbf{0}$ and *positive-semi-definite (negative-semi-definite)* if $\mathbf{x}^T\mathbf{Q}\,\mathbf{x} \ge 0\ (\le 0)$ for all $\mathbf{x}$ such that there is one $\mathbf{x} \ne \mathbf{0}$ satisfying $\mathbf{x}^T\mathbf{Q}\,\mathbf{x} = 0$.

The reader may easily show that

1. If $\mathbf{x}^T\mathbf{Q}\mathbf{x}$ is positive-semi-definite (negative-semi-definite) then it is convex (concave) in $\mathbf{x}$ over all of $\mathbf{R}^n$, and
2. If $\mathbf{x}^T\mathbf{Q}\mathbf{x}$ is positive-definite (negative-definite) then it is strictly convex (strictly concave) in $\mathbf{x}$ over all of $\mathbf{R}^n$.

Non-Linear Programming

General Non-Linear programming Problem

Definition: Let 'z' be a real valued function of 'n' variables defined by $z = f(x_1, x_2, \ldots, x_n)$. —— (a)

Let $\{b_1, b_2, \ldots, b_m\}$ be a set of constraints such that

$$\left.\begin{array}{l} g^1(x_1, x_2, \ldots, x_n)\{\leq, \geq \text{ or } =\}\, b_1 \\ g^2(x_1, x_2, \ldots, x_n)\{\leq, \geq \text{ or } =\}\, b_2 \\ \vdots \\ g^m(x_1, x_2, \ldots, x_n)\{\leq, \geq \text{ or } =\}\, b_m \end{array}\right\} \text{—— (b)}$$

where g^i's are real valued functions of n variables $x_1, x_2, \ldots x_n$.

Finally let $x_j \geq 0$, $j = 1, 2, \ldots, n$ —— (c)

If either $f(x_1, x_2, \ldots, x_n)$ or some $g^i(x_1, x_2, \ldots x_n)$, $i = 1 \ldots m$ or both are non-linear, then the problem of determining the n-type $(x_1, x_2, \ldots, x_n)$ which makes 'z' a maximum or minimum and satisfies (b) and (c), is called a general non-linear programming problem (GNLPP).

Problem of Constrained Maxima and Minima

If the non-linear programming problem is composed of some differentiable objective function and equality side constraints, the optimization may be achieved by the use of Lagrange multipliers.

Constrained optimization with Equality Constraints :

Necessary Conditions for Maximum (Minimum) of Objective function (Two variables Case)

Let us consider the problem of

Maximizing or Minimizing $Z = f(x_1, x_2)$

S.t.c : $g(x_1, x_2) = c$ and $x_1, x_2 \geq 0$

where c is a constant.

We assume that $f(x_1, x_2)$ and $g(x_1, x_2)$ are differentiable w.r.t. x_1 and x_2. Let us introduce differentiable function $h(x_1, x_2)$ differentiable w.r.t. x_1 and x_2 and defined by $h(x_1, x_2) \equiv g(x_1, x_2) - c$.

Then the problem can be restated as

$$\text{Max } Z = f(x_1, x_2)$$

$$\text{s.t.c. } h(x_1, x_2) = 0 \text{ and } x_1, x_2 \geq 0.$$

To find the necessary conditions for a maximum (or minimum) value of Z, a new function is formed by introducing a Lagrange multiplier λ, as

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda h(x_1, x_2).$$

The number 'λ' is an unknown constant, and the function $L(x_1, x_2, \lambda)$ is called the Lagrangian function with Lagrange multiplier λ.

The necessary conditions for a maximum (or) minimum (Stationary value) of $f(x_1, x_2)$ subject to $h(x_1, x_2) = 0$ are thus given by

$$\frac{\partial L(x_1, x_2, \lambda)}{\partial x_1} = 0, \quad \frac{\partial L(x_1, x_2, \lambda)}{\partial x_2} = 0 \text{ and } \frac{\partial L(x_1, x_2, \lambda)}{\partial \lambda} = 0$$

Now, these partial derivatives are given by

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1}$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2}; \text{ and}$$

$$\frac{\partial L}{\partial \lambda} = -h, \text{ where } L = L(x_1, x_2, \lambda)$$

$$f = f(x_1, x_2)$$

$$\text{and } h = h(x_1, x_2).$$

(or) $L_1 = f_1 - \lambda h_1$

$L_2 = f_2 - \lambda h_2$ and $L_\lambda = -h$

The necessary conditions for max (or) min of $f(x_1, x_2)$ are thus given by $f_1 = \lambda h_1$, $f_2 = \lambda h_2$ and $-h(x_1, x_2) = 0$

Note: These necessary conditions become sufficient conditions for a max (min) if the obj. fun. is Concave (Convex) and the side constraints are in the form of equalities.

① Obtain the necessary and sufficient conditions for the optimum solution of the following NLPP

Minimize $z = f(x_1, x_2) = 3e^{2x_1+1} + 2e^{x_2+5}$

s.t.c. $x_1 + x_2 = 7$ and $x_1, x_2 \geq 0$.

Soln:

Let us introduce a new differentiable Lagrangian function $L(x_1, x_2, \lambda)$ defined by

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda(x_1 + x_2 - 7)$$

$$= 3e^{2x_1+1} + 2e^{x_2+5} - \lambda(x_1 + x_2 - 7)$$

Where λ is the Lagrangian multiplier.

Since the objective function $z = f(x_1, x_2)$ is convex and the side constraint an equality one, the necessary and sufficient conditions for the minimum of $f(x_1, x_2)$ are given by

$$\frac{\partial L}{\partial x_1} = 6e^{2x_1+1} - \lambda = 0 \quad (\text{or}) \quad \lambda = 6e^{2x_1+1}$$

$$\frac{\partial L}{\partial x_2} = 2e^{x_2+5} - \lambda = 0 \quad (\text{or}) \quad \lambda = 2e^{x_2+5}$$

$$\frac{\partial L}{\partial \lambda} = -(x_1 + x_2 - 7) = 0 \quad (\text{or}) \quad x_1 + x_2 = 7$$

These imply $6e^{2x_1+1} = 2e^{x_2+5} = 2e^{7-x_1+5}$ $\quad \because x_1 + x_2 = 7$, $x_2 = 7 - x_1$

$$\Rightarrow 3e^{2x_1+1} = e^{7-x_1+5}$$

Taking log on both sides

$$\log 3 + 2x_1 + 1 = 7 - x_1 + 5$$

$$(\text{or}) \quad 3x_1 = 11 - \log 3$$

Thus ... $x_1 = 3.3$

Min z = ~~12,227~~ 18,000

$x_2 = 7 - x_1 = 3.7$

~~...~~

1) A manufacturing Concern produces a product Consisting of two materials, say A_1 and A_2. The production function is estimated as

$$z = f(x_1, x_2) = 3.6x_1 - 0.4x_1^2 + 1.6x_2 - 0.2x_2^2$$

where z represents the quantity (in tons) of the product produced and x_1 and x_2 designate the input amounts of raw materials A_1 and A_2. The Company has Rs. 50,000 to spend on these two raw materials. The unit price of A_1 is Rs. 10,000 and of A_2 is Rs. 5,000. Determine how much input amounts of A_1 and A_2 be decided so as to maximize the production output.

Soln:

Since the Company must operate within available funds, the budgetary Constraint is

$$10{,}000x_1 + 5{,}000x_2 \le 50{,}000$$

(or) $$2x_1 + x_2 \le 10$$

We reduce this inequality Constraint to an equality one by imposing an additional assumption that the Company has to spend every available single paisa on these raw materials. Then the Constraint is $2x_1 + x_2 = 10$.

Thus the NLPP Can be written as,

$$\text{Max } z = f(x_1, x_2) = 3.6x_1 - 0.4x_1^2 + 1.6x_2 - 0.2x_2^2$$

s.t.c. $2x_1 + x_2 = 10$ and $x_1, x_2 \ge 0$.

(or) $$\text{Max } z = 3.6x_1 - 0.4x_1^2 + 1.6x_2 - 0.2x_2^2$$

s.t.c. $h(x_1, x_2) = 2x_1 + x_2 - 10 = 0$ and $x_1, x_2 \ge 0$.

Observe that $f(x_1, x_2)$ and $h(x_1, x_2)$ are both differentiable w.r.t x_1 & x_2. Also $Z = f(x_1, x_2)$ is a Concave function and the said Constraint is an equality Constraint.

Therefore the necessary & sufficient conditions for a maximum of are

$$f_1 = \lambda h_1, \quad f_2 = \lambda h_2 \text{ and } -h(x_1, x_2) = 0.$$

The Lagrangian function $L(x_1, x_2, \lambda)$ is defined by

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda(2x_1 + x_2 - 10)$$

$$= 3.6x_1 - 0.4x_1^2 + 1.6x_2 - 0.2x_2^2 - \lambda(2x_1 + x_2 - 10)$$

$$\frac{\partial L}{\partial x_1} = 3.6 - 0.8x_1 - 2\lambda = 0$$

$$\Rightarrow 3.6 - 0.8x_1 = 2\lambda \quad \text{——— (1)}$$

$$\frac{\partial L}{\partial x_2} = 1.6 - 0.4x_2 - \lambda = 0$$

$$\Rightarrow 1.6 - 0.4x_2 = \lambda \quad \text{——— (2)}$$

$$\frac{\partial L}{\partial \lambda} = -(2x_1 + x_2 - 10) = 0$$

$$\Rightarrow 2x_1 + x_2 = 10 \quad \text{——— (3)}$$

From (1) & (2), we have

$$1.8 - 0.4x_1 = 1.6 - 0.4x_2$$

$$\Rightarrow 0.4x_1 - 0.4x_2 - 0.2 = 0$$

From (3), put $x_2 = 10 - 2x_1$, we get

$$0.4x_1 - 0.4(10 - 2x_1) - 0.2 = 0 \Rightarrow x_1 = 3.5$$

Thus $2x_1 + x_2 = 10 \Rightarrow 2(3.5) + x_2 = 10 \Rightarrow x_2 = 3$.

The ∴ Max $z = f(3.5, 3) = 3.6(3.5) - 0.4(3.5)^2 + 1.6(3) - 0.2(3)^2$

$= 10.7$ (tonnes)

Thus, in order to have a max. production of 10.7 tonnes the company must input 3.5 units of A_1 and 3 units of A_2.

Necessary Conditions for a General NLPP

Consider the general NLPP:

Maximize (or minimize) $z = f(x_1, x_2, \dots, x_n)$

s.t.c. $g^i(x_1, \dots, x_n) = c_i$

and $x_i \geq 0$; $i = 1, 2, \dots, m (< n)$

The problem can then be written in the matrix form as

Max (or Min) $z = f(x)$; $x \in R^n$

s.t.c. $h^i(x) = 0$, $x \geq 0$.

To find the necessary conditions for a max or min of $f(x)$, the Lagrangian function $L(x, \lambda)$, is formed by introducing 'm' Lagrangian multipliers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$.

This function is defined by,

$$L(x, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i h^i(x)$$

Assuming that L, f and h^i are all differentiable partially w.r.t. $x_1, x_2, \dots, x_n$ and $\lambda_1, \lambda_2, \dots, \lambda_m$ the necessary conditions for a max (min) of $f(x)$ are:

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h^i(x)}{\partial x_j} = 0 \quad , \quad j = 1, 2, \dots, n$$

$$\frac{\partial L}{\partial \lambda_i} = -h^i(x) = 0 \quad , \quad i = 1, 2, \dots, m$$

These $m+n$ necessary conditions can be represented in the following form:

$L_j = f_j - \sum_{i=1}^{m} \lambda_i h_j^i = 0$ (or) $f_j = \sum_{i=1}^{m} \lambda_i h_j^i$, $j = 1, 2, \dots, n$

and $L_{\lambda_j} = -h^i = 0$ (or) $h^i = 0$

where $f_j = \frac{\partial f(x)}{\partial x_j}$, $h^i = h^i(x)$ and $h_j^i = \frac{\partial h^i(x)}{\partial x_j}$.

(i) Obtain the set of necessary conditions for the NLPP.

Maximize $z = x_1^2 + 3x_2^2 + 5x_3^2$

S.t.c. $x_1 + x_2 + 3x_3 = 2$

$5x_1 + 2x_2 + x_3 = 5$, $x_1, x_2, x_3 \geq 0$.

Soln:

Here we have $f(x) = x_1^2 + 3x_2^2 + 5x_3^2$

$g^1(x) = x_1 + x_2 + 3x_3$

$g^2(x) = 5x_1 + 2x_2 + x_3$

and $c_1 = 2$, $c_2 = 5$.

Defining $h^i(x) = g^i(x) - c_i$, $i = 1, 2$, we have

$h^1(x) = x_1 + x_2 + 3x_3 - 2$

$h^2(x) = 5x_1 + 2x_2 + x_3 - 5$

For necessary conditions for maximizing $f(x)$, we construct the Lagrangian function

$L(x, \lambda) = f(x) - \lambda_1 h^1(x) - \lambda_2 h^2(x)$

$= x_1^2 + 3x_2^2 + 5x_3^2 - \lambda_1(x_1 + x_2 + 3x_3 - 2) - \lambda_2(5x_1 + 2x_2 + x_3 - 5)$

This yields the following necessary conditions:

$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda_1 - 5\lambda_2 = 0$; $\frac{\partial L}{\partial x_2} = 6x_2 - \lambda_1 - 2\lambda_2 = 0$

$\frac{\partial L}{\partial x_3} = 10x_3 - 3\lambda_1 - \lambda_2 = 0$; $\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + 3x_3 - 2) = 0$

$\frac{\partial L}{\partial \lambda_2} = -(5x_1 + 2x_2 + x_3 - 5) = 0$.

$x_1 = \quad = 0.85$

$x_2 = \quad = 0.21$

$x_3 = \quad = 0.31$

Sufficient Conditions for a General NLPP with one Constraint

Let the Lagrangian function for a general NLPP involving 'n' variables and one constraint be :

$$L(x,\lambda) = f(x) - \lambda h(x)$$

The necessary conditions for a stationary point to be a maximum or minimum are

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial h}{\partial x_j} = 0 \quad , j = 1,2,\dots,n$$

$$\frac{\partial L}{\partial \lambda} = -h(x) = 0 \quad \text{and}$$

the value of λ is obtained by $\lambda = \frac{\partial f/\partial x_j}{\partial h/\partial x_j}$, $j = 1,2,\dots n$.

The sufficient conditions for a maximum or minimum require the evaluation at each stationary point, of $n-1$ principal minors of the determinant given below:

$$\Delta_{n+1} = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \cdots & \frac{\partial h}{\partial x_n} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_n} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial h}{\partial x_n} & \frac{\partial^2 f}{\partial x_n \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} - \lambda \frac{\partial^2 h}{\partial x_n^2} \end{vmatrix}$$

If $\Delta_3 > 0, \Delta_4 < 0, \Delta_5 > 0, \dots$, the signs pattern being alternate, the stationary point is a local maximum.

If $\Delta_3 < 0, \Delta_4 < 0, \dots \Delta_{n+1} < 0$, the sign being always negative, the stationary point is a local minimum.

Q. Solve the non LPP :

Minimize $z = 2x_1^2 - 24x_1 + 2x_2^2 - 8x_2 + 2x_3^2 - 12x_3 + 200$

s.t.c. $x_1 + x_2 + x_3 = 11, \quad x_1, x_2, x_3 \geq 0$.

Soln :

We formulate the Lagrangian function as

$$L(x_1, x_2, x_3, \lambda) = 2x_1^2 - 24x_1 + 2x_2^2 - 8x_2 + 2x_3^2 - 12x_3 + 200 - \lambda(x_1 + x_2 + x_3 - 11)$$

$$\frac{\partial L}{\partial x_1} = 4x_1 - 24 - \lambda = 0 \; ; \quad \frac{\partial L}{\partial x_2} = 4x_2 - 8 - \lambda = 0$$

$$\frac{\partial L}{\partial x_3} = 4x_3 - 12 - \lambda = 0 \; ; \quad \frac{\partial L}{\partial \lambda} = -(x_1 + x_2 + x_3 - 11) = 0.$$

The solution of the simultaneous equns. yields the stationary point

$$x_0 = (x_1, x_2, x_3) = (6, 2, 3) \; ; \quad \lambda = 0$$

The sufficient condition for the stationary point to be a minimum is that the minors Δ_3 and Δ_4 be both negative.

Now, we have

$$\Delta_3 = \Delta_{2+1} = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x_2^2} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 4 \end{vmatrix} = 0(16-0) - 1(4-0) + 1(0-4)$$

$$= 0 - 4 - 4 = -8.$$

$$\Delta_4 = \Delta_{3+1} = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_3} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_3} \\ \frac{\partial h}{\partial x_3} & \frac{\partial^2 f}{\partial x_3 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} - \lambda \frac{\partial^2 h}{\partial x_3^2} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{vmatrix}$$

$$= 0 \begin{vmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 0 & 4 \end{vmatrix} + 1 \begin{vmatrix} 1 & 4 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 4 \end{vmatrix} - 1 \begin{vmatrix} 0 & 4 & 0 \\ 1 & 0 & 4 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= 0 - 1\left[1(16-0) - 0 + 0\right] + 1\left[1(0) - 4(4-0) + 0\right] - 1\left[0 - 4(0-4) + 0\right]$$

$$= 0 - 16 - 16 - 16 = -48$$

$\therefore \Delta_3 < 0$ & $\Delta_4 < 0$. Thus $x_1 = 6$, $x_2 = 2$, $x_3 = 3$ provides the solution to the NLPP with Min $Z = 102$

Scanned with CamScanner

Sufficient Conditions for a General problem with $m(<n)$ Constraints

Introducing the 'm' Lagrange multipliers $\lambda = (\lambda_1, \lambda_2, \dots \lambda_m)$, let the Lagrangian function for a general NLPP with more than one Constraint be:

$$L(x, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i h^i(x)$$

Then the necessary conditions for stationary points of $f(x)$ is given by

$$\frac{\partial L}{\partial x_j} = 0 \text{ and } \frac{\partial L}{\partial \lambda_i} = 0 \qquad i = 1, 2, \dots m;\ j = 1, 2, \dots, n$$

Assume that the function $L(x, \lambda)$, $f(x)$ and $h(x)$ all possess partial derivatives of order one and two w.r.t. the decision variables.

Let, $$V = \left(\frac{\partial^2 L(x, \lambda)}{\partial x_i \partial x_j} \right)_{n \times n}$$

be the matrix of second order partial derivatives of $L(x, \lambda)$ w.r.t. decision variables

$$U = \left[h_j^i(x) \right]_{m \times n}$$

where $h_j^i(x) = \frac{\partial h^i(x)}{\partial x_j}$, $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$.

Define the square matrix

$$H^B = \left[\begin{array}{c|c} O & U \\ \hline U^T & V \end{array} \right]_{(m+n) \times (m+n)}$$

where O is an m×m null matrix. The matrix H^B is called the bordered Hessian matrix.

Then, the sufficient conditions for maximum and minimum stationary points are given below:

Let (x_0, λ_0) for the function $L(x, \lambda)$ be its stationary point. Let H_0^B be the corresponding bordered Hessian matrix computed at this stationary point. Then x_0 is a

(a) maximum point, if starting with principal minor of order $(2m+1)$, the last $(n-m)$ principal minors of H_0^B from an alternating sign pattern starting with $(-1)^{m+n}$; and

(b) minimum point, if starting with principal minor of order $(2m+1)$, the last $(n-m)$ principal minors of H_0^B have the sign of $(-1)^m$.

⊙ Solve the NLPP

Optimize $Z = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$

s.t.c. $x_1 + x_2 + x_3 = 15$

$2x_1 - x_2 + 2x_3 = 20$, $x_1, x_2, x_3 \geq 0$.

Soln:

Here we have

$f(x) = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$

$h^1(x) = g^1(x) - c_1 = x_1 + x_2 + x_3 - 15$

$h^2(x) = g^2(x) - c_2 = 2x_1 - x_2 + 2x_3 - 20$

~~Sufficient~~ Construct the Lagrangian function

$$L(x,\lambda) = f(x) - \lambda_1 h^1(x) - \lambda_2 h^2(x)$$

$$= (4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2) - \lambda_1(x_1 + x_2 + x_3 - 15) - \lambda_2(2x_1 - x_2 + 2x_3 - 20)$$

The stationay point (x_0, λ_0) has thus given the following necessary conditions:

$$\frac{\partial L}{\partial x_1} = 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 = 0 ; \quad \frac{\partial L}{\partial x_2} = 4x_2 - 4x_1 - \lambda_1 + \lambda_2 = 0$$

$$\frac{\partial L}{\partial x_3} = 2x_3 - \lambda_1 - 2\lambda_2 = 0 ; \quad \frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + x_3 - 15) = 0$$

$$\frac{\partial L}{\partial \lambda_2} = -[2x_1 - x_2 + 2x_3 - 20] = 0$$

The solution to these simultaneous equations yields

$$x_0 = (x_1, x_2, x_3) = (33/9, 10/3, 8) \text{ and } \lambda_0 = (\lambda_1, \lambda_2) = (40/9, 52/9)$$

The bordered Hessian matrix at this solution (x_0, λ_0) is given by

$$H^B = \left[\begin{array}{c|c} O & U \\ \hline U^T & V \end{array}\right] \Rightarrow H_0^B = \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ \hline 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{array}\right]$$

$$\left(\frac{\partial^2 f}{\partial x_1^2} - \lambda\frac{\partial^2 h}{\partial x_1^2} \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda\frac{\partial^2 h}{\partial x_1 \partial x_2} \quad \frac{\partial^2 f}{\partial x_1 \partial x_3} - \lambda\frac{\partial^2 h}{\partial x_1 \partial x_3}\right.$$

$$= 8[8-0] + 4(-8+0)$$

$$\left\{ \frac{\partial^2 f}{\partial x_1^2} \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} \quad \frac{\partial^2 f}{\partial x_1 \partial x_3} \right\}$$

Here since $n = 3$ and $m = 2$, therefore $n - m = 1$, $(2m + 1 = 5)$. This means that one needs to check the determinant of H_0^B only it must have the sign of $(-1)^2$.

Now since $\det H_0^B = 96 > 0$, x_0 is a minimum point.

Convex Functions

Defn. 1: Convex function

Let 'S' be a non-empty Convex subset of R^n. A function $f(x)$ on 'S' is said to be convex if for any two vectors x_1 and x_2 in S

$$f[\lambda x_1 + (1-\lambda) x_2] \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

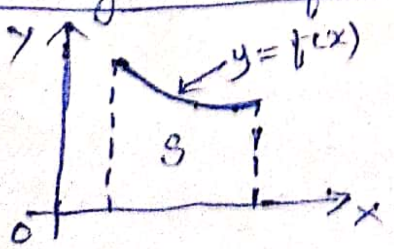
$$0 \leq \lambda \leq 1.$$

Defn. 2: Strictly Convex function

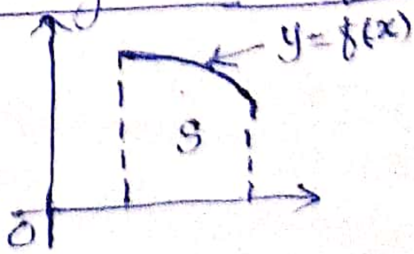
Let 'S' be a non-empty Convex subset of R^n. A function $f(x)$ on 'S' is said to be Strictly convex if for any two different vectors x_1 and x_2 in 'S'

$$f[\lambda x_1 + (1-\lambda) x_2] < \lambda f(x_1) + (1-\lambda) f(x_2)$$

Strictly Convex function



Strictly Concave function



Defn 3: Concave (Strictly Concave) function

A function $f(x)$ on a non-empty subset S of R^n is said to be Concave (Strictly Concave) if $-f(x)$ is Convex (Strictly Convex).