# UNIT-IV

## Initial value problems for Ordinary differential equations:

A number of problems in science and technology can be formulated into differential equations. The analytical methods of solving differential equations are applicable only to a limited class of equations. Quite often differential equations involved in physical problems do not belong to any of these standard types and one has to resort to numerical methods.

Solving an ordinary differential equation means finding an explicit expression for $y$ in terms of a finite number of elementary functions of $x$. Such a solution of a differential equation is known as the finite form of solution. In the absence of such a solution, we have shifted to numerical methods of solution. The differential equation together with the initial conditions is called an initial value problem.

Let us consider the first order differential equation.

$$\frac{dy}{dx} = f(x,y), \text{ given } y(x_0)=y_0 \longrightarrow (1)$$

To study the various numerical methods of solving such equations. In most of these methods, we replace the differential equation by a difference equation and then solve it. These methods yield solutions either as a power series in $x$ from which the values of $y$ can be found by direct substitution, or as a set of values of $x$ and $y$. The method of Taylor Series belongs to the former class of solutions.

In this method, $y$ in ① is approximated by a truncated Series, each term of which is a function of $x$. As such, this is reffered to as single step method. The methods of Euler, Runge-Kutta, Milne, Adams-Bashforth etc. belong to the latter class of solutions. In these methods, the next point is evaluated by performing iterations till sufficient accuracy is achieved. As such, these methods are called Step-by-Step methods.

Taylor's Series method:

Consider the first order differential equation.

$$\frac{dy}{dx} = f(x,y) \longrightarrow ①$$

with $y(x_0)=y_0$. Suppose we want to find the numerical Solution of the equation ①. $y(x)$ can be expanded about the point $x=x_0$ in a Taylor's Series as,

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!}(y'(x_0)) + \frac{(x-x_0)^2}{2!}(y''(x_0)) + \ldots\ldots$$

$$= y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \cdots$$

putting $x_1 = x_0+h$, ($h$ is the step-size) we have

$$y(x_1) = y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots \longrightarrow ②$$

In this equation $y_0', y_0'', y_0''' \cdots$ can be found by using ① and the successive differentiations.

Once $y_1$ has been calculated from ②, $y_1', y_1'' \ldots$ can be calculated from ①. So expanding $y(x)$ in a Taylor Series about $x=x_1$, we get,

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$$

where, $y_2 = y(x_2)$ and $x_2 = x_1 + h$.

Similarly expanding $y(x)$ at a general point $x_n$, we will get,

$$y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \cdots \rightarrow (3)$$

where, $y_n^r$ denotes the $r$th derivative of $y$ with respect to $x$ at the point $(x_n, y_n)$; $n = 0, 1, 2, \ldots$

Problem

1. By Taylor's series expansion, find $y$ at $x = 0.1, 0.2$ correct to 3 significant degits given $\frac{dy}{dx} - 2y = 3e^x$, $y(0) = 0$.

Solution.

Given, $x_0 = 0$, $y_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $h = 0.1$

Now,

$$y' = 2y + 3e^x$$

$$y'' = 2y' + 3e^x$$

$$y''' = 2y'' + 3e^x$$

$$y^{iv} = 2y''' + 3e^x$$

$$y_0' = 2y_0 + 3e^{x_0} = 2(0) + 3e^0 = 3$$

$$y_0'' = 2y_0' + 3e^{x_0} = 2(3) + 3 = 9$$

$$y_0''' = 2y_0'' + 3e^{x_0} = 2(9) + 3 = 21$$

$$y_0^{iv} = 2y_0''' + 3e^{x_0} = 2(21) + 3 = 45$$

∴ By Taylor's Series,

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

$$= 0 + \frac{0.1}{1!}(3) + \frac{(0.1)^2}{2!}(9) + \frac{(0.1)^3}{3!}(21) + \frac{(0.1)^4}{4!}(45)$$

$$y_1 = 0.349$$

$y_1' = 2y_1 + 3e^{x_1} = 2(0.349) + 3e^{0.1} = 4.014$

$y_1'' = 2y_1' + 3e^{x_1} = 2(4.014) + 3e^{0.1} = 11.344$

$y_1''' = 2y_1'' + 3e^{x_1} = 2(11.344) + 3e^{0.1} = 26.003$

$y_1^{iv} = 2y_1''' + 3e^{x_1} = 2(26.003) + 3e^{0.1} = 55.321$

$\therefore$ By Taylor's Series,

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{iv} + \cdots$$

$$= 0.349 + \frac{0.1}{1!}(4.014) + \frac{(0.1)^2}{2!}(11.344) + \frac{(0.1)^3}{3!}(26.003) + \frac{(0.1)^4}{4!}(55.321)$$

$y_2 = 0.812$

$y(0.1) = 0.349$

$y(0.2) = 0.812.$

2. Using Taylor Series method find y at $x = 0.1\,(0.1)\,(0.4)$ given $\frac{dy}{dx} = x^2 - y$, $y(0) = 1$ correct to 4 decimals.

Solution:

Given $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, $x_4 = 0.4$

Now,

$y' = x^2 - y \quad ; \quad y_0' = x_0^2 - y_0 = 0 - 1 = -1$

$y'' = 2x - y' \qquad y_0'' = 2x_0 - y_0' = 0 - (-1) = 1$

$y''' = 2 - y'' \qquad y_0''' = 2 - y_0'' = 2 - 1 = 1$

$y^{iv} = -y''' \qquad y_0^{iv} = -y_0''' = -1$

$\therefore$ By Taylors Series,

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv}$$

$y_1 = 1 + \frac{(0.1)}{1}(-1) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(1) + \frac{(0.1)^4}{24}(-1)$

$= 1 - 0.1 + 0.005 + 0.00016 - 0.0000042$

$y_1 = 0.9052$

$y_1' = x_1^2 - y_1 = 0.01 - 0.9052 = -0.8952.$

$y_1'' = 2x_1 - y_1' = 2(0.1) - (-0.8952) = 1.0952.$

$y_1''' = 2 - y_1'' = 2 - 1.0952 = 0.9048$

$y_1^{iv} = -y_1''' = -0.9048$

$\therefore$ By Taylor's series,

$y_2 = y_1 + \frac{(0.1)}{1!}(-0.8952) + \frac{(0.1)^2}{2!}(1.0952) + \frac{(0.1)^3}{3!}(0.9048) + \frac{(0.1)^4}{4!}(-0.9048)$

$= 0.9052 - 0.08952 + 0.00548 + 0.0001508 - 0.0000038$

$y_2 = 0.8213$

$y_2' = x_2^2 - y_2 = 0.04 - 0.8213 = -0.7813$

$y_2'' = 2x_2 - y_2' = 2(0.2) + 0.7813 = 1.1813$

$y_2''' = 2 - y_2'' = 2 - 1.1813 = 0.8187$

$y_2^{iv} = -y_2''' = -0.8187$

$\therefore$ By Taylor's series,

$y_3 = y_2 + \frac{(0.1)}{1!}(-0.7813) + \frac{(0.1)^2}{2!}(1.1813) + \frac{(0.1)^3}{3!}(0.8187) + \frac{(0.1)^4}{4!}(-0.8187)$

$= 0.8213 - 0.07813 + 0.000591 + 0.0001365 - 0.00000341$

$= 0.8220275 - 0.0781334$

$y_3 = 0.7492$

$y_3' = x_3^2 - y_3 = (0.3)^2 - 0.7492 = -0.6592$

$y_3'' = 2x_3 - y_3' = 2(0.3) + 0.6592 = 1.2592$

$y_3''' = 2 - y_3'' = 2 - 1.2592 = 0.7408$

$y_3^{iv} = -y_3''' = -0.7408$

$\therefore$ By Taylor's series,

$$y_4 = y_3 + \frac{(0.1)}{1!}(-0.6592) + \frac{(0.1)^2}{2!}(1.2592) + \frac{(0.1)^3}{3!}(0.7408) + \frac{(0.1)^4}{4!}(-0.7408)$$

$$= 0.7492 - 0.06592 + 0.006296 + 0.00012347 - 0.000003087$$

$y_4 = 0.6897$

$\therefore$ $y(0.1) = 0.9052$

$y(0.2) = 0.8213$

$y(0.3) = 0.7492$

$y(0.4) = 0.6897$

2. Using Taylor's Series method, find correct to 4 decimals, the value of $y(0.1)$ given $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$.

Solution:

Given, $x_0 = 0$, $y_0 = 1$, $x_1 = 0.1$

Now, $y' = x^2 + y^2$

$y'' = 2x + 2yy'$

$y''' = 2 + 2(yy'' + y'^2) = 2 + 2yy'' + 2y'^2$

$y^{iv} = 2(yy''' + y'y'') + 4y'y'' = 2yy''' + 6y'y''$

$y_0' = x_0^2 + y_0^2 = 0 + 1^2 = 1$

$y_0'' = 2x_0 + 2y_0y_0' = 2(0) + 2(1)(1) = 2$

$$y_0''' = 2 + 2y_0y_0'' + 2y_0'^2 = 2 + 2(1)(2) + 2(1)^2 = 8$$

$$y_0^{iv} = 2y_0y_0''' + 6y_0'y_0'' = 2(1)(8) + 6(1)(2) = 28$$

By Taylor's Series,

$$y_1 = y_0 + \frac{h}{1!}(y_0') + \frac{h^2}{2!}(y_0'') + \frac{h^3}{3!}(y_0''') + \frac{h^4}{4!}(y_0^{IV})$$

$$= 1 + \frac{0.1}{1}(1) + \frac{0.01}{2}(2) + \frac{0.001}{6}(8) + \frac{0.0001}{24}(28)$$

$$= 1 + 0.1 + 0.01 + 0.00133 + 0.0001166$$

$$y_1 = 1.11145$$

$$\therefore y(0.1) = 1.11145.$$

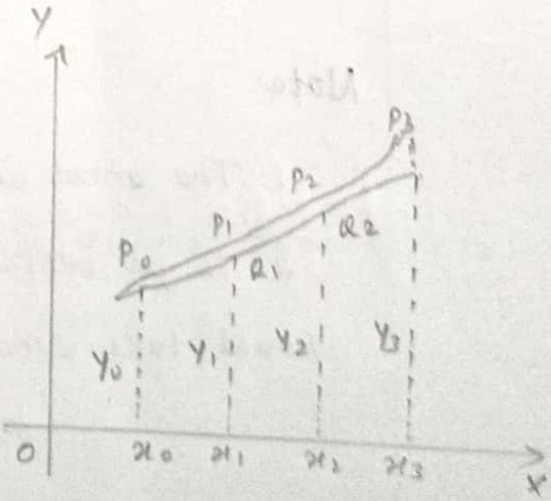
Euler's Method:

In this method, the actual curve of the Solution is approximated by a sequence of short striaght lines, given the initial value problem.

$$\frac{dy}{dx} = f(x,y);\ y(x_0) = y_0$$

Consider the points $x_0; x_1, x_2, \ldots$ where $x_1 = x_0 + h$, $x_2 = x_0 + 2h$ and so on. Let the actual solution of the differential equation be given by the curve below.

We wish to find the value of $y$ on the curve corresponding to $x = x_1$.




The equation of the tangent to the curve at $P_0(x_0, y_0)$ is,

$$y - y_0 = \left(\frac{dy}{dx}\right)_{(x_0, y_0)} (x - x_0)$$

$$= f(x_0, y_0) \cdot (x - x_0)$$

$$y = y_0 + f(x_0, y_0)(x - x_0)$$

We approximate the portion of the curve from $x_0$ to $x_1$ by this tangent.

Hence the approximate value of $y$ on the curve at $x=x_1$ ((i,e) at the point $P_1$) is given by the value of $y$ on the tangent, at $x=x_1$.

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

$$y_1 = y_0 + h f(x_0, y_0)$$

Since $x_1 - x_0 = h$.

Now we consider the tangent to the curve at $P_1$ and draw a line at $Q_1$. Whose slope is that of this tangent.

Then, as above, we get,

$$y_2 = y_1 + h f(x_1, y_1)$$

proceeding in a similar manner, we get Euler's formula (Euler's algorithm).

$$y_{n+1} = y_n + h f(x_n, y_n),\ n = 0, 1, 2, \cdots$$

Thus the actual curve $P_0\, P_1\, P_2\, P_3 \cdots$ is approximated by a sequence $P_0\, Q_1\, Q_2\, Q_3 \cdots$ of short straight lines. We notice that, as the $x$ value increases the new curve deviates more ~~fro~~ and more from the original curve. (i,e) the approximate value of $y$ derivates more from the actual value.

Note:

1. The error in Euler's method is of order $h^2$.

2. To get better accuracy with Euler's method, we should take smaller values of $h$.

Problem:

1. Find $y(1)$ given $y' = xy$, $y(0)=1$, taking $h=0.25$, using Euler's method.

Solution:

Given $f(x,y) = y' = xy$

$x_0 = 0,\ x_1 = 0.25,\ x_2 = 0.5,\ x_3 = 0.75,\ x_4 = 1,\ y_0 = 1,\ h = 0.25$

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$y(0.25) = y_1 = y_0 + hf(x_0, y_0)$$

$$= 1 + 0.25 \times (0 \times 1)$$

$$y(0.25) = 1$$

$$y(0.5) = y_2 = y_1 + hf(x_1, y_1)$$

$$= 1 + 0.25 \times (0.25 \times 1)$$

$$= 1.0625$$

$$y(0.75) = y_3 = y_2 + hf(x_2, y_2)$$

$$= 1.0625 + 0.25(0.5 \times 1.0625)$$

$$= 1.1953$$

$$y(1) = y_4 = y_3 + hf(x_3, y_3)$$

$$= 1.1953 + 0.25(0.75 \times 1.1953)$$

$$= 1.4194$$

2. Using Euler's method solve $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0)=1$ for, $x = (0)(0.02)(0.1)$.

Solution:

Given $f(x,y) = \frac{y-x}{y+x}$

$x_0 = 0,\ x_1 = 0.02,\ x_2 = 0.04,\ x_3 = 0.06,\ x_4 = 0.08,\ x_5 = 0.1$

$y_0 = 1,\ h = 0.02$

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$y(0.02) = y_1 = y_0 + hf(x_0, y_0)$$
$$= 1 + 0.02\, f(0,1)$$
$$= 1 + 0.02\left(\frac{1-0}{1+0}\right)$$
$$= 1.02$$

$$y(0.04) = y_2 = y_1 + hf(x_1, y_1)$$
$$= 1.02 + 0.02 f(0.02, 1.02)$$
$$= 1.02 + 0.02\left(\frac{1.02 - 0.02}{1.02 + 0.02}\right)$$
$$= 1.0392$$

$$y(0.06) = y_3 = y_2 + hf(x_2, y_2)$$
$$= 1.0392 + 0.02\, f(0.04, 1.0392)$$
$$= 1.0392 + 0.02\left(\frac{1.0392 - 0.04}{1.0392 + 0.04}\right)$$
$$= 1.05772$$

$$y(0.08) = y_4 = y_3 + hf(x_3, y_3)$$
$$= 1.05772 + 0.02\, f(0.06, 1.05772)$$
$$= 1.05772 + 0.02\left(\frac{1.05772 - 0.06}{1.05772 + 0.06}\right)$$
$$= 1.07557$$

$$y(0.1) = y_5 = y_4 + hf(x_4, y_4)$$
$$= 1.07557 + 0.02\, f(0.08, 1.07557)$$
$$= 1.07557 + 0.02\left(\frac{1.07557 - 0.08}{1.07557 + 0.08}\right)$$
$$= 1.0928$$

## Modified Euler's method:

In general, we have modified Euler's formula.

$$y_{n+1} = y_n + \frac{1}{2} h \left[ f(x_n, y_n) + f(x_n + h, y_n + h f(x_n, y_n)) \right]$$

1. Using Modified Euler's method find y at $x=0.1$ and $x=0.2$ given $\frac{dy}{dx} = y - \frac{2x}{y}$, $y(0)=1$.

Solution:

By modified Euler's method,

$$y_{n+1} = y_n + \frac{1}{2}h\left[f(x_n, y_n) + f(x_n+h, y_n + hf(x_n, y_n))\right]$$

given $f(x,y) = \frac{dy}{dx} = y - \frac{2x}{y}$

$x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $y_0 = 1$, $h = 0.1$

To find $y(0.1)$

$$f(x_0, y_0) = f(0,1) = 1 - \frac{2\times 0}{1} = 1$$

$$y(0.1) = y_1 = y_0 + \frac{1}{2}h\left[f(x_0,y_0) + f(x_0+h, y_0+hf(x_0,y_0))\right]$$

$$= 1 + \frac{1}{2}(0.1)\left[f(0,1) + f(0.1, 1 + (0.1 \times f(0,1)))\right]$$

$$= 1 + \frac{1}{2}(0.1)\left[1 + f(0.1, 1 + (0.1\times 1))\right]$$

$$= 1 + \frac{0.1}{2}\left[1 + f(0.1, 1.1)\right]$$

$$= 1 + \frac{0.1}{2}\left[1 + \left(1.1 - \frac{2\times 0.1}{1.1}\right)\right]$$

$$= 1 + 0.05\left[1 + 0.9182\right]$$

$$y(0.1) = 1.09591$$

To find $y(0.2)$

$$f(x_1, y_1) = f(0.1, 1.09591)$$

$$= 1.09591 - \left(\frac{2\times 0.1}{1.09591}\right)$$

$$= 0.91341$$

$$y(0.2) = y_1 + \frac{1}{2}h\left[f(x_1,y_1) + f(x_1+h, y_1+hf(x_1,y_1))\right]$$

$$= 1.09591 + \frac{1}{2}\times 0.1\left[0.91341 + f(0.2, 1.09591 + (0.1 \times 0.91341))\right]$$

$$= 1.09591 + \frac{0.1}{2}\left[0.91341 + f(0.2, 1.18725)\right]$$

$$y(0.2) = 1.09591 + 0.05\left[0.91314 + \left(1.18725 - \frac{2\times0.2}{1.18725}\right)\right]$$

$$= 1.09591 + 0.05\,(1.76375)$$

$$= 1.18410$$

$$\therefore y(0.1) = 1.09591$$

$$y(0.2) = 1.18410$$

2. Solve the equation $\frac{dy}{dx} = 1 - y$ with the initial condition $x = 0,\ y = 0$ using,

i) Euler's method

ii) Modified Euler's method and tabulate the solutions at $x = 0.1, 0.2, 0.3, 0.4$. compare with the exact solution.

Solution:

Given, $f(x,y) = 1 - y$

$x_0 = 0,\ x_1 = 0.1,\ x_2 = 0.2,\ x_3 = 0.3,\ x_4 = 0.4$

$y_0 = 0,\ h = 0.1$

i) Euler's method:

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$y(0.1) = y_1 = y_0 + hf(x_0, y_0)$$

$$= 0 + [0.1 \times f(0,0)]$$

$$= 0.1 \times (1-0)$$

$$= 0.1$$

$$y(0.2) = y_2 = y_1 + hf(x_1, y_1)$$

$$= 0.1 + [0.1 \times f(0.1, 0.1)]$$

$$= 0.1 + [0.1 \times (1 - 0.1)]$$

$$= 0.19$$

$y(0.3) = y_3 = y_2 + hf(x_2, y_2)$

$= 0.19 + (0.1 \times f(0.2, 0.19))$

$= 0.19 + (0.1 \times (1 - 0.19))$

$= 0.271$

$y(0.4) = y_4 = y_3 + hf(x_3, y_3)$

$= 0.271 + 0.1 \times f(0.3, 0.271)$

$= 0.271 + 0.1 \times (1 - 0.271)$

$= 0.3439$

ii) Modified Euler's method:

$$y_{n+1} = y_n + \frac{1}{2}h[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))]$$

To find $y(0.1)$

$f(x_0, y_0) = f(0, 0) = 1 - 0 = 1$

$y(0.1) = y_1 = y_0 + \frac{1}{2}h[f(x_0, y_0) + f(x_0 + h, y_0 + hf(x_0, y_0))]$

$= 0 + \frac{1}{2}(0.1)[1 + f(0.1, 0.1)]$

$= 0.05[1 + (1 - 0.1)]$

$= 0.095$

To find $y(0.2)$

$f(x_1, y_1) = f(0.1, 0.095) = 1 - 0.095 = 0.905$

$y(0.2) = y_2 = 0.095 + \frac{1}{2}(0.1)[0.905 + f(0.2, 0.1855)]$

$= 0.095 + 0.05[0.905 + (1 - 0.1855)]$

$= 0.1810$

To find $y(0.3)$

$f(x_2, y_2) = f(0.2, 0.1810) = 1 - 0.1810 = 0.8190$

$y(0.3) = 0.1810 + \frac{1}{2}(0.1)[0.8190 + f(0.3, 0.26290)]$

$= 0.1810 + \frac{1}{2}(0.1)[0.8190 + f(0.3, 0.26290)]$

$y(0.3) = 0.2588$

To find $y(0.4)$

$$f(x_3, y_3) = f(0.3, 0.2588) = 1 - 0.2588 = 0.7412$$

$$y(0.4) = 0.2588 + \frac{1}{2}(0.1)\left[0.7412 + f(0.4, 0.3329)\right]$$

$$= 0.2588 + 0.05\left[0.7412 + (1 - 0.3329)\right]$$

$$= 0.3292$$

Exact solution:

Given $\frac{dy}{dx} = 1 - y$

$$(D+1)y = 1$$

Auxillary equation is $m + 1 = 0$

$\therefore$ complementary function is $Ae^{-x}$.

$$\text{Particular integral} = \frac{1}{D+1} e^{0x}$$

$$= \frac{1}{0+1} e^{0x} = 1$$

$\therefore$ solution is $y = Ae^{-x} + 1 \rightarrow (1)$

given, when $x = 0$, $y = 0$. Substituting in equation (1).

$$0 = Ae^{-0} + 1$$

$$A = -1$$

$$y = -e^{-x} + 1 = 1 - e^{-x}$$

$$y(0) = 1 - e^{-0} = 0$$

$$y(0.1) = 1 - e^{-0.1} = 0.09516$$

$$y(0.2) = 1 - e^{-0.2} = 0.18127$$

$$y(0.3) = 1 - e^{-0.3} = 0.25918$$

$$y(0.4) = 1 - e^{-0.4} = 0.32967$$

The above results are tabulated below.

| $x$ | $y$ (Euler's method) | $y$ (modified Euler's method) | $y$ (Exact solution) |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0.1 | 0.1 | 0.0950 | 0.09516 |
| 0.2 | 0.19 | 0.1810 | 0.18127 |
| 0.3 | 0.271 | 0.2588 | 0.25918 |
| 0.4 | 0.3439 | 0.3292 | 0.32967 |

The Solutions got by modified Euler's method are very close to the exact Solutions.

Runge-Kutta methods:

A Runge-Kutta formula of order $n$ for solving the first order ODE $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ is a formula that gives the value of $y_{n+1}$ in terms of the values of $f(x,y)$ at $n$ different points. The formula coincides with the Taylor Series formula for the same equation upto the terms of $h^n$.

Note:

1. The truncation error in Runge-Kutta method of order $n$ is $\frac{y^{(n+1)}(\xi)\, h^{n+1}}{(n+1)!}$, where $x \le \xi \le x+h$.

2. RK methods do not require computation of higher order derivatives

3. These are self-starting methods.

4. Runge-Kutta formula of first order is the same as Euler's formula.

Second order Runge - Kutta method for first order ODE:

To solve $\frac{dy}{dx} = f(x,y)$ given $y(x_0) = y_0$, $\Delta x = h$, Second order Runge - Kutta formula is given by the equations.

$$K_1 = hf(x,y)$$

$$K_2 = hf\left(x+\frac{h}{2}, y+\frac{K_1}{2}\right)$$

$$\Delta y = K_2$$

$$y(x+h) = y + \Delta y$$

Fourth order Runge-Kutta method for first order ODE.

The Runge-Kutta formula of fourth order for solving the first order ODE $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ is given by the equations.

$$K_1 = hf(x,y)$$

$$K_2 = hf\left(x+\frac{h}{2}, y+\frac{K_1}{2}\right)$$

$$K_3 = hf\left(x+\frac{h}{2}, y+\frac{K_2}{2}\right)$$

$$K_4 = hf(x+h, y+K_3)$$

$$\Delta y = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$y(x+h) = y + \Delta y.$$

Note:

1. This is the most widely used of all the Runge-Kutta formulae, that it is simply reffered to as "Runge-Kutta formula".

2. This method agrees with Taylor's series Solution upto the term in $h^4$.

1. Evaluate $y(0.1)$ using Second order Runge-Kutta method of given $y' = \frac{1}{2}(1+x)y^2$, $y(0) = 1$

Solution:

$$\text{Given, } f(x,y) = \frac{dy}{dx} = \frac{1}{2}(1+x)y^2$$

$$x_0 = 0, \; x_1 = 0.1, \; y_0 = 1, \; h = 0.1$$

Second order Runge-Kutta formula is given by,

$$K_1 = h f(x_0, y_0)$$

$$K_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$

$$\Delta y = K_2$$

$$y_1 = y_0 + \Delta y$$

$$\therefore K_1 = h f(x_0, y_0)$$

$$= 0.1 \times f(0,1)$$

$$= 0.1 \times \frac{1}{2}(1+0)(1^2)$$

$$K_1 = 0.05$$

$$K_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$

$$= 0.1 \times f(0.05, 1.025)$$

$$= 0.1 \times \frac{1}{2}(1+0.05)(1.025)^2$$

$$K_2 = 0.05516$$

$$\Delta y = K_2 = 0.05516$$

$$y(0.1) = y_1 = y_0 + \Delta y$$

$$= 1 + 0.05516$$

$$= 1.05516$$

2. Given $\frac{dy}{dx} - x^2 y = x$, $y(0) = 1$ using Runge-Kutta method of fourth order, find $y$ at $x = 0.1$

Solution:

$$\text{Given } f(x,y) = \frac{dy}{dx} = x^2 y + x$$

$x_0 = 0,\ y_0 = 1,\ x_1 = 0.1,\ h = 0.1$

By Runge-Kutta method of fourth order,

$$K_1 = hf(x_0, y_0)$$
$$= 0.1 \times f(0,1)$$
$$= 0.1((0^2 \times 1) + 0)$$
$$= 0.$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$
$$= 0.1 \times f\left(0 + \frac{0.1}{2}, 1 + \frac{0}{2}\right)$$
$$= 0.1 \times f(0.05, 1)$$
$$= 0.00525$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$
$$= 0.1 \times f\left(0 + \frac{0.1}{2}, 1 + \frac{0.00525}{2}\right)$$
$$= 0.1 \times ((0.05^2 \times 1.00263) + 0.05)$$
$$= 0.00525$$

$$K_4 = hf(x_0 + h, y_0 + K_3)$$
$$= 0.1 \times f(\cancel{0.05}\,(0 + 0.1, 1 + 0.00525))$$
$$= 0.1 \times f(0.1, 1.00525)$$
$$= 0.1 \times (0.1^2 \times 1.00525) + 0.1)$$
$$= 0.01101$$

$$\Delta y = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$
$$= \frac{1}{6}(0 + (2 \times 0.00525) + (2 \times 0.00525) + 0.01101)$$
$$= 0.00534$$

$$\therefore y_1 = y(0.1) = y_0 + \Delta y$$
$$y_1 = 1 + 0.00534 = 1.00534$$

3. Using Runge-Kutta method of 4th order, Solve $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$ with $y(0)=1$, to find y at $x=0.2, 0.4$.

Solution:

Given, $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$

$x_0=0, y_0=1, x_1=0.2, x_2=0.4, h=0.2$

By Runge-Kutta method of fourth order,

$$K_1 = hf(x_0, y_0)$$
$$= 0.2 \times f(0,1)$$
$$= 0.2 \times \left(\frac{1^2-0^2}{1^2+0^2}\right)$$
$$K_1 = 0.2$$

$$K_2 = hf\left(x+\frac{h}{2}, y+\frac{K_1}{2}\right)$$
$$= 0.2 \times f\left(0+\frac{0.2}{2}, 1+\frac{0.2}{2}\right)$$
$$= 0.2 \times f(0.1, 1.1)$$
$$= 0.2 \times \left(\frac{(1.1)^2-(0.1)^2}{(1.1)^2+(0.1)^2}\right)$$
$$K_2 = 0.1967$$

$$K_3 = hf\left(x+\frac{h}{2}, y+\frac{K_2}{2}\right)$$
$$= 0.2 \times f\left(0.1, 1+\frac{0.1967}{2}\right)$$
$$= 0.2 \times f(0.1, 1.0984)$$
$$= 0.2 \times \left(\frac{(1.0984)^2-(0.1)^2}{(1.0984)^2+(0.1)^2}\right)$$
$$K_3 = 0.1967$$

$$K_4 = hf(x+h, y+K_3)$$
$$= 0.2 \times f(0+0.2, 1+0.1967)$$
$$= 0.2 \times f(0.2, 1.1967)$$
$$K_4 = 0.1891$$

$$\Delta y = \frac{1}{6}(K_1 + 2K_2 + 3K_3 + K_4)$$

$$= \frac{1}{6}(0.2 + 0.3934 + 0.3934 + 0.1891)$$

$$\Delta y = 0.1960$$

$$y(0.2) = y + \Delta y = 1 + 0.1960$$

$$y(0.2) = 1.1960$$

$$x_1 = 0.2,\ y_1 = 1.1960,\ h = 0.2.$$

$$K_1 = hf(x_1, y_1)$$

$$= 0.2 \times f(0.2, 1.1960)$$

$$= 0.2 \times \left(\frac{(1.1960)^2 - (0.2)^2}{(1.1960)^2 + (0.2)^2}\right)$$

$$K_1 = 0.1891$$

$$K_2 = 0.2 \times f\left(0.2 + \frac{0.2}{2},\ 1.1960 + \frac{0.1891}{2}\right)$$

$$= 0.2 \times f(0.3, 1.2906)$$

$$= 0.2 \times \left(\frac{(1.2906)^2 - (0.3)^2}{(1.2906)^2 + (0.3)^2}\right)$$

$$K_2 = 0.1797$$

$$K_3 = 0.2 \times f\left(0.3,\ 1.1960 + \frac{0.1797}{2}\right)$$

$$= 0.2 \times f(0.3, 1.2859)$$

$$= 0.2 \times \left(\frac{(1.2859)^2 - (0.3)^2}{(1.2859)^2 + (0.3)^2}\right)$$

$$K_3 = 0.1794$$

$$K_4 = 0.2 \times f(0.2 + 0.2,\ 1.1960 + 0.1794)$$

$$= 0.2 \times f(0.4, 1.3754)$$

$$= 0.2 \times \left(\frac{(1.3754)^2 - (0.4)^2}{(1.3754)^2 + (0.4)^2}\right)$$

$K_4 = 0.1688$

$\Delta y_1 = \frac{1}{6}(0.1891 + 0.3594 + 0.3588 + 0.1688)$

$= \frac{1}{6}(1.0761)$

$\Delta y_1 = 0.1794$

$y(0.4) = y_1 + \Delta y_1$

$= 1.1960 + 0.1794$

$y(0.4) = 1.3754$

$\therefore y(0.2) = 1.1960$

$y(0.4) = 1.3754$

Predictor - Corrector methods:

Consider the initial value problem $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$. To solve this problem using predictor-corrector method, we use two formulae. We predict the value of y using the predictor formula and then correct this value using the corrector formula. Thus in a predictor-corrector method, a crude estimate of y is initially computed and subsequently refined to get a better approximation. Thus predictor-corrector methods are multi-step methods.

These methods usually require the values of y at $x_n, x_{n-1}, x_{n-2}, x_{n-3}$ to compute the value of y at $x_{n+1}$. Hence they are not self-starting methods. Two commonly used predictor-corrector methods are,

1. Milne's method

2. Adam-Bashforth method.

It may be noted that Euler's formula and modified Euler's formula (Self-starting methods) discussed earlier, form a pair of predictor-corrector formulae.

Milne's predictor - Corrector formulae:

To Solve $y' = \frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$, using Milne's method, we require the values of $y$ at $x_0, x_1, x_2, x_3$ to find $y(x_4)$

Let $y(x_{n-3}) = y_{n-3}$, $y(x_{n-2}) = y_{n-2}$

$y(x_{n-1}) = y_{n-1}$, $y(x_n) = y_n$

be Known. The Milne's formulae are:

predictor formula:

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n)$$

Corrector formula:

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1}).$$

Here the value of $y_{n+1,p}$ got using the predictor formula is used to find $y'_{n+1}$ for the correction formula.

Note:

1. When $y_0, y_1, y_2$ and $y_3$ are known, we find $y_4$ using the formulae.

$$y_{4,p} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$$

$$y_{4,c} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

2. We can refine the value of $y_{n+1}$ further by using $y_{n+1}$ again in the corrector formula.

3. We need four consecutive previous values of $y$ to calculate $y_{n+1}$.

4. The error in Milne's method is of order $h^5$.

1. Find $y(0.4)$ by Milne's method for $\frac{dy}{dx} = y - \frac{2x}{y}$ ; $y(0) = 1$, $y(0.1) = 1.0959$, $y(0.2) = 1.1841$, $y(0.3) = 1.2662$.

Solution:

Given $y' = y - \frac{2x}{y}$ , $h = 0.1$

$x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, $x_4 = 0.4$

$y_0 = 1$, $y_1 = 1.0959$, $y_2 = 1.1841$, $y_3 = 1.2662$.

Now,

$$y_1' = y_1 - \frac{2x_1}{y_1} = 1.0959 - \frac{2(0.1)}{1.0959} = 0.9134$$

$$y_2' = y_2 - \frac{2x_2}{y_2} = 1.1841 - \frac{2(0.2)}{1.1841} = 0.8463$$

$$y_3' = y_3 - \frac{2x_3}{y_3} = 1.2662 - \frac{2(0.3)}{1.2662} = 0.7923$$

By Milne's predictor formula,

$$y_{4,p} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$$

$$= 1 + \frac{4(0.1)}{3}(2(0.9134) - 0.8463 + 2(0.7923))$$

$$= 1.3420$$

$$y_{4,p}' = y_4 - \frac{2x_4}{y_4} = 1.3420 - \frac{2(0.4)}{1.3420} = 0.7459$$

By Milne's corrector formula,

$$y_{4,c} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

$$= 1.1841 - \frac{0.1}{3}(0.8463 + 4(0.7923) + 0.7459)$$

$$= 1.3428.$$

$$\therefore y(0.4) = 1.3428.$$

2. Given $\frac{dy}{dx} = xy + y^2$, $y(0) = 1$, $y(0.1) = 1.1169$, $y(0.2) = 1.2773$, $y(0.3) = 1.5049$, evaluate $y(0.4)$ by using Milne's method.

Solution:

Given $y' = xy + y^2$, $h = 0.1$

$x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, $x_4 = 0.4$

$y_0 = 1$, $y_1 = 1.1169$, $y_2 = 1.2773$, $y_3 = 1.5049$

Now,

$$y_1' = x_1y_1 + y_1^2 = (0.1 \times 1.1169) + (1.1169)^2 = 1.3592$$

$$y_2' = x_2y_2 + y_2^2 = (0.2 \times 1.2773) + (1.2773)^2 = 1.8870$$

$$y_3' = x_3y_3 + y_3^2 = (0.3 \times 1.5049) + (1.5049)^2 = 2.7162$$

By Milne's predictor formula,

$$y_{4,p} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$$

$$= 1 + \frac{4(0.1)}{3}(2 \times 1.3592) - 1.8870 + (2 \times 2.7162)$$

$$= 1.83510$$

Now,

$$y'_{4,p} = x_4y_4 + y_4^2 = (0.4 \times 1.8352) + (1.8352)^2 = 4.1020$$

By Milne's corrector formula,

$$y_{4,c} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

$$= 1.2773 + \frac{0.1}{3}(1.8870 + (4 \times 2.7162) + 4.1020)$$

$$= 1.8391$$

$\therefore y(0.4) = 1.8391$.

Adams-Bashforth predictor - Corrector formulae:

To solve $y' = \frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ using Adams-Bashforth method, we require the values of $y$ at $x_{n-3}, x_{n-2}, x_{n-1}, x_n$ to predict the value of $y$ at $x_{n+1}$, Adams Bashforth formula are:

Predictor formula:

$$y_{n+1,p} = y_n + \frac{h}{24}\left[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}\right]$$

Corrector formula:

$$y_{n+1,c} = y_n + \frac{h}{24}\left[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}\right]$$

Note:

If $n=3$, then we have,

$$y_{4,p} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

$$y_{4,c} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$$

1. Given $\frac{dy}{dx} = x^2(1+y)$ and $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$, $y(1.3) = 1.979$, Evaluate $y(1.4)$ by Adams-Bashforth method.

Solution:

Given $\frac{dy}{dx} = x^2(1+y)$, $h = 0.1$

$x_0 = 1$, $x_1 = 1.1$, $x_2 = 1.2$, $x_3 = 1.3$, $x_4 = 1.4$

$y_0 = 1$, $y_1 = 1.233$, $y_2 = 1.548$, $y_3 = 1.979$

$y_0' = x_0^2(1+y_0) = (1)^2(1+1) = 2$

$y_1' = x_1^2(1+y_1) = (1.1)^2(1+1.233) = 2.7019$

$y_2' = x_2^2(1+y_2) = (1.2)^2(1+1.548) = 3.6691$

$y_3' = x_3^2(1+y_3) = (1.3)^2(1+1.979) = 5.0345$

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By Adam's predictor formula,

$$y_{4,p} = y_3 + \frac{h}{24}(55y_3' - 59y_2' + 37y_1' - 9y_0)$$

$$= 1.979 + \frac{0.1}{24}(55(5.0345) - 59(3.6691) + 37(2.7019) - 9(2))$$

$$= 2.5723$$

$$y_4' = x_4^2(1 + y_{4,p})$$

$$= (1.4)^2(1 + 2.5723)$$

$$y_4' = 7.0017$$

By Adam's corrector formula,

$$y_{4,c} = y_3 + \frac{h}{24}(9y_4' + 19y_3' - 5y_2' + y_1')$$

$$= 1.979 + \frac{0.1}{24}(9(7.0017) + 19(5.0345) - 5(3.6691) + 2.7019)$$

$$= 2.5749$$

$$\therefore y(0.4) = 2.5749$$

2. Solve $2y' - x - y = 0$ given $y(0) = 2$, $y(0.5) = 2.636$, $y(1) = 3.595$, $y(1.5) = 4.968$ to get $y(2)$ by Adam's method.

Solution:

Given, $2y' - x - y = 0$, $h = 0.5$

$$y' = \frac{1}{2}(x+y)$$

$x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$, $x_4 = 2$.

$y_0 = 2$, $y_1 = 2.636$, $y_2 = 3.595$, $y_3 = 4.968$

Now,

$$y_0' = \frac{1}{2}(x_0 + y_0) = \frac{1}{2}(0+2) = 1$$

$$y_1' = \frac{1}{2}(x_1 + y_1) = \frac{1}{2}(0.5 + 2.636) = 1.568$$

$$y_2' = \frac{1}{2}(x_2 + y_2) = \frac{1}{2}(1 + 3.595) = 2.2975$$

$$y_3' = \frac{1}{2}(x_3+y_3) = \frac{1}{2}(1.5+4.968) = 3.234$$

By Adam's predictor formula,

$$y_{4,P} = y_3 + \frac{h}{24}(55y_3' - 59y_2' - 37y_1' - 9y_0')$$

$$= 4.968 + \frac{0.5}{24}(55(3.234) - 59(2.2975) + 37(1.568) - 9(1))$$

$$= 6.8708$$

$$y'_{4,P} = \frac{1}{2}(x_4+y_4) = \frac{1}{2}(2+6.8708) = 4.4354$$

By Adams corrector formula,

$$y_{4,C} = y_3 + \frac{h}{24}(9y_4' + 19y_3' - 5y_2' + y_1')$$

$$= 4.968 + \frac{0.5}{24}(9(4.4354) + 19(3.234) - 5(2.2975) + 1.568)$$

$$y_{4,C} = 6.8731$$

$$y(2) = 6.8731$$

**Source** :

Text Books:

1. P.Kandasamy, V.Thilagavathy, K.Gunavathi : "Numerical Methods",S.Chand& Company Ltd,New Delhi, 2016.