UNIT-III.

SOLUTION OF SYSTEM OF LINEAR ALGEBRAIC EQUATIONS

In this Section, methods for solving n linear, Simultaneous algebraic equations in n unknowns are discussed. The coefficient matrix A is then an $n \times n$ Square matrix. Such systems of equations are quite Commonly encountered in Engineering. Cramer's rule, Unfortunately, requires excessive calculations. So, several efficient techniques have been developed to obtain Solutions of large sets of equations of the type $Ax = B$. These can be Classified into direct and indirect or iterative methods. Some of these methods are described in this section.

Direct Methods:

i) Guass Elimination method:

This is a simple and systematic method used to solve linear Simultaneous equations. It is a direct method in which the given set of equations is written in matrix form. The augmented matrix is converted into an equivalent upper triangular matrix using elementry transformations and the solution is got by the method of back-substitution.

Consider a system of three equations in three unknowns. Say

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

We write this in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

(i,e) $AX = B$, where,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The Augmented matrix $[A|B]$ is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

Now we convert the augmented matrix into an upper triangular matrix by the following steps.

1. The first equation is called the pivotal equation and its leading coefficient a_{11} is called the first pivot. If necessary, re-arrange the equations so that $a_{11} \neq 0$. Using the pivot, the elements below it are converted to 0. For this, multiply the first row of the augmented matrix by $-\frac{a_{21}}{a_{11}}$ and add it to the second row. Next, multiply the first row by $-\frac{a_{31}}{a_{11}}$ and it to the third row.

Now we have a matrix of the form.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} & a_{23} & b_2 \\ 0 & a_{32} & a_{33} & b_3 \end{bmatrix}$$

2. Now a'_{22} is the next pivot.

Multiply the second row by $-\frac{a'_{32}}{a'_{22}}$ and add it to the third row. Now the matrix is of the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a''_{33} & b''_3 \end{bmatrix}$$

Which is upper triangular. (i,e) elements below the leading diagonal are all zeroes.

3. Now we rewrite the matrix in equation form as,

$$a_{11}x + a_{12}y + a_{13}z = b_1 \quad \rightarrow (1)$$

$$a'_{22}y + a'_{23}z = b'_2 \quad \rightarrow (2)$$

$$a''_{33}z = b''_3 \quad \rightarrow (3)$$

From equation (3), we get,

$$z = \frac{b''_3}{a''_{33}}$$

Substituting the value of z in equation (2).

$$\text{We get } y = \frac{b'_2 - a'_{23}z}{a'_{22}} = \frac{a''_{33}b'_2 - a'_{23}b''_3}{a'_{22}\,a''_{33}}$$

Now substituting the values of y and z in equation 1 we get the value of x. This procedure is called back substitution.

Pivoting:

When any of the pivot elements is zero, the ~~zero~~ order of the equations is changed. (i,e) the rows are interchanged such that the pivot is non-zero. The interchange of rows in the matrix is called pivoting.

Problem

1. Solve $3x - y = 2$, $x + 3y = 4$ using Gaussian Elimination method.

Solution:

The given system of equation can be written in matrix form as,

$$\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

The Augmented matrix is,

$$\begin{bmatrix} 3 & -1 & 2 \\ 1 & 3 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & \frac{10}{3} & \frac{10}{3} \end{bmatrix} \text{ by } R_2 \to \left(\frac{-1}{3}\right) R_1 + R_2$$

i.e., $3x - y = 2 \quad \to (1)$

$\frac{10}{3} y = \frac{10}{3} \quad \to (2)$

We solve these equations by back substitutions:

From equation (2), we get $y = 1$.

Substituting in Equation (1), we get,

$$3x - 1 = 2$$
$$3x = 3$$
$$x = 1$$

$\therefore$ The solution is $x = 1, y = 1$.

2. Solve the following system by Guass- Elimination method.

$$x + 2y + z = 3$$
$$2x + 3y + 3z = 10$$
$$3x - y + 2z = 13$$

Solution:

The system of equation can be written in matrix form as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 13 \end{bmatrix}$$

The Augmented matrix is,

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{bmatrix}$$

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$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -2 & 4 \end{bmatrix} \text{ by } \begin{array}{l} R_2 \to (-2)R_1 + R_2 \\ R_3 \to (-3)R_1 + R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{bmatrix} \text{ by } R_3 \to -7R_2 + R_3$$

This can be written as,

$$x + 2y + z = 3 \longrightarrow ①$$

$$-y + z = 4 \longrightarrow ②$$

$$-8z = -24 \longrightarrow ③$$

We solve these equations by the method of back-substitution

By equation ③, $z = 3$

Substituting $z = 3$ in equation ②.

$$-y + 3 = 4$$

$$\therefore y = -1$$

Substituting $y = -1$, $z = 3$ in equation ①.

$$x - 2 + 3 = 3$$

$$\therefore x = 2$$

$\therefore$ The Solution is $x = 2$, $y = -1$, $z = 3$.

Guass-Jordan Elimination method (Direct method)

This method is a modification of the above Guass elimination method. In this method, the coefficient matrix A of the System $AX = B$ is brought to a diagonal matrix or unit matrix by making the matrix A not only upper triangular but also lower triangular by making all elements above the leading diagonal of A also as Zereos. By this way, the System $AX = B$ will reduce to the form.

$$\left[\begin{array}{ccccc|c} a_{11} & 0 & 0 & 0 & 0 & b_1 \\ 0 & b_{22} & 0 & 0 & 0 & c_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & d_3 \\ 0 & 0 & 0 & 0 & \alpha_{nn} & k_n \end{array}\right] \longrightarrow (1).$$

From (1),

$$x_n = \frac{k_n}{\alpha_{nn}}, \ldots \ldots x_2 = \frac{c_2}{b_{22}}, \; x_1 = \frac{b_1}{a_{11}}$$

Note:

By this method, the values of $x_1, x_2, \ldots \ldots x_n$ are got immediately without using the process of back substitution.

problem:

1. Solve $10x+y+z=12$
$2x+10y+z=13$
$x+y+5z=7$
by i) Guass Elimination method.
ii) Guass Jordon method.

Solution:

The system of equation can be written in matrix form as,

$$\begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 7 \end{bmatrix}$$

Guass Elimination method:

The Augmented matrix is

$$\begin{bmatrix} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 10 & 1 & 1 & 12 \\ 0 & \frac{49}{5} & \frac{4}{5} & \frac{53}{5} \\ 0 & \frac{9}{10} & \frac{49}{10} & \frac{29}{5} \end{bmatrix} \quad \text{by } R_2 \to \left(-\tfrac{2}{10}\right)R_1 + R_2, \; R_3 \to \left(\tfrac{-1}{10}\right)R_1 + R_3$$

$$\sim \begin{bmatrix} 10 & 1 & 1 & 12 \\ 0 & \frac{49}{5} & \frac{4}{5} & \frac{53}{5} \\ 0 & 0 & \frac{473}{98} & \frac{473}{98} \end{bmatrix} \quad \text{by } R_3 \to \left(\tfrac{-9}{98}\right)R_2 + R_3$$

The system of equations now becomes,

$$10x + y + z = 12 \longrightarrow ①$$

$$\frac{49}{5}y + \frac{4}{5}z = \frac{53}{5} \longrightarrow ②$$

$$\frac{473}{98}z = \frac{473}{98} \longrightarrow ③$$

We solve these equations by back substitution.

From Equation ③, $z = 1$.

Substitute $z = 1$ in equation ②. we get

$$\frac{49}{5}y = \frac{49}{5}$$

$$y = 1$$

Substitute $y = 1$, $z = 1$ in Equation ①.

We get, $10x = 10$

$$x = 1$$

$\therefore$ The solution is $x = 1$, $y = 1$, $z = 1$

Guass Jordon method:

$$\begin{bmatrix} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 10 & 1 & 1 & 12 \\ 0 & \frac{49}{5} & \frac{4}{5} & \frac{53}{5} \\ 0 & \frac{9}{10} & \frac{49}{10} & \frac{29}{5} \end{bmatrix} \text{ by } \begin{array}{l} R_2 \to \left(\frac{-2}{10}\right)R_1 + R_2 \\ R_3 \to \left(\frac{-1}{10}\right)R_1 + R_2 \end{array}$$

$$\sim \begin{bmatrix} 10 & 0 & \frac{45}{49} & \frac{535}{49} \\ 0 & 49/5 & 4/5 & \frac{53}{5} \\ 0 & 0 & \frac{473}{98} & \frac{473}{98} \end{bmatrix} \text{ by } \begin{array}{l} R_1 \to \left(\frac{-5}{49}\right)R_2 + R_1 \\ R_3 \to \left(\frac{-9}{98}\right)R_2 + R_3 \end{array}$$

$$\sim \begin{bmatrix} 10 & 0 & 0 & 10 \\ 0 & \frac{49}{5} & 0 & \frac{49}{5} \\ 0 & 0 & \frac{473}{98} & \frac{473}{98} \end{bmatrix} \text{ by } \begin{array}{l} R_1 \to \left(\frac{-90}{473}\right)R_3 + R_1 \\ R_3 \to \left(\frac{-392}{2365}\right)R_3 + R_2 \end{array}$$

Solving, we get ~~the~~ the system of equations.

$$10x = 10$$

$$\frac{49}{5} y = \frac{49}{5}$$

$$\frac{473}{98} y = \frac{473}{98}$$

Solving we get $x = 1$, $y = 1$, $z = 1$

Note:

In the last matrix, we can perform.

$$R_1 \to R_1 \div 10$$

$$R_2 \to R_2 \div \frac{40}{5}$$

$$R_3 \to R_3 \div \frac{473}{98}$$

and get, $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$

(i.e, $x = 1$
$y = 1$
$z = 1$

Method of Triangularization (or Method of factorization) (Direct method).

This method is also called as decomposition method.

In this method, the coefficient matrix A of the system $AX = B$, is decomposed or factorized into the product of a lower triangular matrix L and an upper triangular matrix U. We will explain this method in the case of three equations in three unknowns.

Consider the system of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \quad \to (1)$$

This System is equivalent to $AX = B$.

$$\text{Where, } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \rightarrow (2)$$

Now we will factorize A as the product of lower triangular matrix.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

and an upper triangular matrix.

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \text{ So that,}$$

Let. $$LUX = B \rightarrow (3)$$

$$UX = Y \rightarrow (4)$$

and hence, $$LY = B \rightarrow (5).$$

$$\text{That is, } \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \rightarrow (5)$$

$\therefore y_1 = b, \; l_{21} y_1 + y_2 = b_2, \; l_{31} y_1 + l_{32} y_2 + y_3 = b_3$

$\therefore$ By forward Substitution y_1, y_2, y_3 can be found out if L is Known from (4).

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$u_{11} x_1 + u_{12} x_2 + u_{13} x_3 = y_1$$

$$u_{22} x_2 + u_{23} x_3 = y_2$$

$$u_{33} x_3 = y_3$$

From these, x_1, x_2, x_3 can be Solved by back Substitution.

Since y_1, y_2, y_3 are Known if U is Known. Now L and U can be found from, $LU = A$.

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Equating corresponding coefficients we get nine equations in nine unknowns. From these 9 equations, we can solve for 3l's and 6u's.

That is, L and U are known. Hence X is found out. Going into details. We get $u_{11} = a_{11}$, $u_{12} = a_{12}$, $u_{13} = a_{13}$. That is the elements in the first row of U are same as the elements in the first of A.

Also, $l_{21}u_{11} = a_{21}$, $l_{21}u_{12} + u_{22} = a_{22}$, $l_{21}u_{13} + u_{23} = a_{23}$.

$$\therefore l_{21} = \frac{a_{21}}{a_{11}}, \quad u_{22} = a_{22} - \frac{a_{21}}{a_{11}} \cdot a_{12} \text{ and } u_{23} = a_{23} - \frac{a_{21}}{a_{11}} \cdot a_{13}$$

Again, $l_{31}u_{11} = a_{31}$, $l_{31}u_{12} + l_{32}u_{22} = a_{32}$ and

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33}.$$

$$\text{Solving, } l_{31} = \frac{a_{31}}{a_{11}}, \quad l_{32} = \frac{a_{32} - \frac{a_{31}}{a_{11}} \cdot a_{12}}{a_{22} - \frac{a_{21}}{a_{11}} \cdot a_{12}}$$

$$u_{33} = a_{33} - \frac{a_{31}}{a_{11}} \cdot a_{13} - \left[\frac{a_{32} - \frac{a_{31}}{a_{11}} \cdot a_{12}}{a_{22} - \frac{a_{21}}{a_{11}} \cdot a_{12}} \right] \left[a_{23} - \frac{a_{21}}{a_{11}} \cdot a_{13} \right]$$

Therefore L and U are known.

Note: In selecting L and U we can also take as

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \text{ So that } A = LU.$$

Problem:

1. Solve by Traingularization method, the following system.

$x+5y+z=14,\ 2x+y+3z=13,\ 3x+y+4z=17.$

Solution:

This is equivalent to.

$$\begin{bmatrix}1&5&1\\2&1&3\\3&1&4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}14\\13\\17\end{bmatrix}$$

$$AX=B$$

Let, $LU=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}=\begin{bmatrix}1&5&1\\2&1&3\\3&1&4\end{bmatrix}$

By seeing, we can write. $u_{11}=1,\ u_{12}=5,\ u_{13}=1.$

$$\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}\begin{bmatrix}1&5&1\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}=\begin{bmatrix}1&5&1\\2&1&3\\3&1&4\end{bmatrix}$$

Hence, $l_{21}=2,\ 5l_{21}+u_{22}=1,\ l_{21}+u_{23}=3$

$\therefore l_{21}=2,\ u_{22}=-9,\ u_{23}=1$

Again $l_{31}=3,\ 5l_{31}+l_{32}u_{22}=1,\ l_{31}+l_{32}u_{23}+u_{33}=4$

$l_{32}=\frac{1-15}{-9}=\frac{14}{9}$; $u_{33}=4-3-\frac{14}{9}=\frac{-5}{9}$

$LUX=B$ implies, $LY=B$, where $UX=Y$.

$LY=B$ gives

$$\begin{bmatrix}1&0&0\\2&1&0\\3&\frac{14}{9}&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}14\\13\\17\end{bmatrix}$$

$y_1=14,\ 2y_1+y_2=13,\ 3y_1+\frac{14}{9}y_2+y_3=17$

$y_1=14,\ y_2=-15,\ y_3=\frac{-5}{3}$

$UX=Y$ implies,

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$$\begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -\frac{5}{9} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ -15 \\ -\frac{5}{3} \end{bmatrix}$$

$$x+5y+z=14$$

$$-9y+z=-15$$

$$-\frac{5}{9}z=-\frac{5}{3}$$

$$z=3$$

$$-9y+3=-15$$

$$y=\frac{-18}{-9}2$$

$$y=2$$

$$x+5(2)+3=14$$

$$x=14-13$$

$$x=1$$

$\therefore$ The solution is $x=1, y=2, z=3$.

2 Solve the following system by triangularization method:

- $x+y+z=1,\ 4x+3y-z=6,\ 3x+5y+3z=4.$

Solution:

This is equivalent to,

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$$AX=B.$$

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

By seeing, we can write,

$$u_{11}=1,\ u_{12}=1,\ u_{13}=1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

$l_{21} = 4$; $l_{21} + u_{22} = 3$; $l_{21} + u_{23} = -1$

$l_{21} + u_{22} = 3 \Rightarrow 4 + u_{22} = 3 \Rightarrow u_{22} = -1$

$l_{21} + u_{23} = -1 \Rightarrow 4 + u_{23} = -1 \Rightarrow u_{23} = -5$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

$l_{31} = 3$; $l_{31} - l_{32} = 5$; $l_{31} - 5l_{32} + u_{33} = 3$

$l_{31} - l_{32} = 5 \Rightarrow 3 - l_{32} = 5 \Rightarrow -l_{32} = 2 \Rightarrow l_{32} = -2$

$l_{31} - 5l_{32} + u_{33} \Rightarrow 3 - 5(-2) + u_{33} \Rightarrow u_{33} = -10$

$LY = B$ implies,

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$y_1 = 1$

$4y_1 + y_2 = 6$

$3y_1 - 2y_2 + y_3 = 4$

$\Rightarrow 4(1) + y_2 = 6 \Rightarrow y_2 = 2$

$\Rightarrow 3(1) - 2(2) + y_3 = 4 \Rightarrow y_3 = 5$

$Ux = y$ implies,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$x + y + z = 1$

$-y - 5z = 2$

$-10z = 5$

$$z = \frac{-5}{10}$$

$$\boxed{z = \frac{-1}{2}}$$

$$-y - 5z = 2$$

$$-y - 5\left(\frac{-1}{2}\right) = 2$$

$$-y + \frac{5}{2} = 2$$

$$-y = 2 - \frac{5}{2}$$

$$-y = \frac{-1}{2}$$

$$\Rightarrow y = \frac{1}{2}$$

$$x + y + z = 1$$

$$x + \frac{1}{2} - \frac{1}{2} = 1$$

$$x = 1$$

$\therefore$ The Solution is $x = 1$, $y = \frac{1}{2}$, $z = \frac{-1}{2}$.

Iterative Methods (Indirect methods).

The direct methods discussed in the previous Section are not suitable for Solving large problems (n more than 25) which are quite commonly encountered in engineering practice. Approximate iterative techniques can be used in Such cases.

Sufficient condition:

Each equation of the system must possess one large Co-efficients and the large coefficient must be attached to a different unknown in that equation. This condition will be Satisfied if the large coefficients are along the leading diagonal of the coefficient matrix.

The system of equations.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Can be solved by iterative method if,

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

(i.e) The absolute values of the leading diagonal elements of the coefficient matrix A of the system $AX = B$ are greater than the sum of the absolute values of the other coefficients of that row. (i.e) The system is diagonally dominant.

Guass - Jacobi method:

Consider $a_1x + b_1y + c_1z = d_1$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Let, $|a_1| > |b_1| + |c_1|$

$$|b_2| > |a_2| + |c_2|$$

$$|c_3| > |a_3| + |b_3|$$

Then, $x = \frac{1}{a_1}(d_1 - b_1y - c_1z)$

$$y = \frac{1}{b_2}(d_2 - a_2x - c_2z)$$

$$z = \frac{1}{c_3}(d_3 - a_3x - b_3y)$$

Use initial values $x^{(0)}, y^{(0)}, z^{(0)}$ and get $x^{(1)}, y^{(1)}, z^{(1)}$. Use these values and get $x^{(2)}, y^{(2)}, z^{(2)}$. Proceed till we get the desired accuracy. If initial values are not known, we use 0,0,0.

Problem 1:

Solve the following system of equation by Guass-Jacobi method:

$$10x+2y+z=9$$
$$x+10y-z=-22$$
$$-2x+3y+10z=22$$

Solution.

The given system of equation is diagonally dominant. So we rewrite the equation as,

$$x=\frac{1}{10}(9-2y-z)$$
$$y=\frac{1}{10}(-22-x+z)$$
$$z=\frac{1}{10}(22+2x-3y)$$

We start with initial values $x=0$, $y=0$, $z=0$ and iterate, using the values obtained in the previous step.

First Iteration:

$$x^{(1)}=\frac{1}{10}(9-2(0)-0)=0.9$$
$$y^{(1)}=\frac{1}{10}(-22-0+0)=-2.2$$
$$z^{(1)}=\frac{1}{10}(22+2(0)-3(0))=2.2$$

We use these values in the next iteration.

Second Iteration:

$$x^{(2)}=\frac{1}{10}(9-2(-2.2)-2.2)=1.12$$
$$y^{(2)}=\frac{1}{10}(-22-0.9-2.2)=-2.07$$
$$z^{(2)}=\frac{1}{10}(22-2(0.9)-3(-2.2))=3.04$$

Third Iteration:

$$x^{(3)}=\frac{1}{10}(9-2(-2.07)-3.04)=1.01$$
$$y^{(3)}=\frac{1}{10}(-22-1.12+3.04)=-2.008$$
$$z^{(3)}=\frac{1}{10}(22+2(1.12)-3(-2.07))=3.045$$

Similarly we do further iterations till we get the desired accuracy. The values are tabulated in the following table

x	y	z
0	0	0
0.9	-2.2	2.2
1.12	-2.07	3.04
1.01	-2.008	3.045
0.9971	-1.9965	3.0044
0.9989	-1.9993	2.9984
1.0000	-2.0000	2.9996
1.0000	-2.0000	3.0000

$\therefore$ The required solution is $x=1, y=-2, z=3$.

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2. Solve the following system of equations correct to 3 decimal places, using Guass-Jacobi method.

$$-x+y+10z=35.61$$

$$x+10y+z=20.08$$

$$10x+y-z=11.19$$

Solution:

The given System of equations is not diagonally dominant. So we rewrite the equation is,

$$10x+y-z=11.19$$

$$x+10y+z=20.08$$

$$-x+y+10z=35.61$$

Now,

$$x=\frac{1}{10}(11.19-y+z)$$

$$y=\frac{1}{10}(20.08-x-z)$$

$z = \frac{1}{10}(35.61 + x - y)$

x	y	z
0	0	0
1.119	2.008	3.561
1.2743	1.54	3.4721
1.3122	1.5334	3.5344
1.3191	1.5233	3.5388
1.3206	1.5222	3.5406
1.3208	1.522	3.541
1.321	1.522	3.541
1.321	1.522	3.541

$\therefore$ The required solution is $x = 1.321$, $y = 1.522$, $z = 3.541$

3. Solve the following system of equations correct to ~~3~~ decimal places, using Guass-Jacobi method.

Guass-Seidel method of Iteration:

This is only a refinement of Guass-Jacobi method.

As before,

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

We start with the initial values $y^{(0)}$, $z^{(0)}$ for y and z and get $x^{(1)}$ from the first equation. That is,

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1 y^{(0)} - c_1 z^{(0)}).$$

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While using the second equation, we use $z^{(0)}$ for z and $x^{(1)}$ for x instead of $x^{(0)}$ as in the Jacobi's method, we get,

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2 x^{(1)} - c_2 z^{(0)}).$$

Now, having known $x^{(1)}$ and $y^{(1)}$, use $x^{(1)}$ for x and $y^{(1)}$ for y in the third equation, we get,

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3 x^{(1)} - b_3 y^{(1)}).$$

In finding the values of the unknowns, we use the latest available values on the right hand side. If $x^{(r)}, y^{(r)}, z^{(r)}$ are the rth iterates, then the iteration scheme will be,

$$x^{(r+1)} = \frac{1}{a_1}(d_1 - b_1 y^{(r)} - c_1 z^{(r)}).$$

$$y^{(r+1)} = \frac{1}{b_2}(d_2 - a_2 x^{(r+1)} - c_2 z^{(r)})$$

$$z^{(r+1)} = \frac{1}{c_3}(d_3 - a_3 x^{(r+1)} - b_3 y^{(r+1)})$$

This process of iteration is continued untill the convergence is assured. As the current values of unknowns at each stage of iteration are used in getting the values of unknowns, the convergence

Note:

1. Since the current values of the unknowns at each stage of iteration are used to find the values of the unknowns, the convergence in Guass-Seidel method is faster than that in Guass-Jacobi method. The rate of convergence in Guass-Seidel method is nearly two-times that of Guass-Jacobi method.

2. Iteration method is a self-correcting method. (ie) any error made in computation is corrected in the subsequent iteration

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1. Solve the system of equations using Gauss-Seidel method.

$$27x + 6y - z = 85$$

$$6x + 15y + 2z = 72$$

$$x + y + 54z = 110.$$

Solution:

The given system of equation is diagonally dominant; So we solve for x, y, z as follows

$$x = \frac{1}{27}(85 - 6y + z)$$

$$y = \frac{1}{15}(72 - 6x - 2z)$$

$$z = \frac{1}{54}(110 - x - y)$$

We take initial values $y = 0$, $z = 0$ and iterate, always using the latest available values. The results are given in the following table.

x	y	z
	0	0
3.14815	3.54074	1.91317
2.43218	3.57204	1.92585
2.42569	3.57294	1.92595
2.42549	3.57301	1.92595
2.42548	3.57301	1.92595
2.42548	3.57301	1.92595

Hence $x = 2.4255$, $y = 3.5730$, $z = 1.9260$.

2. Solve by Gauss-Seidel method:

$$2x + y + 6z = 9$$

$$8x + 3y + 2z = 13$$

$$x + 5y + z = 7$$

Solution:

Since the given system of equations is not diagonally dominant, we rewrite the equations as follows:

$$8x+3y+2z=13$$

$$x+5y+z=7$$

$$2x+y+6z=9$$

Hence, $x=\frac{1}{8}(13-3y-2z)$

$y=\frac{1}{5}(7-x-y)$

$z=\frac{1}{6}(9-2x-y)$

x	y	z
	0	0
1.62500	1.07500	0.77917
1.02708	1.03875	0.98452
0.98934	1.00523	1.00268
0.99737	0.99999	1.00088
0.99978	0.99987	1.00010
1.00002	0.99998	1.00000

Hence the solution is $x=1, y=1, z=1$.

3. Solve the following system by Gruass-Jacobi and Gruass Seidel methods:

$$10x-5y-2z=3$$

$$4x-10y+3z=-3$$

$$x+6y+10z=-3$$

Solution:

Since the given system of equations is diagonally dominant, we rewrite the equation as follows:

Hence, $x = \frac{1}{10}(3+5y+2z)$

$y = \frac{1}{10}(-3+4x+3z)$

$z = \frac{1}{10}(-3-x-6y)$

i) Guass - Jacobi method:

x	y	z
0	0	0
0.3	0.3	-0.3
0.39	0.33	-0.51
0.363	0.303	-0.537
0.3441	0.2841	-0.5181
0.3384	0.2822	-0.5048
0.340	0.284	-5.503
0.341	0.285	-0.504
0.342	0.285	-0.505
0.342	0.285	-0.505

$x = 0.342,\ y = 0.285,\ z = -0.505$

ii) Guass Seidel method:

x	y	z
	0	0
0.3	0.42	-0.582
0.394	0.283	-0.509
0.340	0.283	-0.504
0.341	0.285	-0.505
0.342	0.285	-0.505
0.342	0.285	-0.505

$\therefore$ Hence the Solution is $x = 0.342,\ y = 0.285,\ z = -0.505$

Inverse of a matrix by using Gruass - Jordon method.

We can use Gruass - Jordon method to find the inverse of a non-singular matrix A.

Step 1 : From the augmented matrix (A, I), where I is the identify matrix of the same order as A.

Step 2 : Transform (A, I) into the form (I, B) using elementary row operations.

Step 3 : Then $A^{-1} = B$.

Problem :

1. Find the inverse of the given matrix by Gruass - Jordon method:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

Solution :

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The Augmented matrix is,

$$(A, I) = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 7/4 & -5/4 & \frac{1}{4} \\ -5/4 & 3/4 & 1/4 \\ -1 & -1 & 1 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 + \frac{1}{4} R_3 \\ R_2 \to R_2 + \frac{1}{4} R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/4 & -5/4 & \frac{1}{4} \\ 0 & 1 & 0 & -5/4 & 3/4 & 1/4 \\ 0 & 0 & 1 & 1/4 & 1/4 & -1/4 \end{array}\right] \quad R_3 \to \left(\tfrac{-1}{4}\right)R_3$$

$$\sim (I, A^{-1})$$

$$\therefore A^{-1} = \begin{bmatrix} \frac{7}{4} & -\frac{5}{4} & \frac{1}{4} \\ -\frac{5}{4} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 7 & -5 & 1 \\ -5 & 3 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

2. Using Guass-Jordon method, find the inverse of the matrix

$$\begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}$$

Solution:

$$\text{Let } A = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The augmented matrix is,

$$[A, I] = \left[\begin{array}{ccc|ccc} 8 & -4 & 0 & 1 & 0 & 0 \\ -4 & 8 & -4 & 0 & 1 & 0 \\ 0 & -4 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 8 & -4 & 0 & 1 & 0 & 0 \\ 0 & 6 & -4 & \frac{1}{2} & 1 & 0 \\ 0 & -4 & 8 & 0 & 0 & 1 \end{array}\right] \quad R_2 \to R_2 + \tfrac{1}{2}R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 8 & 0 & -\frac{8}{3} & \frac{4}{3} & \frac{2}{3} & 0 \\ 0 & 6 & -4 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{16}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right] \quad \begin{array}{l} R_1 \to \frac{2}{3}R_2 + R_1 \\ R_3 \to \frac{2}{3}R_2 + R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 8 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & 6 & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{16}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right] \quad \begin{array}{l} R_1 \to \frac{1}{2}R_3 + R_1 \\ R_2 \to \frac{3}{4}R_3 + R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc} \frac{3}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{16} & \frac{1}{8} & \frac{3}{16} \end{array}\right] \quad \begin{array}{l} R_1 \to \frac{1}{8} R_1 \\ R_2 \to \frac{1}{6} R_2 \\ R_3 \to \frac{3}{16} R_3 \end{array}$$

$$\sim [A, I, A^{-1}]$$

$$\therefore A^{-1} = \begin{bmatrix} \frac{3}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{16} & \frac{1}{8} & \frac{3}{16} \end{bmatrix}$$

Source :

Text Books:

1. P.Kandasamy, V.Thilagavathy, K.Gunavathi : "Numerical Methods",S.Chand& Company Ltd,New Delhi, 2016.