UNIT-II

Numerical Solution of Algebraic and transcendental equations:

Numerical Analysis

Analytical Solution	Numerical solution
(Exact solution)	(Approximate Solution)

Numerical Solution of:

* Algebraic and Transcendental equations.
* Interpolation
* Numerical Differentiation and Integration.
* ODE's and PDE's etc.

Solution of Algebraic and Transcendental equations.

* An important problem in scientific and engineering work is to find the roots of an equation form $f(x)=0$
* If $f(x)=0$ is a quadratic expression then we have a simple formula for the roots of $f(x)=0$.
* For Example, if $f(x)=ax^2+bx+c=0$, then $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ are the roots of the above equation.
* The functions may be algebraic or transcendental.

Algebraic Equations:

$$a_0 x^n + a_1 x^{n-1} + \dots\dots + a_n = 0.$$

$(a_0, a_1, \dots\dots a_n$ are real numbers, $a_0 \neq 0$ and $n \geq 1$ are a positive integer)

Example, $x^3 - 2x^2 - 9x + 18 = 0$.

Transcendental Equations:

Equations involving exponential, lograthmic, trigonometric functions.

Example, $4x - e^x = 0$

$3x = \cos x + 1$

$x \log_{10} x - 1.2 = 0.$

Root of an Equation:

Any value 'a' for which $f(a) = 0$ is called a root or Solution of the equation $f(x) = 0$.

Geometrically, the point at which a curve $y = f(x)$ intersects the x-axis, is a root of the equation $f(x) = 0$.

Fundemental Theorem of Algebra:

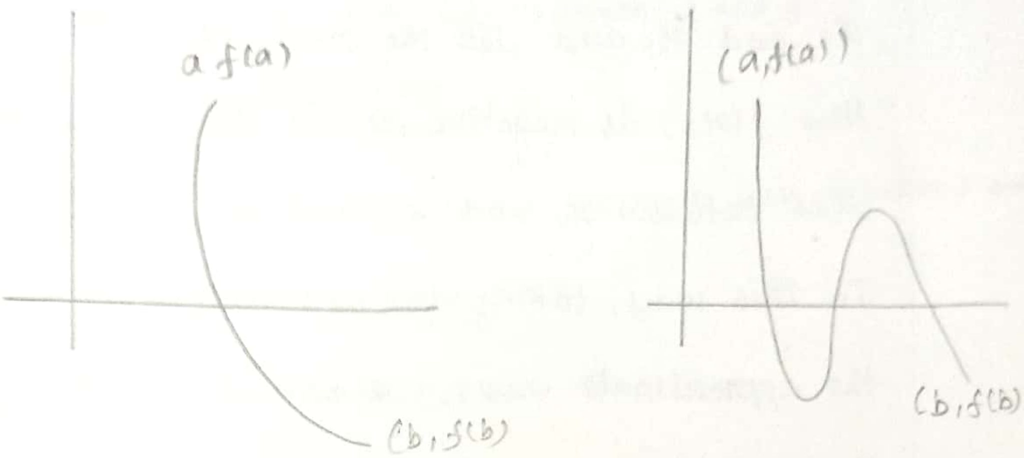
Every equation has a root, real or imaginary.

Note:

1. Every algebraic equation of degree n has exactly 'n' roots.

2. The number and nature of roots of a transcendal equation are not known.

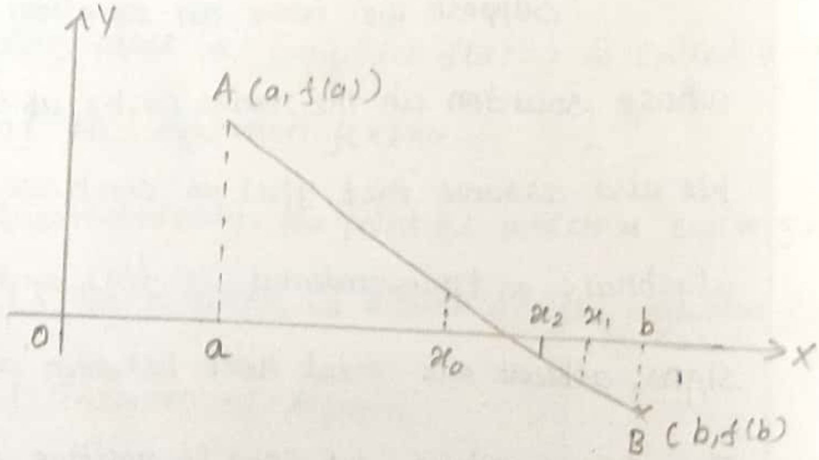
3. If $f(x)$ is continue in the interval (a, b) and if $f(a)$ and $f(b)$ are of opposite signs, then the equation $f(x)$ will have at-least one real root between a and b.



BISECTION METHOD (OR) BOLZANO'S METHOD (OR) INTERVAL HALVING METHOD:

Suppose we have an equation of the form $f(x)=0$ whose solution in the range (a, b) is to be Searched. We also assume that $f(x)$ is continuous and it can be algebraic or transcendental. If $f(a)$ and $f(b)$ are of opposite signs, atleast one real root between 'a' and 'b' Should exist. For conveneince, let $f(a)$ be positive and $f(b)$ be negative. Then atleast one root exists between 'a' and 'b'. As a first approximation, we assume that root to be $x_0 = \frac{a+b}{2}$ (mid point of the ends of the range). Now, find the sign of $f(x_0)$. If $f(x_0)$ is negative, the root lies between a and x_0. If $f(x_0)$ is positive, the root lies between x_0 and b. Any one of this is true. Suppose $f(x_0)$ is positive as

Shown in the Figure, then the root lies between x_0 and b and take the root as $x_1 = \frac{x_0+b}{2}$. Now $f(x_1)$ is negative (as in the Figure). Hence the root lies between x_0 and x_1 and let the root be (approximate), $x_2 = \frac{x_0+x_1}{2}$. Now $f(x_2)$ is negative as in the Figure. Then the root lies between x_0 and x_2 and let $x_3 = \frac{x_0+x_2}{2}$ and so on. In this way, taking the mid-point of the range as the approximate root, we form a sequence of approximate roots $x_0, x_1, x_2, \ldots\ldots$ whose limit of convergence is the exact root. However, depending on the precision required, we stop the processes after some steps. Though simple, the convergence of this method is slow but sure.



Problem:

1. Assuming that a root of $x^3 - 9x + 1 = 0$ lies in the interval $(2, 3)$. Find that root by bisection method.

Solution:

Let $f(x) = x^3 - 9x + 1$

$f(0) = (0)^3 - 9(0) + 1 = 1 = +ve$

$f(1) = 1^3 - 9(1) + 1 = -ve$

$f(2) = (2)^3 - 9(2) + 1 = 9 - 18 = -9 = -ve.$

$f(3) = (3)^3 - 9(3) + 1 = 28 - 29 = -ve.$

$f(4) = (4)^3 - 9(4) + 1 = +ve.$

Therefore, the root lies between 2 and 4.

$$\text{Let } x_0 = \frac{2+4}{2} = 3$$

Now, $f(3) = +ve$, hence the root lies between 2 and 3.

$$x_1 = \frac{2+3}{2} = 2.5$$

$$f(x_1) = f(2.5) = -ve$$

The root lies between 2.5 and 3.

$$x_2 = \frac{2.5+3}{2} = 2.75$$

$$f(2.75) = -ve.$$

The root lies between 2.75 and 3.

$$x_3 = \frac{2.75+3}{2} = 2.875$$

$$f(x_3) = f(2.875) = -ve.$$

The root lies between 2.875 and 3.

$$x_4 = \frac{2.875+3}{2} = 2.9375$$

$$f(2.9375) = -ve.$$

The root lies between 2.9375 and 3.

$$x_5 = \frac{2.9375+3}{2} = 2.9688.$$

$$f(2.9688) = +ve$$

The root lies between 2.9688 and 2.9375

$$x_6 = \frac{2.9375+2.9688}{2} = 2.9532$$

$$f(2.9532) = +ve.$$

The root lies between 2.9375 and 2.9532.

$$x_7 = \frac{2.9375 + 2.9532}{2} = 2.9454$$

$f(2.9454) = +ve.$

The root lies between 2.9375 and 2.9454

$$x_8 = \frac{2.9375 + 2.9454}{2} = 2.9415$$

$f(2.9415) = -ve$

The root lies between 2.9415 and 2.9454

$$x_9 = \frac{2.9415 + 2.9454}{2} = 2.9435$$

$f(2.9435) = +ve.$

The root lies between 2.9415 and 2.9435

$$x_{10} = \frac{2.9415 + \cancel{2.9445}\ 2.9435}{2} = 2.9425$$

$f(2.9425) = -ve.$

The root lies between 2.9425 and 2.9435

$$x_{11} = \frac{2.9425 + 2.9435}{2} = 2.9430$$

$f(2.9430) = +ve$

The root lies between 2.9425 and 2.9430

$$x_{12} = \frac{2.9425 + 2.9430}{2} = 2.94275$$

$\cancel{f}\ x_{13} = 2.942875$

Approximate root is 2.9429.

2. Find the positive root of $x - \cos x = 0$ by bisection method correct to 4 decimal.

Solution:

let $f(x) = x - \cos x$

$f(0) = 0 - \cos(0) = -1 = -ve$

$f(1) = 1 - \cos(1) = 0.4597 = +ve.$

The root lies between 0 and 1

$$x_0 = \frac{0+1}{2} = 0.5$$

$f(0.5) = -0.3776 = -ve$

The root lies between 0.5 and 1

$$x_1 = \frac{0.5+1}{2} = 0.75$$

$f(0.75) = 0.018311 = +ve$

The root lies between 0.5 and 0.75

$$x_2 = \frac{0.5+0.75}{2} = 0.625$$

$f(0.625) = -0.18596 = -ve$

The root lies between 0.625 and 0.75

$$x_3 = \frac{0.625+0.75}{2} = 0.6875$$

$f(0.6875) = -0.085335 = -ve$

The root lies between 0.6875 and 0.75

$$x_4 = \frac{0.6875+0.75}{2} = 0.71875$$

$f(0.71875) = -0.033879 = -ve$

The root lies between 0.71875 and 0.75

$$x_5 = \frac{0.71875+0.75}{2} = 0.73438$$

$f(0.73438) = -0.007866 = -ve$

The root lies between 0.73438 and 0.75

$$x_6 = \frac{0.73438+0.75}{2} = 0.74219$$

$f(0.74219) = 0.0051999 = +ve$

The root lies between 0.73438 and 0.74219

$$x_7 = \frac{0.73438 + 0.74219}{2} = 0.73829$$

$f(0.73829) = -0.0013305 = -ve$

The root lies between 0.73829 and 0.74219

$$x_8 = \frac{0.73829 + 0.74219}{2} = 0.7402$$

$f(0.7402) = 0.0018663 = +ve$

The root lies between 0.73829 and 0.7402

$$x_9 = \frac{0.73829 + 0.7402}{2} = 0.73925$$

$f(0.73925) = 0.00027593 = +ve$

The root lies between 0.73829 and 0.73925

$$x_{10} = \frac{0.73829 + 0.73925}{2} = 0.7388$$

$f(0.7388) = 0.000083 = +ve.$

~~∴ The equation of~~

∴ The root of the equation is 0.7388.

REGULA FALSI METHOD (OR THE METHOD OF FALSE POSITION

Consider the equation $f(x)=0$ and let $f(a)$ and $f(b)$ be of opposite signs. Also, let $a < b$. The curve $y = f(x)$ will meet the x-axis at some point between $A(a, f(a))$ and $B(b, f(b))$. The equation of the chord joining the two points $A(a, f(a))$ and $B(b, f(b))$ is $\frac{y - f(a)}{x - a} = \frac{f(a) - f(b)}{a - b}$.

The x-coordinate of the point of intersection of this chord with the x-axis gives an approximate value for the root of $f(x)=0$. Setting $y=0$ in the chord equation. we get,

$$\frac{-f(a)}{x-a} = \frac{f(a)-f(b)}{a-b}$$

$$x[f(a)-f(b)] - af(a) + af(b) = -af(a) + bf(a)$$

$$x[f(a)-f(b)] = bf(a) - af(b)$$

$$x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)}$$

This value of x_1 gives an approximate value of the root of $f(x)=0$. $(a < x_1 < b)$.

Now $f(x_1)$ and $f(a)$ are of opposite signs or $f(x_1)$ and $f(b)$ are of opposite signs.

If $f(x_1).f(a) < 0$, then x_2 lies between x_1 and a.

Hence, $x_2 = \dfrac{a.f(x_1) - x_1 f(a)}{f(x_1)-f(a)}$.

In the same way, we get $x_3, x_4, \ldots\ldots$

This sequence $x_1, x_2, \ldots\ldots x_3$ will converge to the required root. In practice, we get x_i and x_{i+1} such that $|x_i - x_{i+1}| < \epsilon$, the required accuracy.

Problem 1:

Solve for a positive root of $x^3-4x+1=0$ by Regula-falsi-method:

Solution:

Let $f(x) = x^3-4x+1=0$

$f(1) = -2 = -ve$

$f(2) = 1 = +ve$

$f(0) = 1 = +ve$

$\therefore$ A root lies between 0 and 1

Another root lies between 1 and 2.

We shall find the root that lies between 0 and 1.

Here $a = 0, b = 1$.

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0 \times f(1) - 1 \times f(0)}{f(1) - f(0)}$$

$$x_1 = \frac{-1}{-2-1} = 0.33333$$

$$f(x_1) = f\left(\tfrac{1}{3}\right) = \tfrac{1}{27} - \tfrac{4}{3} + 1$$

$$= -0.2963$$

Now $f(0)$ and $f\left(\tfrac{1}{3}\right)$ are opposite in sign.

Hence the root lies between 0 and $\tfrac{1}{3}$.

$$x_2 = \frac{0 \times f\left(\tfrac{1}{3}\right) - \tfrac{1}{3} f(0)}{f\left(\tfrac{1}{3}\right) - f(0)}$$

$$= \frac{-\tfrac{1}{3}}{-1.2963}$$

$$x_2 = 0.25714$$

Now $f(x_2) = f(0.25714) = -0.011588 = -ve$.

$\therefore$ The root lies between 0 and 0.25714

$$x_3 = \frac{0 \times f(0.25714) - 0.25714 f(0)}{f(0.25714) - f(0)}$$

$$= \frac{-0.25714}{-1.011558}$$

$$x_3 = 0.25420$$

$$f(x_3) = f(0.25420) = -0.0003742$$

The root lies between 0 and 0.25420

$$x_4 = \frac{0 \times f(0.25420) - 0.25420 \times f(0)}{f(0.25420) - f(0)}$$

$$= \frac{-0.25420}{-1.0003742}$$

$$x_4 = 0.25410$$

$$f(x_4) = f(0.25410) = -0.000012936$$

The root lies between 0 and 0.25410

$$x_5 = \frac{0 \times f(0.25410) - 0.25410 \times f(0)}{f(0.25410) - f(0)}$$

$$= \frac{-0.25410}{-1.000012936}$$

$$x_5 = 0.25410.$$

∴ The root of the equation is 0.25410.

ITERATION METHOD (Fixed point iteration method).

To find the real root of an equation.

$$f(x) = 0 \rightarrow ①.$$

We rewrite it as,

$$x = \phi(x) \rightarrow ②$$

Equation ① and ② are equivalent, so their roots are the same. The root of equation ② is got as the abscission of the point of intersection of the curves $y = x$ and $y = \phi(x)$ and it is called the fixed point.

Procedure:

We first find an initial approximate value x_0 of the required root. Now using equation ②, we find a better approximation x_1 as $x_1 = \phi(x_0)$. We continue the iteration by finding,

$$x_2 = \phi(x_1)$$
$$x_3 = \phi(x_2)$$
$$\cdots \cdots$$

If the sequence $x_0, x_1, x_2, \ldots\ldots$ converges to a limit α then α, is the required root of the equation $f(x) = 0$.

Note:

The process converges only when the absolute value of the slope of $y = \phi(x)$ curve is the less than the slope of $y = x$ curve

(i,e) when $|\phi'(x)| < 1$.

Also if the slope of $y = \phi(x)$ is closer to zero, convergence will be faster.

Fixed point theorem:

If $\phi(x)$ and $\phi'(x)$ are continuous in the interval I in which the root α of the equation $x = \phi(x)$ lies, then the sequence of approximations $x_0, x_1, x_2, \ldots$ converge to the root α if,

i) $|\phi'(x)| < 1$ for all x in I.

ii) The initial approximation x_0 is choosen in I.

Problem:

1. Find the real root of the equation $\cos x = 3x - 1$ correct to 4 decimal places by iteration method.

Solution:

(Note: Use 'radian" mode in the Calculator for problems involving trigonometric function).

Let $f(x) = \cos x - 3x + 1 = 0$.

$f(0) = \cos(0) - 3(0) + 1 = 2 > 0$

$f(1) = \cos(1) - 3(1) + 1 = -1.4597 > 0$.

Hence there is a root of the equation between 0 and 1.

We rewrite the equation as,

$$3x = 1 + \cos x$$

$$x = \frac{1 + \cos x}{3} = \phi(x)$$

$$\phi'(x) = \frac{1}{3}(-\sin x)$$

$$|\phi'(0)| = 0$$

$$|\phi'(0.25)| = 0.0825$$

$$|\phi'(0.5)| = 0.1598$$

$$|\phi'(0.75)| = 0.2272$$

$$|\phi'(1)| = 0.2805$$

In fact $|\phi'(x)| = \dfrac{|\sin x|}{3} < 1$ for all x.

So we may use iteration method.

Let $x_0 = 0.5$

$x_1 = 0.62586$

$x_2 = 0.60349$

$x_3 = 0.60779$

$x_4 = 0.60697$

$x_5 = 0.60713$

$x_6 = 0.60710$

$x_7 = 0.60710$

$\therefore$ Hence the required root is 0.60710

2 Solve by iteration method $x^3+x^2=100$ for its real root.

Solution:

$$f(x) = x^3+x^3-100=0.$$

$$f(0) = -100<0$$

$$f(1) = -98<0$$

$$f(2) = -88<0$$

$$f(3) = -64<0$$

$$f(4) = -20<0$$

$$f(5) = 50<0$$

Hence there is a root of $f(x)=0$ that lies between 4 and 5. The given equation is rewritten as,

$$x^2(x+1)=100$$

$$x^2 = \frac{100}{x+1}$$

$$x = \frac{10}{\sqrt{x+1}} = \phi(x)$$

$$\therefore \phi'(x) = 10 \times \left(\frac{-1}{2}\right) \times (x+1)^{-3/2}$$

$$\phi'(x) = \frac{-5}{(x+1)^{3/2}}$$

$$|\phi'(x)| = \frac{5}{|(x+1)^{3/2}|} < 1 \text{ for all } x \in (4,5)$$

So we can use iteration method.

Let $x_0 = 4.5$

$x_1 = 4.26401$

$x_2 = 4.35854$

$x_3 = 4.31993$

$x_4 = 4.33558$

$x_5 = 4.32922$

$x_6 = 4.33180$

$x_7 = 4.33075$

$x_8 = 4.33118$

$x_9 = 4.33100$

$\therefore$ Hence the required root is 4.331

3. Find the negative root of the equation $x^3 - 2x + 5 = 0$ by iteration method.

Solution:

$$f(x) = x^3 - 2x + 5 = 0.$$

$$f(0) = 5 > 0$$

$$f(-1) = 6 > 0$$

$$f(-2) = 1 > 0$$

$$f(-3) = -16 < 0$$

Hence there is a root between -2 and -3.

We rewrite the given equation is,

$$x^3 = 2x - 5$$

$$x = (2x-5)^{\frac{1}{3}} = \phi(x)$$

$$\phi'(x) = \frac{1}{3}(2x-5)^{-2/3} \times 2$$

$$|\phi'(-2)| = 0.1541 < 1$$

$$|\phi'(-2.25)| = 0.1486 < 1$$

$$|\phi'(-2.5)| = 0.1436 < 1$$

$$|\phi'(2.75)| = 0.1390 < 1$$

$$|\phi'(-3)| = 0.1348 < 1$$

$|\phi'(x)| < 1$ for all $x \in (-3, -2)$

Let $x_0 = -2$

$x_1 = -2.08008$

$x_2 = -2.9235$

$x_3 = -2.9422$

$x_4 = -2.09450$

$x_5 = -2.09454$

$x_6 = -2.09455$

$x_7 = -2.09455$

Hence the required root is -2.09455.

NEWTON RAPHSON METHOD (NEWTON ITERATION METHOD).

Method of Tangents.

When the derivative of $f(x)$ is a simple expression and can be easily found, the roots of $f(x)$ can be calculated rapidly using this method.

Derivation:

Let $x = x_0$ be an approximate root of the equation $f(x)=0$ and let $x = x_1$ be the exact root ($\because f(x_1)=0$) such that $x - x_0 = h$ is small.

$$\therefore x_1 = x_0 + h \longrightarrow ①$$

Since $f(x)=0$ we have $f(x_0+h)=0$

By Taylor's Series expensions,

$$f(x_0) = \frac{h}{1!} f'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots = 0$$

Since h is small, we omit $h^2, h^3, \ldots$

$$\therefore f(x_0) + h f'(x_0) = 0$$

$$\therefore h = \frac{-f(x_0)}{f'(x_0)} \text{ resp approximately.}$$

$$\text{By ①, } x_1 = x_0 - \frac{f(x_0)}{f'(x_1)}$$

This gives a better approximation than x_0.

Similarly, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

Generally, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n = 0, 1, 2, \ldots$

This is Newton-Raphson's formula.

$\therefore$ Newton method converges if,

$$\left| \frac{f(x)\, f''(x)}{|f'(x)|^2} \right| < 1$$

(i,e) $|f(x) f''(x)| < [f'(x)]^2$ in the interval.

Note.

The error at any stage is proportional to the square of the error in the previous stage. So the order of convergence of Newton-Raphson method is two, (i,e)., the convergence is quadratic.

Problem:

1. By using Newton-Raphson method. Find the root of $x^4 - x - 10 = 0$, which the nearer to $x = 2$, correct upto three decimal places.

Solution:

Let $f(x) = x^4 - x - 10$

$f'(x) = 4x^3 - 1$

$f(1) = 1 - 1 - 10 = -10 < 0$

$f(2) = 16 - 2 - 10 = 4 > 0$

There is a root between 1 and 2 given that the root is nearer to $x = 2$. By Newton-Raphson method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left(\frac{x_n^4 - x_n - 10}{4x_n^3 - 1}\right)$$

$$= \frac{4x_n^4 - x_n - x_n^4 + x_n + 10}{4x_n^3 - 1}$$

$$x_{n+1} = \frac{3x_n^4 + 10}{4x_n^3 - 1}$$

Let $x_0 = 2$,

$$x_1 = \frac{3x_0^4 + 10}{3} = \frac{3(2^4)+10}{4(2^3)-1} = 1.87097$$

$$x_2 = \frac{3x_1^4 + 10}{4x_1^3 - 1} = \frac{3(1.87097)^4 + 10}{4(1.87097)-1} = 1.85578$$

$$x_3 = \frac{3x_2^4 + 10}{4x_2^3 - 1} = \frac{3(1.85578)^4 + 10}{4(1.85578)^3 - 1} = 1.85558$$

$$x_4 = \frac{3x_3^4 + 10}{4x_3^3 - 1} = \frac{3(1.85558)^4 + 10}{4(1.85558)^3 - 1} = 1.85558.$$

$x_3 = x_4 = 1.85558 \simeq 1.856.$

$\therefore$ The root, correct to three decimal places is

1.856.

2. Find by Newton-Raphson method, the real root of the equation $3x = \cos x + 1$.

Solution:

Let $f(x) = 3x - \cos x - 1$

$f'(x) = 3 + \sin x$

$f(0) = 0 - 1 - 1 = -2 < 0$

$f(1) = 3 - \cos(1) - 1 = 1.459 > 0$

$\therefore$ A root lies between 0 and 1.

By Newton-Raphson method.

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$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left(\frac{3x_n - \cos x_n - 1}{3 + \sin x_n} \right)$$

$$= \frac{3x_n + x_n \sin x_n - 3x_n + \cos x_n + 1}{3 + \sin x_n}$$

Let $x_0 = 1$

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = 0.62002$$

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} = 0.60712$$

$$x_3 = \frac{x_2 \sin x_2 + \cos x_2 + 1}{3 + \sin x_2} = 0.60710$$

$$x_4 = \frac{x_3 \sin x_3 + \cos x_3 + 1}{3 + \sin x_3} = 0.60710$$

$\therefore$ The required root is 0.6071

3. Using Newton - Raphson method, establish the formula, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$, to calculate the square root of N. and find the square root of 5 correct to four places of decimals.

Solution:

Let $x = \sqrt{N}$

$\therefore x^2 = N$

$x^2 - N = 0$. The solution of this equation is $\sqrt{N}$

Let $f(x) = x^2 - N$.

$f'(x) = 2x$.

By Newton - Raphson method.

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$$

$$x_{n+1} = x_n - \left(\frac{x_n^2 - N}{2x_n}\right)$$

$$= \frac{2x_n^2 - x_n^2 + N}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + N}{2x_n}$$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

$\therefore$ $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$ is the iterative formula to Calculate $\sqrt{N}$.

To find the square root of 5, take $N=5$. Now, $\sqrt{5}$ lies between 2 and 3.

Take $x_0 = 2$.

$$x_1 = \frac{1}{2}\left(x_0 + \frac{5}{x_0}\right) = \frac{1}{2}\left(2 + \frac{5}{2}\right) = 2.25$$

$$x_2 = \frac{1}{2}\left(2.25 + \frac{5}{2.25}\right) = 2.23611$$

$$x_3 = \frac{1}{2}\left(2.23611 + \frac{5}{2.23611}\right) = 2.23607$$

$$x_4 = \frac{1}{2}\left(2.23607 + \frac{5}{2.23607}\right) = 2.23607$$

$\therefore$ Hence $\sqrt{5}$ is approximately equal to 2.2361.

4. Find an iterative formula to find the reciprocal of a given number N by Newton-Raphson method and find the value of $\frac{1}{19}$.

Solution:

Let $x = \frac{1}{N}$

$N = \frac{1}{x}$.

Take $f(x) = \frac{1}{x} - N = 0$. The solution of this equation gives the value of $\frac{1}{N}$.

$$f'(x) = \frac{-1}{x^2}$$

By Newton - Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left(\frac{\frac{1}{x_n} - N}{-1/x_n^2} \right)$$

$$= x_n + x_n^2 \left(\frac{1}{x_n} - N \right)$$

$$= x_n + x_n - N x_n^2$$

$$= 2x_n - N x_n^2$$

$$= x_n (2 - N x_n)$$

$x_{n+1} = x_n (2 - N x_n)$ is the interactive formula to find the reciprocal of N.

Take $N = 19$

We know that, $\frac{1}{20} = 0.05$

Take, $x_0 = 0.05$

$$x_1 = x_0 (2 - N x_0)$$

$$= 0.05 (2 - 19 \times 0.05)$$

$$x_1 = 0.05250$$

$$x_2 = 0.05263$$

$$x_3 = 0.05263$$

Hence the appropriate value of $\frac{1}{19}$ is 0.05263.

HORNER'S METHOD :

This numerical method is employed to determine both the commensurable and the uncommensurable real roots of a polynomial equation. Firstly, we find the integral part of the root and then by every iteration we find each decimal place value in succession.

Suppose a positive root of $f(x)=0$ lies between a and $a+1$ (where a is an integer).

Let that root be $a.a_1, a_2 a_3 a_4 \ldots$

First, diminish the roots of $f(x)=0$ by the integral part 'a' and let $\phi_1(x)=0$ posses the root $0.a_1 a_2 a_3 a_4 \ldots$

~~First~~ Secondly, multiply the roots of $\phi_1(x)=0$ by 10 and let $\phi_2(x)=0$ possess $a_1 . a_2 a_3 a_4 \ldots$ as a root.

Thirdly, find the value of a_1 and then diminish the roots by a_1 and let $\phi_3(x)=0$ possess a root ~~0.a₂a₃~~ $0.a_2 a_3 a_4 \ldots$

Now repeating the ~~possess~~ process we find $a_2, a_3, a_4 \ldots$ each time.

Problem 1:

Find the positive root between 1 and 2 which satisfies $x^3-3x+1=0$ to 3 decimal places.

Solution:

$$\text{Let } f(x) = x^3-3x+1=0.$$

$$f(0) = 0-3(0)+1 = 1$$

$$f(1) = (1)^3-3(1)+1 = -1 = -ve.$$

$$f(2) = (2)^3-3(2)+1 = 3 = +ve.$$

Therefore a root lies between 1 and 2. Let it be $1.a_1 a_2 a_3 \ldots$

The integeral part is 1.

$\therefore$ Diminish the root of $f(x)=0$ by 1.

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	1	0	-3	1
1		1	1	-2
	1	1	-2	$\boxed{-1}$
1		1	2	
	1	2	$\boxed{0}$	
1		1		
	1	$\boxed{3}$		

$\phi_1(x) = x^3 + 3x^2 - 1 = 0$ has the root $0.a_1a_2a_3\ldots$

Multiply the roots of $\phi_2(x) = 0$ by 10.

$\therefore \phi_2(x) = x^3 + 30x^2 - 1000 = 0$ has root $a_1a_2a_3\ldots$

$\phi_3(5) = -ve$, $\phi_3(6) = +ve$.

$a_1 = 5$

$\therefore \phi_2(5) = 0$ has a root $5.a_2a_3\ldots$

Diminish the roots of $\phi_3(x) = 0$ by 5.

	1	30	0	-1000
5		5	175	875
	1	35	175	$\boxed{-125}$
5		5	200	
	1	40	$\boxed{375}$	
5		5		
	1	$\boxed{45}$		

$\therefore x^3 + 45x^2 + 375x - 125 = 0$ has a root $0.a_2a_3\ldots$

$\therefore$ Multiply the root by 10.

$\phi_3(x) = x^3 + 450x^2 + 37500x - 125000 = 0$ has a root $a_2a_3a_4$.

Now $\phi_3(3) = -ve$, $\phi_4(4) = +ve$.

$a_2 = 3$

$\phi_3(x)=0$ has a root $3.03a_4$.....

Now diminish the roots of $\phi_3(x)=0$ by 3

3	1	450	37500	-125000
		3	1359	116577
3	1	453	38859	-8423
		3	1368	
3	1	456	40227	
		3		
	1	459		

$\phi_4(x)=x^3+459x^2+40227x-8423=0$ has the root $0.a_3a_4$.....

$\therefore$ Multiply the roots of $\phi_4(x)=0$ by 10.

$\therefore$ $\phi_5(x)=x^3+4590x^2+4022700x-8423000=0$ has the root $a_3.a_4a_5$......

~~Now diminish~~ $\phi_5(2)=-ve$

$\phi_5(3)=-ve.$

$a_3=2$

Now diminish the roots of $\phi_5(x)=0$ by 2.

2	1	4590	4022700	-8423000
		2	9184	8063768
2	1	4592	4031884	-359232
		2	9188	
2	1	4594	4041072	
		2		
	1	4596		

$\therefore$ $\phi_6(x)=x^3+4596x^2+4041072-359232=0$ has the root $0.a_4a_5$.

$\therefore$ The required root is 1.532.

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Source :

Text Books:

1. P.Kandasamy, V.Thilagavathy, K.Gunavathi : "Numerical Methods",S.Chand& Company Ltd,New Delhi, 2016.