6·9. FACTORIAL EXPERIMENTS

So far we have considered the testing of a number of treatments, not necessarily related to each other, in Randomised Blocks, or Latin Squares or Graeco Latin Squares. In industrial applications frequently we know that several factors may affect the characteristics in which we are interested, and we wish to estimate the effects of each of the factors and how the effect of one factor varies over the level of the other factors. For example, the yield of a chemical process may be affected by several factors such as the levels of pressure, temperature, rate of agitation, and proportions of reactants, etc. One might try to test each of the factors separately, holding all other factors constant in a given experiment, but with a little thought it might be clear that such an experiment might not give the information required. The logical procedure would be to vary all factors simultaneously, *i.e.*, within the framework of the same experiment. When we do so, we have what is now widely known as a *factorial experiment*.

The factorial experiments are particularly useful in experimental situations which require the examination of the effects of varying two or more factors. In a complete exploration of such a situation it is not sufficient to vary one factor at a time, but that all combinations of the different factor levels must be examine in order to elucidate the effect of each factor and the possible ways in which each factor may be modified by the variation of the others. In the analysis of the experimental results, the effect of each factor can be determined with the same accuracy as if only one factor had been varied at a time and the interaction effects between the factors can also be evaluated.

In the foregoing experiments performed either in C.R.D. or R.B.D. or L.S.D., we were primarily concerned with the comparison and estimation of the effects of a *single* set of treatments like varieties of wheat, manure of different methods of cultivation etc. Such experiments which deal with one factor only may be called *simple experiments*. In *factorial experiment*'* , as the adjective factorial indicates, the effects of several factors of variation are studied and investigated simultaneously, the treatments being all the combinations of different factors under study. In these experiments an attempt is made to estimate the effects of each of the factors and also the interaction effects, *i.e.*, the variation in the effect of one factor as a result to different levels of other factors.

As a simple illustration let us consider two fertilizers, say, Potash (K) and Nitrogen (N). Let us suppose that there are p different varieties of Potash and q different varieties of Nitrogen. p and q are termed as the *levels* of the factors potash and Nitrogen respectively. To find the effectiveness of various treatments, *viz.*, different levels of Potash or Nitrogen we might conduct two simple experiments, one for Potash and the other for Nitrogen. A series of experiments in which only one factor is varied at a time would be both lengthy and costly and might still be unstatisfactory because of systematic changes in the general background conditions. Moreover, these simple experiments do not give us any information regarding the dependence or independence of one factor on the other, *i.e.*, they do not tell us anything about the interaction effect (NK). The only alternative is to try to investigate the variations in several factors simultaneously by conducting the above experiment as a $p \times q$ factorial experiment, where p and q are the levels of various factors under consideration. In general, if the levels of various factors are equal then s^n factorial experiment means an experiments with n factors, each at s levels where n is any positive integer greater than or equal to 2, *e.g.*,

* Prior to R.A. Fisher (1926), factorial experiments were called *'complex experiments'*.

2^3-experiment means an experiment with 3 factors at 2 levels each and 3^2-experiment means an experiment with 2 factors at 3 levels each.

Advantages of Factorial Experiment.

1. It increases the scope of the experiment and its inductive value and it does so mainly by giving information not only on the main factors but on their interactions.

2. The various levels of one factor constitute replications of other factors and increase the amount of information obtained on all factors.

3. When there are no interactions, the factorial design gives the maximum efficiency in the estimate of the effects.

4. When interactions exist, their nature being unknown a factorial design is necessary to avoid misleading conclusions.

5. In the factorial design the effect of a factor is estimated at several levels of other factors and the conclusions hold over a wide range of conditions.

Basic Ideas and Notations in the 2^n-Factorial Experiment. Let us first consider the design of the form 2^n in which there are n factors, each at two levels. Levels may be quite literally two quantitative levels or concentrations of, say, a fertilizer or it may mean two qualitative alternatives like two species of a plant. In some cases one level is simply the control group, *i.e.*, the absence of the factor and the other is its presence.

In order to develop extended notation to present the analysis of the design in a concise form, let us start, for simplicity with a 2^2-factorial design.

6·9·1. 2^2-Factorial Design. Here we have two factors each at two levels (0,1), say, so that there are $2 \times 2 = 4$ treatment combinations in all. Following the notations due to Yates, let the capital letters A and B indicate the names of the two factors under study and let the small letters a and b denote one of the two levels of each of the corresponding factors and this will be called the second level. The first level of A and B is generally expressed by the absence of the corresponding letter in the treatment combinations. The four treatment combinations can be enumerated as follows :

a_0b_0 or '1' : Factors A and B, both at first level.
a_1b_0 or a : A at second level and B at first level.
a_0b_1 or b : A at first level and B at second level.
a_1b_1 or ab : A and B both at second level.

These four treatment combinations can be compared by laying out the experiment in (*i*) R.B.D., with r replicates (say), each replicate containing 4 units, or (*ii*) 4×4 L.S.D., and ANOVA can be carried out accordingly. In the above cases there are 3 *d.f.* associated with the *treatment effects*. In factorial experiment our main objective is to carry out separate tests for the main effects A, B and the interaction AB, splitting the treatment S.S. with 3 *d.f.* into three orthogonal components each with 1 *d.f.* and each associated either with the main effects A and B or the intersection AB.

Main Effects and Interactions. Suppose the factorial experiment with $2^2 = 4$ treatments is conducted in r-blocks or *replicates* as they are often called. Let [1], [a], [b] and [ab] denote the total yields of the r-units (plots) receiving the treatments 1, a, b and ab respectively and let the corresponding mean values obtained on dividing these totals by r be denoted by (1), (a), (b) and (ab) respectively. The letters A, B and AB when they refer to numbers will represent the main effects due to the factors A and B and their interaction AB respectively.

The effect of factor A can be represented by the difference between mean yields obtained at each level.

Thus the effect of factor A at the first level b_0 of $B = (a_1b_0) - (a_0b_0) = (a) - (1)$ ... (6·212)

Similarly, the effect of A at the second level b_1 of $B = (a_1b_1) - (a_0b_1) = (ab) - (b)$ ...(6·212a)

These two effects in (6·212) and (6·212a) are termed as the *simple effects of the factor A.*

The average observed effect of A over the two levels of B is called the *main effect* due to A and is defined by :

$$A = \frac{1}{2}\left[(ab) - (b) + (a) - (1)\right] \quad \text{...(6·213)}$$

A simplified form of (6·213) is given by :

$$A = \frac{1}{2}(a - 1)(b + 1) \quad \text{... (6·213a)}$$

where the right-hand side is to be expanded algebraically and then the treatment combinations are to be replaced by treatment means.

Arguing similarly we shall get the *main effect* due to factor B as

$$B = \frac{1}{2}\left[(ab) - (a) + (b) - (1)\right] \quad \text{... (6·214)}$$

or

$$B = \frac{1}{2}(a + 1)(b - 1) \quad \text{... (6·214a)}$$

where, again, the right-hand side is to be expanded algebraically and the treatment combinations are to be replaced by their means.

The interaction of two factors is the failure of the levels of one factor, say, A to retain the same order and magnitude of performance throughout all levels of the second factor, say, B. If the two factors act independently of one another, we should expect the true effect of one to be same at either level of other. In other words, we should expect that the two expressions observed in (6·212) and (6·212a) were really the estimates of the same thing. On the other hand, if the two factors are not independent, the two expressions in (6·212) and (6·212a) will not be same and the difference of these two numbers is, therefore, a measure of the extent to which the factors interact and we write the *two-factor interaction or the first order interaction* between the factors A and B as :

$$AB = \frac{1}{2}\left[(ab) - (b) - (a) + (1)\right] \quad \text{... (6·215)}$$

or

$$AB = \frac{1}{2}(a - 1)(b - 1) \quad \text{.. (6·215a)}$$

where, as usual the R.H.S. is to be expanded algebraically and the treatment combinations are to be replaced by the corresponding treatment means.

***Remarks* 1.** From (6·214), we get an expression for the interaction of factor B with the factor A as

$$BA = \frac{1}{2}\left[(ab) - (a) - (b) + (1)\right] \quad \text{... (6·216)}$$

$$= \frac{1}{2}(a - 1)(b - 1) \quad \text{... (6·216a)}$$

which are same as the expressions in (6·215) and (6·216a). Hence, the interaction AB is same as the interaction AB which implies that the interaction does not depend on the order of the factors.

2. Contrast (*Definition*). *A linear combination* $\sum_{i=1}^{k} c_i t_i$ of k treatment means t_i $(i = 1, 2, ..., k)$ is called a *contrast* (*or a comparison*) of treatment means if $\sum_{i=1}^{k} c_i = 0$. In other words, *contrast* is a linear combination of treatment means such that the sum of the coefficients is zero.

Orthogonal Contrasts (*Definition*). *Two contrasts of k-treatment means* t_i, $(i = 1, 2, ..., k)$, *viz.,*

$$\sum_{i=1}^{k} c_i t_i, \quad \sum_{i=1}^{k} c_i = 0 \qquad \text{... (6·217)}$$

and

$$\sum_{i=1}^{k} d_i t_i, \quad \sum_{i=1}^{k} d_i = 0$$

are said to be orthogonal if $\sum_{i=1}^{k} c_i d_i = 0$. ... (6·217*a*)

In other words, the contrasts are orthogonal if the sum of the product of the coefficients of corresponding treatment means is zero.

Thus from (6·213), (6·214) and (6·215) it can be easily seen that the main effects *A* and *B* as well as the interaction *AB* are contrasts of the treatment means. We also observe that for the main effects *A* and *B*, the sum of the product of the coefficients of corresponding treatment means is :

$$\frac{1}{2}\times\frac{1}{2}+\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)\times\left(-\frac{1}{2}\right)+\left(-\frac{1}{2}\right)\times\left(-\frac{1}{2}\right)=0$$

i.e., the main effects *A* and *B* are orthogonal contrasts. It can be similarly verified that the contrasts *A*, *B* and *AB* are orthogonal to each other.

3. Let *M* denote the mean yield of the four treatment combinations. Then

$$M = \frac{1}{4}\left[(ab) + (a) + (b) + (1)\right]$$

$$= \frac{1}{4}(a + 1)(b + 1) \qquad \text{... (6·218)}$$

as usual. The general mean *M* and the three mutually orthogonal contrasts, *viz.*, the main effects *A* and *B*, and the interaction *AB* are summarised in terms of the means of the treatment combinations in the *Table 6·38*.

TABLE 6·38 : TABLE OF SIGNS AND DIVISORS GIVING *M*, THE MAIN EFFECTS AND INTERACTIONS IN TERMS OF INDIVIDUAL TREATMENT MEANS

Factorial effects	*Treatment Mean* (1)	(*a*)	(*b*)	(*ab*)	*Divisor*
M	+	+	+	+	4
A	–	+	–	+	2
B	–	–	+	+	2
AB	+	–	–	+	2

The signs for the mean effects in *Table 6·38* can be easily written according to the following rule : "Give a plus sign to each of the treatment means whenever the corresponding factor is at the second level, otherwise give a negative sign. Thus, in terms of the notations *adopted*, for the main effects, plus sign is to be given to the treatment combinations which contain the corresponding small letter otherwise a negative sign is to be given. For a two-factor interaction, the signs to be attached to various treatment means are obtained by combining (multiplying) the signs of the corresponding main effects.

4. The rows of the above table enable us to express the factorial effects in terms of the treatment means. On solving the equations (6·213) to (6·215) and (6·218) for (*a*), (*b*) , etc., we shall find that the columns of the *Table 6·38* enable us to express the treatment means in terms of the factorial effects. For example, we shall get

$$(ab) = M + \tfrac{1}{2}\left[A + B + AB\right] \qquad \text{... (6·219)}$$

$$(a) = M + \tfrac{1}{2}\left[A - B - AB\right] \qquad \text{... (6·219a)}$$

and so on, the only point to be noted is that the factor $\frac{1}{2}$ occurs with all the terms except *M*.

The *Table 6·38* also enables us to express the simple effects in terms of the factorial effects, *e.g.*,

(*a*) – (1) = Simple effect of *A* when *B* is at first level = *A* – *AB* (From *Table 6·38*) ... (6·220)

(*ab*) – (*b*) = Simple effect of *A* when *B* is at second level = *A* + *AB* (From *Table 6·38*) ... (6·220*b*)

Thus we see that if the interactions are absent, *i.e.*, if *AB* = 0 then

$$A = (a) - (1) = (ab) - b) \qquad \text{... (6·220b)}$$

i.e., the simple effects are equal to the main or factorial effects. Thus the interaction *AB* may be interpreted as a measure of the error committed in estimating the effect of *A* if the factors *A* and *B* are assumed to be independent.

5. From the expression (6·213a), (6·214a) and (6·215a), we observe that a minus sign appears in any factor on the R.H.S. if the letter is present on the left.

Statistical Analysis of 2^2-design. Factorial experiments are conducted either in C.R.D. or R.B.D. or L.S.D. and thus they can be analysed in the usual manner except that in this case the treatment S.S. is split into three orthogonal components each with 1 *d.f.* It has already been pointed out that the main effects A and B, and the interaction AB are mutually orthogonal contrasts of treatment means. The S.S. due to the factorial effects A, B and AB is obtained by multiplying the squares of the factorial effects by a suitable quantity. In practice, these effects are usually computed from the treatment totals $[a]$, $[b]$, $[ab]$ etc., rather than from the treatment means (a), (b), etc. Factorial effect totals are given by the expressions :

$$\left.\begin{aligned} [A] &= [ab] - [b] + [a] - [1] \\ [B] &= [ab] + [b] - [a] - [1] \\ [AB] &= [ab] - [a] - [b] - [1] \end{aligned}\right\} \qquad \dots (6·221)$$

The S.S. due to any factorial effect is obtained on multiplying the square of the effect total by the factor $(1/4r)$, where r is the common replication number (*c.f.* Remark 7 below). Thus

$$\left.\begin{aligned} &\text{S.S. due to main effect of } A = [A]^2/4r \\ &\text{S.S. due to main effect of } B = [B]^2/4r \\ \text{and}\quad &\text{S.S. due to interaction } AB = [AB]^2/4r, \end{aligned}\right\} \qquad \dots (6·222)$$

each with 1 *d.f.*

TABLE 6·39 : ANOVA TABLE FOR FIXED EFFECT MODEL TWO FACTOR (2^2) EXPERIMENT IN R.B.D. IN 'r' REPLICATES

Source of Variation	*d.f.*	*S.S.*	*M.S.S.*	*Variance Ratio 'F'*
Blocks (Replicates)	$r-1$	S_R^2	$s_R^2 = \dfrac{S_R^2}{(r-1)}$	$F_R = s_R^2 / s_E^2$
Main effect A	1	$S_A^2 = [A]^2/4r$	$s_A^2 = S_A^2$	$F_A = s_A^2 / s_E^2$
Main effect B	1	$S_B^2 = [B]^2/4r$	$s_B^2 = S_B^2$	$F_B = s_B^2 / s_E^2$
Interaction $A \times B$	1	$S_{AB}^2 = [AB]^2 / 4r$	$s_{AB}^2 = S_{AB}^2$	$F_{AB} = s_{AB}^2 / s_E^2$
Error	$3(r-1)$	S_E^2 = By subtraction	$s_E^2 = S_E^2 / [3(r-1)]$	
Total	$4r-1$	$\sum_i \sum_j (y_{ij} - \bar{y}_{..})^2$		

Here each of the statistics F_A, F_B and F_{AB} follows central F-distribution with $[1, 3(r-1)]$ *d.f.* If for any factorial effect, calculated F is greater than tabulated F for $[1, 3(r-1)]$ *d.f.* and at certain level of significance 'say' α, then the null hypothesis H_0 of the presence of the factorial effect is rejected, otherwise H_0 may be accepted.

Remarks.

6. The S.S. due to factorial effects A, B and AB, each with 1 *d.f.* will add up to the treatment S.S. with 3 *d.f.*

7. If T_i is the treatment total for the ith treatment replicated r_i times; $i = 1, 2, \dots, k$ then the contrast of the treatment totals is given by the linear combination

$$u = \sum_{i=1}^{k} c_i T_i, \text{ where } \sum_{i=1}^{k} c_i r_i = 0 \qquad \dots (6·223)$$

S.S. due to the contrast u is given by :

$$S_u^2 = \frac{u^2}{\sum r_i c_i^2} \quad \ldots (6\cdot223a)$$

Thus the S.S. due to the main effect A, where $[A]$ is a contrast of treatment totals *c.f.* (6·221) is given by :

$$S_A^2 = \frac{[A]^2}{\sum_{i=1}^{4} r.1} = \frac{[A]^2}{4r} \quad \ldots (6\cdot223b)$$

Similarly we can obtain the expression for the S.S. due to B and AB as given in (6·222).

8. The factorial effects can be expressed in terms of treatment totals as follows :

$$\text{Main effect of } A = \frac{1}{2}\Big[(ab)-(b)+(a)-(1)\Big] = \frac{1}{2}\Big[\frac{[ab]}{r}-\frac{[b]}{r}+\frac{[a]}{r}-\frac{[1]}{r}\Big]$$

$$= \frac{1}{2r}\Big[[ab]-[b]+[a]-[1]\Big] = \frac{[A]}{2r}$$

Similarly, we can obtain expressions for B and AB. Thus

$$\left.\begin{aligned} \text{Main effect of } A &= [A]/2r \\ \text{Main effect of } B &= [B]/2r \\ \text{Main effect of } AB &= [AB]/2r \end{aligned}\right\} \quad \ldots (6\cdot224)$$

9. We can test the significance of the factorial effects directly from the factorial effect totals without applying the F-test as in ANOVA Table.

The standard error for any factorial effect total $[A]$, $[B]$ or $[AB]$ is $\sqrt{4r\sigma_e^2}$ and the S.E. of any factorial effect mean is $\sqrt{\sigma_e^2/r}$, where σ_e^2 is estimated by the error mean S.S., *viz.*, s_E^2. Then the significant value for the factorial effect totals significant at 5% level of significance is given by :

$$d = t_{0\cdot025}\ (\text{for error } d.f.) \times \text{S.E. (factorial effect total)}\ t_{3(r-1)}\,(0\cdot025) \times \sqrt{4r\, s_E^2} \quad \ldots (6\cdot225)$$

whereas for a factorial effect mean, the significant value is given by :

$$d_1 = t_{3(r-1)}(0\cdot025)\ \sqrt{s_E^2/r}\,. \quad \ldots (6\cdot225a)$$

6·9·2. Yates' Method of computing Factorial Effect Totals. For the calculation of various factorial effect totals for 2^n-factorial experiments ***F. Yates*** developed a special computational rule which enables us to avoid specific algebraic formulae, *e.g.*, the expressions in (6·221) for 2^2-factorial experiment. Yates' method consists in the following steps :

1. In the first column we write the treatment combinations. It is an essential part of the procedure that the treatment combinations be written in a standard systematic order as explained below :

"Starting with the treatment combination I, each factor is introduced in turn and is then followed by all combinations of itself with the treatment combinations previously written down, *e.g.*, for 2^2-experiment with factors A and B, the order of treatment combinations will be 1, a, b, ab and for 2^3 factorial experiment with factors A, B, and C, the order of treatment combinations will be 1, a, b, ab, c, ac, bc, abc, and so on. [For details of 2^3-experiment, notations, etc. see § 6·9·3.]

2. Against each treatment combination, write the corresponding total yields from all the replicates.

3. The entries in the third column can be split into two halves. The first half is obtained by writing down in order, the pairwise sums of the values in column 2 and the second half is obtained by writing in the same order the pairwise differences of the values in second column. It is to be remembered that the first member is to be subtracted from the second member of a pair.

4. To complete the next (4th) column, the whole of the procedure as explained in step 3 is repeated on column 3, and for 2^3-design, the 5th column is derived from 4th in a similar manner.

Thus for a 2^n-factorial experiment there will be n cycles of this "sum and difference" procedure. The first term in the last, *viz.*, $(n + 2)$th column always given the grand total (G) while the other entries in the last column are the totals of the main effects or the interactions corresponding to the treatment combinations in the first column of the table. In the *Tables 6·40 and 6·41*, we illustrate Yates' Method for 2^2 and 2^3 factorial experiments respectively.

TABLE 6·40 : YATES' METHOD FOR A 2^2-EXPERIMENT

Treatment Combination (1)	*Total Yield from all replicates* (2)	(3)	(4)	*Effect Totals*
'1'	[1]	[1] + [a]	[1] + [a] + [b] + [ab]	Grand Total
a	[a]	[b] + [ab]	[ab] – [b] + [a] – [1]	[A]
b	[b]	[a] – [1]	[ab] + [b] – [a] – [1]	[B]
ab	[ab]	[ab} – [b]	[ab] – [b] – [a] + [1]	[AB]

TABLE 6·41 : YATES' METHOD FOR A 2^3-EXPERIMENT

Treatment Combination (1)	*Treatment Totals* (2)	(3)	(4)	(5)	*Effect Totals*
'1'	[1]	$[1] + [a] = u_1$ (say)	$u_1 + u_2 = v_1$	$v_1 + v_2 = w_1$	Grand Total
a	[a]	$[b] + [ab] = u_2$ (say)	$u_3 + u_4 = v_2$	$v_3 + v_4 = w_2$	[A]
b	[b]	$[c] + [ac] = u_3$ (say)	$u_5 + u_6 = v_3$	$v_5 + v_6 = w_3$	[B]
ab	[ab]	$[bc] + [abc] = u_4$ (say)	$u_7 + u_8 = v_4$	$v_7 + v_8 = w_4$	[AB]
c	[c]	$[a] - [1] = u_5$ (say)	$u_2 - u_1 = v_5$	$v_2 - v_1 = w_5$	[C]
ac	[ac]	$[ab] - [b] = u_6$ (say)	$u_4 - u_3 = v_6$	$v_4 - v_3 = w_6$	[AC]
bc	[bc]	$[ac] - [c] = u_7$ (say)	$u_6 - u_5 - v_7$	$v_6 - v_5 = w_7$	[BC]
abc	[abc]	$[abc] - [bc] = u_8$ (say)	$u_8 - u_7 = v_8$	$v_8 - v_7 = w_8$	[ABC]

TABLE 6·42

Block	*Yields (lbs per plot)*			
I	(1) 23	k 25	p 22	kp 38
II	p 40	(1) 26	k 36	kp 38
III	(1) 29	k 20	pk 30	p 20
IV	kp 34	k 31	p 24	(1) 28

Example 6·9. *An experiment was planned to study the effect of sulphate of potash and super phosphate on the yield of potatoes. All the combinations of 2 levels of super phosphate [0 cent (p_0) and 5 cent (p_1)/acre] and two levels of sulphate of potash [0 cent (k_0) and 5 cent (k_1/acre] were studied in a randomised block design with 4 replications for each. The (1/70) yields [lb. per plot = (1/70) acre obtained are given in Table 6·42.*

Analyse the data and give your conclusions.

Solution. Taking deviation from $y = 29$, we re-arrange the data in *Table 6·42* in the Table 6·45 for computations of the S.S. due to treatments and blocks :

Treatment Combination	*Blocks* I	II	III	IV	*Treatment Totals,* T_i	T_i^2
(1)	−6	−3	0	−1	−10	100
k	−4	7	−9	2	−4	16
p	−7	11	−9	−5	−10	100
kp	9	9	1	5	24	576
(Block Totals) B_j	−8	24	−17	1	↓ $G = 0$ →	
B_j^2	64	576	289	1		

H_0 : The data is homogeneous with respect to the blocks and the treatments.

$N = 4 \times 4 = 16$; $G = 0$; $R.S.S. = \sum_i \sum_j y_{ij}^2 = 660$

$$\text{C.F.} = \text{Correction Factor} = \frac{G^2}{N} = \frac{(0)^2}{16} = 0$$

$$\text{Total S.S.} = RSS - \text{C.F.} = 660 - 0 = 660$$

$$\text{Block S.S.} = \tfrac{1}{4}\sum_j B_j^2 - \text{C.F.} = \frac{64 + 576 + 289 + 1}{4} = \frac{930}{4} = 232{\cdot}50$$

$$\text{Treatment S.S.} = \tfrac{1}{4}\sum_i T_i^2 - \text{C.F.} = \frac{100 + 16 + 100 + 576}{4} = \frac{792}{4} = 198$$

$$\text{Error S.S.} = 660 - (232{\cdot}50 + 198{\cdot}0) = 229{\cdot}50$$

We now compute the factorial effect totals by Yates Method.

TABLE 6·44 : YATES' METHOD FOR 2^2-EXPERIMENT

Treatment Combination (1)	*Total Yield from all blocks* (2)	(3)	*Factorial effects totals* (4)	*S.S.* (5) = $(4)^2/4r$
'I'	−10	−14	0 = G	$(0)^2/16 = 0 = C.F.$
k	−4	14	40 = [K]	$(40)^2/16 = 100 = S_K^2$
p	−10	6	28 = [P]	$(28)^2/16 = 49 = S_P^2$
kp	24	34	28 = [KP]	$(28)^2/16 = 49 = S_{KP}^2$

TABLE 6·45 : ANALYSIS OF VARIANCE TABLE FOR THE 2^2-EXPERIMENT

Source of variation	*d.f.*	*S.S.*	*M.S.S.*	*Variance Ratio F*	*Tabulated F* 5%	1%
Blocks	3	232·5	77·5	3·04	3·86	6·99
Treatments	3	198·0	66·0	2·59	3·86	6·99

K	1	100	100	3·92	5·12	10·56
P	1	49	49	1·92	5·12	10·56
KP	1	49	49	1·92	5·12	10·56
Error	9	229·5	25·5	—	—	—
Totals	15	660	—	—	—	—

As in each of the cases, the computed value of F is less than the corresponding tabulated (critical) value, there are no significant main or interaction effects present in the experiment. The blocks as well as treatments do not differ significantly.

Since the blocks do not differ significantly, we conclude that there is no special contribution from fluctuations in soil fertility and thus the division of the whole experimental area into blocks does not result in any gain in accuracy.

Remark. It may be noted that

$$S_K^2 + S_P^2 + S_{KP}^2 = 100 + 49 + 49 = 198 = \text{Treatment S.S.},$$

as it should be.

6·9·3. 2^3-Factorial Experiment. In 2^3-experiment we consider three factors, say, A, B, and C each at two levels, say, (a_0, a_1), (b_0, b_1) and (c_0, c_1) respectively, so that there are $2^3 = 8$ treatment combinations in all. Extending the notations due to Yates for a 2^2-experiment, let the corresponding small letters a, b and c denote the second level of each of the corresponding factors. The first level of each factor A, B and C is signified by the absence of the corresponding letter in the treatment combinations. The eight treatment combinations in a standard order are

$$\text{'1'}, a, b, ab, c, ac, bc, abc,$$

where, for example

$$1 = a_0b_0c_0, \quad a = a_1b_0c_0, \quad ab = a_1b_1c_0, \quad abc = a_1b_1c_1, \text{ etc.}$$

2^3-factorial experiment can be performed as a *C.R.D.* with 8 treatments, or *R.B.D.* with r replicates (say), each replicate containing 8 treatments of L.S.D. with $m = 8$ and data can be analysed accordingly. In 2^3-experiment we split up the treatment S.S. with 7 *d.f.* into 7 orthogonal components corresponding to the three main effects A, B and C, three first order (or two factor) interactions AB, AC, and BC and one second order interaction (or three factor interaction) ABC, each carrying 1 *d.f.* As in the case of 2^2-experiment A, B, AB, BC, etc., when they refer to numbers will represent the corresponding factorial effects.

Remark. The second order interaction ABC is a slightly difficult concept to grasp than the first order interactions and it can be interpreted in various ways. For example, we might interpret it as the difference in the interaction AB calculated at each of the two levels of C. However, in practice three factor interactions are usually small relative to the main effects and the two factor interactions and for practical purposes quite frequently they can be neglected. Very rarely, cases arise where second order interactions are important.

Main Effects and Interactions. Following the same notations for treatment totals and treatment means as in 2^2-factorial experiment, the simple effect of A, (say), is given by the differences in the mean yields of A as a result of increasing the factor A from the level a_0 to a_1, at other levels of the factors B and C.

Level of B	*Level of C*	*Simple effect of A*
b_0	c_0	$(a_1b_0c_0) - (a_0b_0c_0) = (a) \quad - (1)$
b_1	c_0	$(a_1b_1c_0) - (a_0b_1c_0) = (ab) \quad - (b)$
b_0	c_1	$(a_1b_0c_1) - (a_0b_0c_1) = (ac) \quad - (c)$
b_1	c_1	$(a_1b_1c_1) - (a_0b_1c_1) = (abc) - (bc)$

... (6·226)

The main effect of A is defined as the average of these 4 simple effects. Thus

$$A = \frac{1}{4}\Big[(abc) - (bc) + (ac) - (c) + (ab) - (b) + (a) - (1)\Big] \quad \ldots (6{\cdot}227)$$

$$= \frac{1}{4}\Big[\{(abc) + (ac) + (ab) + (a)\} - \{(bc) + (c) + (b) + (1)\}\Big]$$

$$= \frac{1}{4}(a-1)(b+1)(c+1), \quad \ldots (6{\cdot}227a)$$

where, as usual the right-hand side is to be expanded algebraically and then the treatment combinations are to be replaced by the corresponding treatment means. Thus main effect is $\frac{1}{4}$ of the difference between the total of mean yields from the plots to which the factor A is applied at second level (a_1) and the first level (a_0). Similarly the main effects of the factors B and C can be obtained to give

$$B = \frac{1}{4}\Big[\{(abc) + (ab) + (bc) + b\} - \{(a) + (c) + (ac) + (1)\}\Big] \quad \ldots (6{\cdot}228)$$

$$= \frac{1}{4}(a+1)(b-1)(c+1) \quad \ldots (6{\cdot}228a)$$

$$C = \frac{1}{4}\Big[\{(abc) + (bc) + (ac) + (c)\} - \{(a) + (b) + (ab) + (1)\}\Big] \quad \ldots (6{\cdot}229)$$

$$= \frac{1}{4}(a+1)(b+1)(c-1) \quad \ldots (6{\cdot}229a)$$

The average effect of A (one level of C) at the level b_0 of B

$$= \frac{1}{2}\Big[(ac) - (c) + (a) - (1)\Big] \quad \text{[From (6·226)]}$$

The average effect of A (one level of C) at the level b_1 of B

$$= \frac{1}{2}\Big[(abc) - (bc) + (ab) - (b)\Big], \quad \text{[From (6·226)]}$$

If the factors A and B were independent then we would expect that the average effect of A will remain the same at either level of B. In this case mean of these two average effects will given the main effect of A, as obtained in (6·227). But if the factors A and B are not independent, then a measure of their interaction AB is given by half of the difference between the average effect of A at the second and the first level of B.

Symbolically, we have

$$AB = \frac{1}{4}\Big[\{(abc) - (bc) + (ab) - (b)\} - \{(ac) - (c) + (a) - (1)\}\Big]$$

$$= \frac{1}{4}\Big[\{(abc) + (ab) + (c) + (1)\} - \{(bc) + (b) + (ac) + (a)\}\Big]$$

$$= \frac{1}{4}(a-1)(b-1)(c+1), \quad \ldots (6{\cdot}230)$$

as usual. Similarly we can obtain expressions for the interactions BC and AC as given below.

$$\left.\begin{aligned} BC &= \frac{1}{4}(a+1)(b-1)(c-1) \\ AC &= \frac{1}{4}(a-1)(b+1)(c-1) \end{aligned}\right\} \quad \ldots (6{\cdot}230a)$$

From (6·226), we obtain the expressions for the interaction of AB at the level c_0 and c_1 of the factor C as follows :

Interaction of AB at level c_0 of $C = \frac{1}{2}\Big[(ab) - (b) - (a) + (1)\Big]$

Interaction of AB at level c_1 of $C = \frac{1}{2}\Big[(abc) - (bc) - (ac) + (c)\Big]$

Thus interaction effect of AB with C, *i.e.*, the interaction ABC is given by half the difference of the fist expression from the second expression. Thus symbolically

$$AB = \frac{1}{4}\left[(abc) - (bc) - (ac) + (c) - (ab) + (b) + (a) - (1)\right]$$

$$= \frac{1}{4}(a-1)(b-1)(c-1) \qquad \text{... (6·231)}$$

as usual.

***Remarks* 1.** From the expression of various factorial effects, *viz.*, A, B, C, AB, BC, AC and ABC in equations (6·227) to (6·231), we see that a minus sign appears in any factor on the R.H.S. if it is present on the left.

2. It can be easily seen that all the seven factorial effects, *viz.*, main effects A, B and C, first order interactions AB, AC and BC, and second order interaction ABC are mutually orthogonal contrasts of the treatment means.

3. It can be easily seen from the symmetry of the results that

$$\left.\begin{aligned} &AB = BA,\ AC = CA,\ BC = CB \\ \text{and} \quad &ABC = ACB = BCA = BAC = CAB = CBA \end{aligned}\right\} \qquad \text{...(6·232)}$$

4. If we write M for the mean yield of all the eight treatment combinations, then

$$M = \frac{1}{8}\left[(abc) + (ab) + (ac) + (bc) + (a) + (b) + (c) + (1)\right]$$

$$= \frac{1}{8}(a+1)(b+1)(c+1) \qquad \text{... (6·233)}$$

The *Table 6·44* gives the divisors and the signs with which the means of various treatment combinations are to be combined to obtain the general mean M and the 7 mutually orthogonal contrasts, *viz.*, the factorial effects A, B, C, AB, AC, BC and ABC.

TABLE 6·44 : TABLE OF SIGNS AND DIVISORS GIVING M AND FACTORIAL EFFECTS IN TERMS OF TREATMENT MEANS FOR 2^3-DESIGN

Factorial Effect	*Treatment mean* (1)	(a)	(b)	(ab)	(c)	(ac)	(bc)	(abc)	*Divisor*
M	+	+	+	+	+	+	+	+	8
A	−	+	−	+	−	+	−	+	4
B	−	−	+	+	−	−	+	+	4
C	−	−	−	−	+	+	+	+	4
AB	+	−	−	+	+	−	−	+	4
AC	+	−	+	−	−	+	−	+	4
BC	+	+	−	−	−	−	+	+	4
ABC	−	+	+	−	+	−	−	+	4

The rule for completing this table for the main effects and the first order interactions is same as given in 2^2-design, *viz.*, give a plus to the treatment combinations which contain the small letter(s) corresponding to the factorial effects. The signs for the second order interaction can be obtained on combining (multiplying) the signs of A, B and C or BC and A, or AC and B.

Model of 2^3-Design. If y_{ijkl} is the response observed at ith level of A, jth level of B, kth level of C in the lth replicate then the linear model for a 2^3-experiment becomes :

$$y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \rho_l + \varepsilon_{ijkl}, \quad (i, j, k = 0, 1 \,;\, l = 1, 2, \ldots, r) \quad \ldots(6\cdot234)$$

where μ is the general mean, α_i, β_j and γ_k are the effects of the ith level of A, jth level of B and kth level of C respectively, $(\alpha\beta)_{ij}$ and $(\alpha\gamma)_{ik}$ are the interaction effect of ith level of A with jth level of B and kth level of C respectively, $(\beta\gamma)_{jk}$ is the interaction effect of jth level of B and kth level of C, $(\alpha\beta\gamma)_{ijk}$ is the interaction effect of ith level of A with jth level of B and kth level of C, ρ_l is the effect due to the lth replicate. ε_{ijkl}, which represent the error effect due to chance are *i.i.d.* $N(0, \sigma_e^2)$. The above parameters are subject to the following restrictions :

$$\sum_{i=0}^{1} \alpha_i = \sum_{j=0}^{1} \beta_j = \sum_{k=0}^{1} \gamma_k = \sum_{l=0}^{r} \rho_l = 0$$

and the sums

$$\Sigma\,(\alpha\beta)_{ij}, \qquad \Sigma(\beta\gamma)_{jk}, \qquad \Sigma(\alpha\gamma)_{ik} \quad \text{and} \quad \Sigma(\alpha\beta\gamma)_{ijk}$$

are equal to zero respectively, when summed over either subscript for all the values of the remaining subscripts.

Statistical Analysis of **2^3-Design.** By using the *Table 6·44* of divisors and signs of a 2^3-factorial experiment, the various factorial effect totals can be expressed as mutually orthogonal contrasts of the 8 treatment totals. Thus, *e.g.*,

$$\begin{aligned} [A] &= [abc] - [bc] + [ac] - [c] + [ab] - [b] + [a] - [1] \\ [AC] &= [abc] - [bc] + [ac] - [c] - [ab] + [b] - [a] + [1] \end{aligned} \quad \ldots(6\cdot235)$$

and so on. Another convenient way usually used for numerical computations of finding the factorial effect totals is the Yates' method as discussed in § 6·9·2.

In the analysis of 2^3-design we split the treatment S.S. with 7 *d.f.* into 7 mutually orthogonal components corresponding to seven factorial effects, each carrying 1 *d.f.* Obviously, the factorial effect totals are contrasts of the treatment totals and hence on using the result in (6·223*a*) [*c.f.* Remark 7 § 6·9·1], the S.S. due to any of the factorial effect is given by :

$$\frac{[\quad]^2}{\sum_{i=1}^{8} r.1} = \frac{[\quad]^2}{8r} \quad \ldots(6\cdot236)$$

i.e., S.S. due to any factorial effect, main or interaction is obtained on multiplying the square of the factorial effect total by $1/(8r)$, where r is the common replication number. Thus, for example

$$\begin{aligned} \text{S.S. due to main effect } A &= \frac{[A]^2}{8r} \text{ with 1 } d.f., \\ \text{S.S. due to interaction } BC &= \frac{[BC]^2}{8r} \text{ with 1 } d.f., \end{aligned} \quad \ldots(6\cdot236a)$$

and so on.

ANOVA can now be carried out as given in the following table :

TABLE 6·45 : ANOVA TABLE FOR A 2^3-EXPERIMENT IN '*r*' RANDOMISED BLOCKS

Source of Variation	*d.f.*	*S.S.*	*M.S.S.* $= \frac{S.S.}{d.f.}$	*Variance Ratio 'F'*
Replications (Blocks)	$(r-1)$	S_R^2	$s_R^2 = \frac{S_R^2}{r-1}$	$F_R = \frac{S_R^2}{S_E^2} \sim F[r-1, 7(r-1)]$
Main Effects				
A	1	$S_A^2 = [A]^2/8r$	$s_A^2 = S_A^2$	$F_A = s_A^2/s_E^2 \sim F[1, 7(r-1)]$
B	1	$S_B^2 = [B]^2/8r$	$s_B^2 = S_B^2$	$F_B = s_B^2/s_E^2 \sim F[1, 7(r-1)]$
C	1	$S_C^2 = [C]^2/8r$	$s_c^2 = S_C^2$	$F_C = s_C^2/s_E^2 \sim F[1, 7(r-1)]$
1st Order Interactions				
AB	1	$S_{AB}^2 = [AC]^2/8r$	$s_{AB}^2 = S_{AB}^2$	$F_{AB} = s_{AB}^2/s_E^2 \sim F[1, 7(r-1)]$
AC	1	$S_{AC}^2 = [AC]^2/8r$	$s_{AC}^2 = S_{AC}^2$	$F_{AC} = s_{AC}^2/s_E^2 \sim F[1, 7(r-1)]$
BC	1	$S_{BC}^2 = [BC]^2/8r$	$s_{BC}^2 = S_{BC}^2$	$F_{BC} = s_{BC}^2/s_E^2 \sim F[1, 7(r-1)]$
2nd Order Interaction				
ABC	1	$S_{ABC}^2 = [ABC]^2/8r$	$s_{ABC}^2 = S_{ABC}^2$	$F_{ABC} = s_{ABC}^2/s_E^2 \sim F[1, 7(r-1)]$
Error	$7(r-1)$	S_E^2 = By subtraction	$s_E^2 = \frac{S_E^2}{7(r-1)}$	
Total	$r.2^2 - 1 = 8r - 1$			

The hypothesis of the presence of a factorial effect is rejected at $\alpha\%$ level of significance if the corresponding calculated F-statistic in the *Table 6·45* is greater than tabulated F_α; 1,7 $(r-1)$ otherwise the hypothesis may be accepted.

Remarks 5. The S.S. due to the seven factorial effects (main and interactions), each with 1*d.f.* will add up to S.S. due to treatments with 7*d.f.*

6. The significance of the various factorial effects can be tested directly from the factorial effect totals by using *t*-test based on error *d.f.* as explained below.

The S.E. of any factorial total in a 2^3-experiment is given by :

$$(2^3 . r\sigma_e^2)^{1/2} = (8r\,\sigma_e^2)^{1/2}$$

which is estimated by S.E. of each factorial total $= (8r\, s_E^2)^{1/2}$, where s_E^2 is the error mean S.S.

Form the expressions for the factorial effects, the variance of each main effect or interaction is given by :

$$\text{Variance of each factorial effect} = \frac{1}{4^2 r^2} \times (8r\,\sigma_e^2) = \frac{\sigma_e^2}{2r}.$$

Hence, the S.E. for any factorial effect is estimated by :

$$\text{S.E. of each factorial effect} = \left(\frac{\hat{\sigma}_e^2}{2r}\right)^{1/2} = \left(\frac{s_E^2}{2r}\right)^{1/2} \quad \ldots (6{\cdot}237a)$$

From (6·237), we obtain the significant value for any factorial effect total at α% level of significance as :

$$d\alpha = t_{error\ d.f.}(\alpha/2) \times (8r\, s_E^{\,2})^{1/2} \qquad \ldots (6\cdot237b)$$

The factorial effects will be significant at α% level of significance if the corresponding factorial effect totals exceed the value of d_α in (6·237b).

7. Proceeding exactly as in 2^2-design, we obtain the factorial effects in terms of the treatment totals as given below :

$$\left.\begin{array}{ll} \text{Main effect of } A = [A]/4r & \text{Interaction } AB = [AB]/4r \\ \text{Main effect of } B = [B]/4r & \text{Interaction } AC = [AC]/4r \\ \text{Main effect of } C = [C]/4r & \text{Interaction } BC = [BC]/4r \end{array}\right\} \qquad \ldots (6\cdot238)$$

6·9·4. 2^n-Factorial Experiment. The results and the notations of 2^2 and 2^3 experiments can be generalised to the case of 2^n experiment. Here we consider n factors each at 2 levels. Suppose $A, B, C, D, \ldots, K$ are the factors each at two levels (0, 1). Corresponding small letters $a, b, c, d, \ldots, k$ denote the corresponding factors at the second level, the first level of any factor being signified by the absence of the corresponding small letter. The treatment combinations, in standard order, can be written as :

1, a, b, ab, c, ac, bc, abc, d, ad, bd, abd, cd, acd, bcd, $abcd$, e, ae, be, abe, ce, ace, bce, $abce$, de, ade, bde, $abde$, cde, $acde$, $bcde$, $abcde$, etc.

For 2^n-experiment, the various factorial effects are enumerated as follows :

Main effects	: ${}^nC_1$ in number
Two-factor interactions	: ${}^nC_2$ in number
Three-factor interactions	: ${}^nC_3$ in number
⋮	⋮
n factor interaction	: ${}^nC_n$ in number

Hence, the total number of factorial effects in 2^n-experiment are :

$$ {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = [{}^nC_0 + {}^nC_1 + \ldots + {}^nC_n] - 1$$

$$= (1+1)^n - 1 = 2^n - 1. \qquad \ldots (6\cdot239)$$

Main Effects and Interactions. As in the case of 2^2 and 2^3-experiments the results for the main effects and interactions can be generalised to the case 2^n-experiment. Thus, for n factors $A, B, C, D, \ldots, K$, the main effects and interactions are given by the expression :

$$\frac{1}{2^{n-1}}\left[(a \pm 1)(b \pm 1)(c \pm 1)(d \pm 1)\ldots(k \pm 1)\right] \qquad \ldots (6\cdot240)$$

the corresponding sign in each factor being taken as negative if the corresponding factor is contained in the factorial effect whose value we want. As usual, the R.H.S. is to be expanded algebraically and then the treatment combinations are to be replaced by the corresponding treatment means. The factorial effect totals can be obtained every conveniently from treatment totals by the generalisation of F. Yates' method as explained in § 6·9·2 for 2^2 and 2^3 experiments. As pointed out there, for 2^n experiment we shall need n cycles of the 'sum and difference' procedure.'

Analysis of 2^n design. It will be seen that all the factorial effects (main and interaction) are mutually orthogonal contrasts of treatment totals. Hence, having obtained the factorial effect totals by Yates' technique, the S.S. due to each factorial effect is given by :

$$\frac{[\quad]^2}{\sum\limits_{i=1}^{2^n} r.\,1^2} = \frac{[\quad]^2}{r.2^n} \qquad \ldots (6\cdot241)$$

where [ ] is the factorial effect total.

TABLE 6·46 : ANOVA TABLE FOR 2^n EXPERIMENT IN *r* RANDOMISED BLOCKS

Source of Variation	*d.f.*	*S.S.*	*M.S.S*
Blocks	$r-1$	$S_R^2 = \frac{\Sigma B_i^2}{2^n} - \text{C.F.}$	$s_R^2 = \frac{S_R^2}{r-1}$
Treatments	$2^n - 1$	$S_T^2 = \frac{\Sigma T_i^2}{r} - \text{C.F.}$	$s_T^2 = \frac{s_T^2}{2^n - 1}$
Main effects			
A	1	$S_A^2 = [A^2/r.\ 2^n$	$s_A^2 = S_A^2$
B	1	$S_B^2 = [B]^2/r.2^n$	$s_B^2 = S_B^2$
⋮	⋮	⋮	⋮
K	1	$S_K^2 = [K]^2 / r.2^n$	$s_K^2 = S_K^2$
Two-factor Interactions			
AB	1	$S_{AB}^2 = [AB]^2 / r.\ 2^n$	$s_{AB}^2 = S_{AB}^2$
AC	1	$S_{AC}^2 = [AC]^2 / r.\ 2^n$	$s_{AC}^2 = S_{AC}^2$
BC	1	$S_{BC}^2 = [BC]^2 / r.2^n$	$s_{BC}^2 = S_{BC}^2$
⋮	⋮	⋮	⋮
Three-factor Interactions			
ABC	1	$S_{ABC}^2 = [ABC]^2 / r.2^n$	$s_{ABC}^2 = S_{ABC}^2$
ACD	1	$S_{ACD}^2 = [ACD]^2 / r.2^n$	$s_{ACD}^2 = S_{ACD}^2$
⋮	⋮	⋮	⋮
n -factor interaction *ABCD...K*	1	$S_{AB...K}^2 = [AB...K]^2 / r.2^n$	$s_{AB...K}^2 = S_{AB...K}^2$
Error	$(r-1)(2^n-1)$	S_E^2 = By subtraction	$s_E^2 = \frac{S_E^2}{(r-1)(2^n-1)}$
Total	$r.\ 2^{n-1}$	Raw S.S. – C.F.	

The block effects and the factorial effects (main and interactions) can be tested for significance by comparing their mean S.S. with error S.S.

Remarks 1. S.S. due to $(2^n - 1)$ mutually orthogonal factorial effects each with 1 *d.f.* will add up to treatment S.S.

2. The main effects and the interactions can be obtained in terms of factorial totals as follows :

$$\text{Factorial effect (Main or Interaction)} = \frac{\text{Factorial effect total}}{r.2^{n-1}} \quad \ldots (6{\cdot}242)$$

Example 6·10. *The Table 6·47 gives the layout and the results of a 2^3 factorial design laid out in four replicates. The purpose of the experiment is to determine the effect of different kinds of fertilizers Nitrogen (N), Potash (K) and Phosphate (P) on potato crop yield.*

TABLE 6·47 : 2^3-FACTORIAL EXPERIMENT LAID OUT IN 4 BLOCKS

Block 1				Block 2				Block 3				Block 4			
nk	*kp*	*p*	*np*	*kp*	*p*	*k*	*nk*	*p*	*1*	*np*	*kp*	*np*	*nk*	*n*	*p*
291	391	312	373	407	324	272	306	323	87	324	423	361	272	103	324
1	*k*	*n*	*nkp*	*n*	*nkp*	*np*	*1*	*nk*	*k*	*n*	*nkp*	*k*	*1*	*nkp*	*kp*
101	265	106	450	89	449	338	106	334	279	128	471	302	131	437	435

The block totals, the treatment total and the grand total are summarised in Table 6·48.

Solution. H_0 : *Blocks as well as treatments are homogeneous.* Block effects are eliminated by carrying out the analysis of the above design as an R.B.D. for eight treatment combinations and four blocks. The initial calculations are, therefore, as follows :

TABLE 6·48 : BLOCK AND TREATMENT TOTALS

Block Totals	1 : 2,289	*Treatment Totals*	'1'	425
	2 : 2,291		*n*	426
	3 : 2,369		*k*	1,118
	4 : 2,375		*nk*	1,203
			p	1,283
			np	1,396
			kp	1,666
			nkp	1,807

Total number of observations = 4 × 8 = 32; G = Grand Total 9,324

$$\text{Correction Factor} = \frac{G^2}{32} = 27{,}16{,}780{\cdot}5$$

$$\text{Total S.S.} = \text{R.S.S.} - \text{C.F.}$$
$$= [(291)^2 + (391)^2 + \ldots + (437)^2 + (445)^2] - 27{,}16{,}780{\cdot}5$$
$$= 3{,}182{,}118{\cdot}0 - 27{,}16{,}780.5 = 4{,}65{,}337{\cdot}5$$

$$\text{Replicate (Block) S.S.} = \frac{1}{8}\sum_{j=1}^{4} B_j^2 - \text{C.F.} = \frac{2{,}17{,}40{,}988}{8} - 27{,}16{,}780{\cdot}5 = 843{\cdot}0$$

$$\text{Treatment S.S.} = \frac{1}{4}\sum_{i=1}^{8} T_i^2 - \text{C.F.} = \frac{1{,}26{,}94{,}944}{4} - 27{,}16{,}780{\cdot}5 = 4{,}56{,}995{\cdot}5$$

$$\text{Error S.S.} = 4{,}65{,}337{\cdot}5 - (4{,}56955{\cdot}5 + 843{\cdot}0) = 7{,}539$$

We shall now break up the treatment S.S. with 7 *d.f.* into 7 orthogonal components each with 1 *d.f.* For this, we use Yates method for finding the various factorial effect totals and their S.S.

TABLE 6·49 : YATES' METHOD FOR FACTORIAL EFFECT TOTALS FOR 2^3-EXPERIMENT

Treatment Combination	*Total Yield*	(1)	(2)	(3)	*Effect Totals*	S.S.
'1'	425	851	3,172	9,324	*G*	$\frac{G^2}{32} = 27{,}16{,}780{\cdot}5$
n	426	2,321	6,152	340	[*N*]	$\frac{[N]^2}{32} = 3{,}612{\cdot}5$
k	1,118	2,679	86	2,264	[*K*]	$\frac{[K]^2}{32} = 1{,}60{,}178{\cdot}0$

nk	1,203	3,473	254	112	[*NK*]	$\frac{[NK]}{32} = 392{\cdot}0$
p	1,283	1	1,470	2,980	[*P*]	$\frac{[P]^2}{32} = 2{,}77{,}512{\cdot}5$
np	1,396	85	794	168	[*NP*]	$\frac{[NP]^2}{32} = 882{\cdot}0$
kp	1,666	113	84	– 676	[*KP*]	$\frac{[KP]^2}{32} = 14{,}280{\cdot}5$
nkp	1,807	141	28	– 56	[*NKP*]	$\frac{[NKP]^2}{32} = 98{\cdot}0$

TABLE 6·50 : ANOVA TABLE FOR 2^3-FACTORIAL EXPERIMENT

Source of Variation	*d.f.*	*S.S.*	*M.S.S.*	*Variance Ratio (F)*	*Tabulated*
Replicates	3	843·0	281·0	< 1	$F_{3,\,21}$ (0·05) = 3·70
Treatment	7	4,56,955·5	65,729·35	181·83*	$F_{7,21}$ (0·05) = 2·50
N	1	3,612·5	3,612·50	10·06*	$F_{1,\,21}$ (0·05) = 4·32
K	1	1,60,178·00	1,60,178·00	446·10*	4·32
NK	1	392·0	392·00	1·09	4·32
P	1	2,77,512·5	2,77,512·5	773·01*	4·32
NP	1	882·0	882·0	2·45	4·32
KP	1	14,280·5	14,280·5	39·7*	4·32
NKP	1	98·0	98·0	0·27	4·32
Error	21	7,539·0	359·00		
Total	31	4,65,337·5			

From the, ANOVA *Table 6·50*, we find that

(*i*) Replicates or blocks are homogeneous.

(*ii*) Treatments differ significantly. Among the factorial effects, all the main effects *N*, *K* and *P*, and the interaction *KP* are significant.

***Remarks* 1.** One way of interpreting the positive interaction of *NK*, is that Nitrogen and Potash do not act independently of one another, the presence of both enhances their individual effects. Similar interpretation can be given to the interaction *NP*. It may be noted that the interaction effect *KP* is negative so that when Potash and Phosphate operate, the full joint benefit of each is not achieved.

2. The significance of the various factorial effects, main and interactions can be tested directly from the factorial effect totals as follows :

$$\text{S.E. for any factorial total} = \sqrt{r.2^3\, s_E^{\,2}} = 4\sqrt{2 \times 359} = 107{\cdot}16$$

∴ Significant value for any factorial effect total, say, at 5% level of significance is :

$$d = t_{21}(0{\cdot}025) \times 107{\cdot}16 = 2{\cdot}080 \times 107{\cdot}16 = 222{\cdot}89$$

Comparing this value with the factorial effect totals (modules values in columns 3 and 4 of *Table 6·49*, we find that all the main effects, *viz.*, *N*, *K* and *P* and the interaction *KP* are significant, the other factorial effects being non-significant, a result which was obtained from the ANOVA *Table 6·50*.

6·10. CONFOUNDING IN FACTORIAL DESIGNS

In factorial experiments, as the number of factors and the levels at which they are employed increase, the total number of treatment combinations increases rather rapidly and consequently the block size has to be enlarged. In many experiments even a single replication of each treatment combination may require far too many experimental units. For example, for a 2^{10} factorial experiment, a complete factorial would require 1,024 units. A large scale experiment of this magnitude may involve a number of blocks and different treatments. The heterogeneity introduced as a consequence of the size of experiment results in extraneous variation which will add to experimental error. As a consequence of increase in the block size or handling such a huge experiment, the purpose of local control (one of the basic principles of a good design) is defeated due to the following two reasons :

(*i*) It is sometimes impracticable to get one complete replicate units which are relatively homogeneous, and

(*ii*) The greater heterogeneity is introduced in the experimental error and reduces the discriminating power of the tests of significance (t, F tests), thus vitiating the conclusions to be drawn.

Hence the precision of a factorial experiment is adversely affected if the treatment combinations are large in number. In order to maintain homogeneity within the blocks, the experimenter must either cut down the number of factors (which, of course, will mean loss of information) or use an incomplete factorial which investigates the main effect of the factors and their more important interactions under uniform conditions by suitably sub-dividing the experimental material into smaller homogeneous blocks. The heterogeneity of blocks is allowed to affect only interactions which are likely to be unimportant.

The process by which unimportant comparisons are deliberately confused or mixed up or entangled with the block comparisons, for the purpose of assessing more important comparisons with greater precision is called *Confounding*. Confounding may also be defined as the technique of reducing the size of a replication over a number of blocks at the cost of losing some information on some effect which is not of much practical importance.

The device of confounding consists in subdividing the replicate into two or more equal subgroups (blocks) and the various treatment combinations into two or more groups of equal size following certain *rules* by which we sacrifice some information on certain higher order interactions and allocating the treatment combinations of any group to any block at random.

Important Remark (*Orthogonality and Confounding*). Let X_i ; $i = 1, 2, \ldots, n$ be *i.i.d.* *r.v.*'s distributed as $N(0, \sigma^2)$. Let us consider their two orthogonal contrasts U and V defined as follows :

$$\left.\begin{aligned} U &= \sum_{i=1}^{n} \lambda_i X_i\,, \quad \sum_{i=1}^{n} \lambda_i = 0 \\ V &= \sum_{i=1}^{n} \mu_i X_i\,, \quad \sum_{i=1}^{n} \mu_i = 0 \end{aligned}\right\} \quad \ldots (6{\cdot}243)$$

$$\text{s.t.} \quad \sum_{i=1}^{n} \lambda_i \mu_i = 0 \quad \ldots (6{\cdot}243a)$$

$$\text{Cov}(U, V) = E(UV) - E(U)\,E(V) = E\left[\left(\sum_i \lambda_i X_i\right)\left(\sum_i \mu_i X_i\right)\right] \quad [\because E(X_i) = 0]$$

$$= \sum_{i=1}^{n} \lambda_i \mu_i E(X_i^2) = \sigma^2 \sum_i \lambda_i \mu_i = 0$$

$[\because X_i\text{'s are } i.i.d.\ N(0, \sigma^2) \text{ and using (6·243a)}]$

Thus U and V are independently and normally distributed. Hence, if the orthogonal contrasts U and V are used to estimate certain effects, the errors in these estimates will not be related and we say that these estimates are orthogonal. According to F. Yates, *orthogonality of a design is the property which assures that different effects will be capable of separate estimation and testing without any entanglement.* In an orthogonal design there are no problems of independent estimation of various effects and their tests of significance. For example, C.R.D., R.B.D. and L.S.D. are orthogonal designs. *The deliberate introduction of non-orthogonality in a design, in order to get better estimates and tests on important comparisons is called confounding.*

The artifice of confounding consists in mixing up inseparably or entangling the effects of unimportant interactions with the block effects. In confounding, between block information termed as *inter-block information* which is contained in block comparisons, is ignored and can be obtained by estimating variance between blocks within replicates which are treated alike.

6·10·1. Confounding in 2^3-Experiment. In a 2^3-experiment, the eight treatment combinations require 8 units of homogeneous material each to form a block. If we decide to use blocks of 4 units (plots) each then a full replication will require only two blocks. In this case 8 treatment combinations are divided into two groups of 4 treatments each in a special way so as to confound any one of the less important interactions with blocks and these groups are allocated at random in the two blocks.

For example, let us consider confounding the highest order interaction ABC. We know that interaction effect ABC is given by

$$ABC = \frac{1}{4}\left[(abc) - (bc) - (ac) + (c) - (ab) + (b) + (a) - (1)\right]$$

$$= \frac{1}{4}\left[(abc) + (a) + (b) + (c) - (ab) - (ac) - (bc) - (1)\right] \quad \text{... (6·244)}$$

Thus in order to confound the interaction ABC with blocks *all the treatment combinations with positive sign are allocated at random in one block and those with negative signs in the other block.* Thus, the arrangement in *Table 6·51* gives ABC confounded with blocks and hence we lose information on ABC.

TABLE 6·51 : ABC CONFOUNDED WITH BLOCKS

Replicate	Block 1 :	(1)	(ab)	(ac)	(bc)
	Block 2 :	(a)	(b)	(c)	(abc)

From (6·244), we also observe that the contrast estimating ABC also contains block effects, effect of block 1 minus the effect of block 2. The other six factorial effects which are also contrasts, *viz.*, A, B, C, AB, AC, BC each contain two treatments in block 1 (or 2) with positive signs and two with negative signs so that they are orthogonal with block totals and hence these differences are not influenced by differences among blocks and can thus be estimated and tested as usual without any difficulty.

For carrying out the statistical analysis, *the various factorial effects and their S.S. are estimated in the usual manner by using Yates' method with the modification that neither the S.S. due to the confounded interaction is computed nor it is included in the ANOVA Table.* This confounded component is contained in the $(2r - 1)$ *d.f.* (in case or r replicates for the above experiment) due to blocks. The *d.f.* in the ANOVA Table will be as given in *Table 6·52.*

TABLE 6·52

Source of variation	*d.f.*
Blocks	$(2r-1)$
A	1
B	1
C	1
AB	1
AC	1
BC	1
Error	$6(r-1)$
Total	$8r-1$

Error S.S. is obtained as usual by subtraction, *i.e.*, $S_E^2 = T.S.S. - S_A^2 - S_B^2 - S_C^2 - S_{AB}^2 - S_{AC}^2 - S_{BC}^2$

***Remarks* 1.** In the above arrangement of treatment combinations in two blocks of 4 units each we find that although interaction ABC is confounded with blocks, all other factorial effects are measured within blocks and are not confounded with blocks. Effect totals of confounded effects are also termed as *Interblock Comparisons* and those of unconfounded effects as *Intrablock Comparisons.*

2. In similar manner we can confound any interaction with blocks. For example, the interaction AC is measured by

$$AC = \tfrac{1}{4}\left[(a-1)(b+1)(c-1)\right]$$

$$= \tfrac{1}{4}\left[(abc) + (ac) + (b) + (1) - (a) - (c) - (ab) - (bc)\right] \quad \ldots (6{\cdot}244a)$$

Hence the interaction AC will be confounded with blocks if the 8 treatment combinations are divided into two groups with combinations abc, ac, b, 1 [all with plus sign in (6·244*a*)] in one group and those with negative signs, *viz.*, a, c, ab, bc in the other group. The arrangement in *Table 6·53* confounds AC with blocks.

TABLE 6·53 : AC CONFOUNDED WITH BLOCKS

Replicate	Block 1 :	(abc)	(b)	(ac)	(1)
	Block 2 :	(c)	(ab)	(ab)	(bc)

2. The above rule can be generalised to confound any interaction effect with blocks in a 2^n-design.

Example 6·11. *Analyse the following* 2^3 *completely confounded factorial design :*

Replicate 1

Block I	'1'	(nk)	(np)	(kp)
	101	291	373	391
Block II	(nkp)	(n)	(k)	(p)
	450	106	265	312

Replicate 2

Block III	'1'	(nk)	(np)	(kp)
	106	306	338	407
Block IV	(nkp)	(n)	(k)	(p)
	449	89	272	324

Replicate 3

Block V	'1'	(nk)	(np)	(kp)
	87	334	324	423
Block VI	(nkp)	(n)	(k)	(p)
	471	128	279	323

Replicate 4

Block VII	'1'	(nk)	(np)	(kp)
	131	272	361	445
Block VIII	(nkp)	(n)	(k)	(p)
	437	103	302	324

(N = Nitrogen ; P = Phosphate ; K = Potash)

Solution. Since in the above 2^3 factorial experiment, replicate has been divided into blocks of 4 plots each, it is a 2^3 confounded design. A careful examination of the treatment combinations in different blocks reveals that in each replicate, the interaction *NPK* has been confounded. [Note that in each replicate, the treatment combinations in the block containing 'I' have no or an even number' of treatments common with *npk*.]. Hence the above design is a 2^3 factorial with the interaction *NPK* completely confounded with blocks.

The *S.S.* due to the six unconfounded factorial effects, *viz.*, the main effects *N*, *P* and *K* and the first order interactions *NP*, *KP* and *NK* are obtained by Yates' technique as usual.

TABLE 6·54 : YATES' METHOD FOR FACTORIAL EFFECTS AND S.S.

Treatment	*Totals*	*1*	*2*	*3*	*Effects*	*S.S.* = [*Effect Totals*]²/32
I	425	851	3,172	9,324	*G*	*C.F.* = 27,16,780·5
n	426	2,321	6,152	340	[*N*]	S_N^2 = 3,612·5
k	1,118	2,679	86	2,264	[*K*]	S_K^2 = 1,60,178·0
nk	1,203	3,473	254	112	[*NK*]	S_{NK}^2 = 392·0
p	1,283	1	1,470	2,980	[*P*]	S_P^2 = 2,77,512·5
np	1,396	85	794	168	[*NP*]	S_{NP}^2 = 882·0
kp	1,666	113	84	−676	[*KP*]	S_{KP}^2 = 14,280·5
nkp	1,807	141	28	−56	[*NKP*]	Not estimable

$\therefore$ *S.S.* due to treatments $= S_N^2 + S_K^2 + S_P^2 + S_{NP}^2 + S_{NK}^2 + S_{KP}^2 = 4{,}56{,}857{\cdot}5$

(Since *NPK* is completely confounded with blocks, its effects enter into the error S.S.)

$$R.S.S. = 31{,}82{,}118{\cdot}0 \quad ; \quad C.F. = \frac{G^2}{8 \times 4} = \frac{(9324)^2}{32} = 27{,}16{,}780{\cdot}5$$

Total *S.S.* = *R.S.S.* — *C.F.* = 31,82,118·0 − 27,16,780·5 = 4,65,337·5

$$\text{Block } S.S. = \frac{1}{4}\left[(1{,}156)^2 + (1{,}133)^2 + (1{,}157)^2 + (1{,}134)^2 + (1{,}168)^2 + (1{,}201)^2 + (1{,}209)^2 + (1{,}166)^2\right] - C.F.$$

$$= \frac{1{,}08{,}72{,}492}{4} - 27{,}16{,}780{\cdot}5 = 1{,}342{\cdot}5$$

$\therefore$ Error *S.S.* = Total *S.S.* − *S.S.* due to Blocks − *S.S.* due to treatments
= 465337·5 − 1342·5 − 456857·5 = 7137·5

TABLE 6·55 : ANALYSIS OF VARIANCE TABLE

Source of Variation	*d.f.*	*S.S.*	*M.S.S.*	*Variance Ratio*
Block	7	1,342·5	191·8	< 1
Treatments	6	4,56,857·5	7,360·5	29·11
N	1	3,612·5	3,612·5	9·11
K	1	1,60,178·0	1,60,178·0	403·90
P	1	2,77,512·5	2,77,512·5	699·90
NK	1	392·0	392·0	0·98
NP	1	882·0	882·0	2·20
KP	1	14,280·5	14,280·5	36·01
Error	18	7,137·5	396·5	
Total	31	4,65,337·5		

Since calculated value of F for blocks is less than 2·59, the tabulated value of F for (7, 31) d.f. at 5% 'level of significance', we fail to reject the null hypothesis.

H_0 : *Confounding is not effective.*

Hence, we conclude that confounding is not effective.

6·10·2. Partial Confounding. Let us consider a 2^3 factorial experiment (where each replicate is divided into two blocks of 4 units each) replicated r times, (say). In such a case the experimenter is absolutely free to confound any factorial effect in any replicate. In other words, it is not necessary to confound the same interaction effect in all the replicates and several factorial effects may be confounded in one single experiment. For example, the following plan (2^3 experiment replicated 4 times) in *Table 6·56* confounds the interactions *ABC, AB, BC* and *AC* in the replications *I, II, III* and *IV* respectively.

TABLE 6·56 : PARTIALLY CONFOUNDED 2^3 DESIGN

Block	*Rep I*		*Rep II*		*Rep III*		*Rep IV*	
	1	2	3	4	5	6	7	8
	abc	*ab*	*abc*	*ac*	*abc*	*ab*	*abc*	*ab*
	a	*ac*	*ab*	*bc*	*bc*	*ac*	*ac*	*bc*
	b	*bc*	*c*	*a*	*a*	*b*	*b*	*a*
	c	(1)	(1)	*b*	(1)	*c*	(1)	*c*
Interaction Confounded	*ABC*		*AB*		*BC*		*AC*	

In the above arrangement, the main effects A, B and C are orthogonal with block totals and are entirely free from block effects. The interaction ABC is completely confounded with blocks in replicate 1, but in the other three replicates, the ABC is orthogonal with blocks and consequently an estimate of ABC may bé obtained from replicates II, III and IV. Similarly it is possible to recover information on the other confounded interactions AB (from replicates I, III, IV), AC (from replicates I, II, III) and BC (from replicates I, II, IV).

When an interaction is confounded in one replicate and not in another, the experiment is said to be *Partially Confounded.* Since the partially confounded interactions are estimated from only a portion of the observations, they are determined with a lower degree of precision than the other effects.

Analysis of Partially Confounded 2^3-Experiment. Let us suppose that a number of repetitions, say r, of the above pattern or layout are performed such that the positions of the replications, blocks within replications, and plots within blocks are randomised. The analysis of 2^3-partially confounded design differs from that of the ordinary 2^3-factorial experiment replicated 4 times only in the calculation of the partially confounded interactions, each interaction being estimated only from the three replicates in which the given interaction is not confounded. Thus, the S.S. for the interaction AB, say, is calculated from the replications I, III and IV, the divisor for $(AB)^2$ being 24 instead of 32. Similarly we can obtain the S.S. for the interaction AC (from replicates I, II, III) and BC (from replicates I, II, IV). The S.S. from blocks and for the unconfounded effects (Main effects A, B and C) are obtained in the usual manner.

Remarks **1.** The above confounding scheme provides full information regarding the unfounded effects A, B and C and partial (3/4th) information regarding the confounded interactions AB, AC, BC and ABC (since only 3 of the 4 replicates provide estimates of AB, AC, BC and ABC).

2. *Analysis of 2^3 partially confounded design with 4 replications and 'r' such repetitions.* Let us suppose that a number of repetitions, say, *r*, or the above pattern or layout are performed such that the positions of the replications, blocks within replications and plots within blocks are randomised. Then the structure of the ANOVA Table will be as given in *Table 6·57*.

TABLE 6·57 : ANOVA TABLE FOR *AB*, *AC*, *BC* AND *ABC* PARTIALLY CONFOUNDED WITH '*r*' REPETITIONS IN 2^3-DESIGN

Source of Variation	*d.f.*	*Sum of Squares*
(*a*) Block	$8r-1$	$\frac{1}{4}\sum(\text{total of block})^2 - \frac{(\text{Grand total})^2}{32r}$
(*i*) Replicates	$4r-1$	$\frac{1}{8}\sum(\text{total of replicate})^2 - \frac{(\text{Grand total})^2}{32r}$
(*ii*) Blocks within replicates	$4r$	(By difference)
(*b*) *A*	1	$[A]^2/32r$
B	1	$[B]^2/32r$
C	1	$[C]^2/32r$
AB	1	$[AB]^2/24r$
AC	1	$[AC]^2/24r$
BC	1	$[BC]^2/24r$
ABC	1	$[ABC]^2/24r$
Error	$24r-7$	(By difference)
	$32r-1$	Total S.S.

3. ***Calculation of S.S. due to Confounded Effects.*** It has already been explained that S.S. for confounded effects are to be obtained from those replications only in which the given effect is not confounded. From practical point of view, these S.S. can be obtained from the table of *Yates' Method for all the four replications* by applying some *adjusting factor* (A.F.) to the confounded effects. The adjusting factor for any confounded effect is computed as follows :

(*i*) Note the replication in which the given effect is confounded.

(*ii*) Note the sign of (1) in the corresponding algebraic expression of the effect to be confounded.

If the sign is positive then

A.F. = [Total of the block containing (1) of replicate in which the effect is confounded].

– [Total of the block not containing (1) of the replicate in which the effect is confounded]

$= T_1 - T_2$, say ...(6·245)

If the sign is negative, then

$\text{A.F.} = T_2 - T_1$ ...(2·245*a*)

This adjusting factor will be subtracted from the factorial effects totals of the confounded effects obtained from Yates Method for all the 4 replicates.

Example 6·12. *Analyse the following 2^3-Factorial experiment in blocks of 4 plots, involving three fertilisers N, P and K, each at two levels.*

Replicate I

Block 1	*np* 101	*npk* 111	*(1)* 75	*k* 55
Block 2	*p* 88	*n* 90	*pk* 115	*nk* 75

Replicate II

Block 3	*(1)* 125	*npk* 95	*nk* 80	*p* 100
Block 4	*n* 53	*npk* 76	*p* 65	*k* 82

Replicate III

Block 5	*pk* 75	*nk* 100	*(1)* 55	*np* 92
Block 6	*np* 53	*npk* 76	*p* 65	*k* 82

Solution. Since each replicate has been divided into 2 blocks, one effect has been confounded in each replicate. Replicate 1 confounds *NP*, replicate II confounds *NK* and *NPK* has been confounded in replicate III.

H_0 : The data is homogeneous with respect to blocks and treatments.

Taking deviations from 87, we prepare the following *Table 6·58* to compute the total *S.S.* and *S.S.* for Blocks.

TABLE 6·58 : CALCULATIONS FOR VARIOUS S.S.

Treatment Combination	*Replicate I*		*Replicate II*		*Replicate II*		*Treatment Totals*
	Block 1	*Block 2*	*Block 3*	*Block 4*	*Block 5*	*Block 6*	
(1)	−12	—	38	—	−32	—	−6
n	—	3	—	−7	—	−34	−38
p	—	1	13	—	—	−22	−8
np	14	—	—	28	5	—	47
k	−32	—	—	8	—	−5	−29
nk	—	−12	−7	—	13	—	−6
pk	—	28	—	3	−12	—	19
npk	24	—	8	—	—	−11	21
Block totals (B_i)	−6	20	52	32	−26	−72	$G = 0$
B_i^2	36	400	2,704	1,024	676	5,184	$\Sigma B_i^2 = 10{,}024$

$$\text{Correction Factor} = \frac{G^2}{3 \times 8} = 0 \quad ; \quad R.S.S. = (-12)^2 + (38)^2 + \ldots + 8^2 + (-11)^2 = 8{,}658$$

$$\text{Total } S.S. \;= R.S.W. - C.F. \;= 8{,}658$$

$$S.S. \text{ due to Blocks } = \sum_i \frac{B_i^2}{4} - C.F. = \frac{10{,}024}{4} = 2{,}506$$

The *S.S.* due to interactions *NP*, *NK* and *NPK* are not estimable directly from the table of Y ates' method, but they will be estimated indirectly.

TABLE 6·59 : YATES' METHOD FOR 2³ PARTIALLY CONFOUNDED EXPERIMENT

Treatment Combination	*Total Yield*	*Yates' Operations*			$S.S. = \dfrac{[\;]^2}{8 \times 3}$
		I	*II*	*(Factorial Effects) III*	
'1'	−6	−44	−5	0 = *G*	
n	−38	39	5	48 = [*N*]	$S_N^2 = 96{\cdot}00$
p	−8	−35	23	158 = [*P*]	$S_P^2 = 1{,}040{\cdot}17$
np	47	40	25	66 = [*NP*]	Not estimable
k	−29	−32	83	10 = [*K*]	$S_K^2 = 4{\cdot}17$
nk	−6	55	75	2 = [*NK*]	Not estimable
pk	19	23	87	−8 = [*PK*]	$S_{PK}^2 = 2{\cdot}67$
npk	21	2	−21	−108 = [*NPK*]	Not estimable

Interaction, which is confounded in replicate 1, is estimated by :

$$NP = \tfrac{1}{4}\left[(n-1)(p-1)(k+1)\right]$$

Here the sign of '1' is positive. Hence, the adjusting factor (*A.F.*) for *NP* which is to be obtained from replicate 1 [*c.f.* equation (6·245) is given by :

$$A.F. \text{ for } NP = (101 + 111 + 75 + 55) - (88 + 90 + 115 + 75) = 342 - 368 = -26$$

$\therefore$ Adjusted effect total for *NP* becomes : $[NP^*] = [NP] - (-26) = 66 + 26 = 92$

Similarly $A.F.$ for $NK = 400 - 380 = 20$

and $A.F.$ for $NPK = 276 - 322 = -46$ [Note that the sign of 1 in the estimate of *NPK* is −1.]

Hence, adjusted effect totals for *NK* and *NPK* are :

$$[NK^*] = 2 - 20 = -18 \quad \text{and} \quad [NPK^*] = -108 - (-46) = -62$$

$$S^2_{NP} = S.S. \text{ due to } NP = \frac{1}{2\times 8}[NP^*]^2 = \frac{(92)^2}{16} = 529$$

$$S^2_{NK} = S.S. \text{ due to } NK = \frac{1}{2\times 8}[NK^*]^2 = \frac{(-18)^2}{16} = 20{\cdot}25$$

$$S^2_{NPK} = S.S. \text{ due to } NPK = \frac{1}{2\times 8}[NP^*]^2 = \frac{(-62)^2}{16} = 240{\cdot}25$$

$$\text{Treatment } S.S. = S^2_N = S^2_P + S^2_K + S^2_{NP} + S^2_{NK} + S^2_{PK} + S^2_{NPK} = 1{,}932{\cdot}51$$

$\therefore$ Error $S.S. = T.S.S. - S.S.$ Blocks $- S.S.$ Treatments

$$= 8{,}658{\cdot}00 - 2{,}506 - 1{,}932{\cdot}75 = 4{,}219{\cdot}25.$$

TABLE 6·60 : ANOVA TABLE FOR THE PARTIALLY CONFOUNDED 2^3 EXPERIMENT

Source of Variation	*d.f.*	*Sum of Squares*	*M.S.S.*	*Variance Ration F*	*Tabulated Value of F*
Blocks	5	2,506·00	501·2	1·31	
Treatments	7	1,932·51	276·07	< 1	$F_{0.05}(5, 11) = 3{\cdot}2$
N	1	96·00	96·00	< 1	$F_{0.01}(5, 11) = 5{\cdot}32$
P	1	1,040·17	1,040·12	2·71	$F_{0.05}(1, 11) = 4{\cdot}84$
NP	1	529·00	529·00	1·38	$F_{0.01}(1, 11) = 6{\cdot}08$
K	1	4·17	4·17	< 1	
NK	1	20·25	20·25	< 1	
PK	1	2·67	2·67	< 1	
NPK	1	240·25	240·25	< 1	
Error	11	4,219·25	383·57		
Total	23	8,658			

From *Table 6·60* it is seen that effect due to blocks, main effects due to factor *N*, *P*, and *K* or interactions are not significant.

6·10·3. Advantages and Disadvantages of Confounding. The only and the greatest advantage of confounding scheme lies in the fact that it reduces the experimental error considerably by stratifying the experimental material into homogeneous su-bsets or sub-groups. The removal of the variation among incomplete blocks (freed from treatments) within replicates often results in smaller error mean square as compared with a randomised complete block design, thus making the comparisons among some treatments more precise. The following are the *disadvantages* of confounding :

1. The confounded contrasts are replicated fewer times than are the other contrasts and as such there is loss of information on them and they can be estimated with a lower degree of precisions as the number of replications for them is reduced. In the confounding scheme, the increased precision is obtained at the cost of sacrifice of information (partial or complete) on certain relatively non-important interactions. It may be pointed out here that an indiscriminate use of confounding may result in complete or partial loss of information on the contrasts or comparisons of greatest importance. As such the experimenter should confound only those treatment combinations or contrasts which are of relatively less or no importance at all.

2. The algebraic calculations are usually more difficult and the statistical analysis is complex, specifically when some of the units (observations) are missing.

3. A number of problems arise if the treatments interact with blocks.

6·11. A 2^n-FACTORIAL EXPERIMENT IN 2^k-BLOCKS PER REPLICATE

In *Example 6·13* we considered a 2^3-factorial design in 2 blocks (of equal sizes) per replicate. In such a layout, we confound one factorial effect in a replicate and this is generally the higher order interaction.

Let us now consider a 2^n-factorial experiment conducted in 2^k-blocks ($k = 2, 3, \ldots$) of equal sizes per replicate.

1. Total number of experimental units $= 2^n$.

Total number of blocks $= 2^k$.

$\therefore$ Number of units (plots) in each block $= \dfrac{2^n}{2^k} = 2^{n-k}$.

Thus, we have 2^{n-k} treatment combinations in each block and these are assigned at random within the units of each block. *In each replicate, there are 2^k block totals, giving rise to $(2^k - 1)$ orthogonal block contrasts, which will be orthogonal to the $(2^k - 1)$ treatment contrasts in the replicate.*

2. Generalised Interaction. The interaction obtained on multiplying the symbols in two effects/interactions together and equating the square of any letter equal to unity is called the *generalised interaction* of the given effects. For example, for any two effects X and Y, the generalised interactions are given in *Table 6·61*.

We have the following general rule in confounding :

"If any two effects / interactions are completely confounded with blocks, then so is their generalised interaction, defined above."

TABLE 6·61 : GENERALISED INTERACTIONS

X	Y	*Generalised Interaction*
A	BC	$A \times BC = ABC$
AB	CD	$AB \times CD = ABCD$
ABC	BCD	$ABC \times BCD = AB^2C^2D = AD$
ABC	CD	$ABC \times CD = ABC^2D = ABD$
ABC	ACD	$ABC \times ACD = A^2BC^2D = BD$
ABD	BCD	$ABD \times BCD = AB^2CD^2 = AC$
ABC	CDE	$ABC \times CDE = ABC^2DE = ABDE$

Source :

1. S.C. Gupta and V.K. Kapoor : Fundamental of Applied Statistics – Sultan Chand & Sons, Fourth Edition, 2015.
2. Panneer Selvam: Design And Analysis of Experiments, Prentice Hall.