Unit III

Likelihood Ratio Test

The likelihood Ratio test is a general method of test construction applicable to almost all parametric type of testing problems where it is possible to write down the likelihood fun. of the sample.

This method is similar to the method of maximum likelihood in the theory of point estimation. The rational behind this method is the same as that behind the maximum likelihood method.

Consider the hypothesis testing problem of testing $H_0 : \theta \in \Theta_0$ against the alternative hypothesis $H_1 : \theta \in \Theta_1$, where Θ_0 and Θ_1 are disjoint subsets of the parameter space so that the relevant parameter space for this testing problem is $\Theta_0 \cup \Theta_1 = \Theta$.

Definition of LRT:

For testing $H_0 : \theta \in \Theta_0$ against $H_1 : \theta \in \Theta_1$, the Ratio.

$$\frac{\sup_{\theta \in \Theta_0} L(\theta/x)}{\sup_{\theta \in \Theta} L(\theta/x)} = \lambda(x)$$

is called the LR for testing H_0 against H_1

From the definition of the maximum likelihood estimators, it is clear that,

$$\lambda(x) = \frac{L(\tilde{\theta}/x)}{L(\hat{\theta}/x)}$$

where, $\tilde{\theta}$ and $\hat{\theta}$ are the MLE of θ under H_0 and H_1 and $H_0 \cup H_1$ respectively.
The ratio $\lambda(x)$ is also called the MLR for testing H_0 against H_1,

Since,

$$\Theta_0 \subset \Theta_0 \cup \Theta_1 \Rightarrow \sup_{\theta \in \Theta_0} L(\theta/x) \leq \sup_{\theta \in \Theta} L(\theta/x)$$

This ratio cannot exceed unity.

Further, both the numerator and the denominator of $\lambda(x)$ are non-negative
So, always. $0 \leq \lambda(x) \leq 1$.

LRT for Binomial distn:

$$x \sim B(n,p)$$

Consider the $H_0: p \leq p_0$ Vs $H_1: p > p_0$

$$\therefore \lambda(x) = \frac{\sup_{p \leq p_0} \binom{n}{x} p^x \cdot (1-p)^{n-x}}{\sup_{0<p<1} \binom{n}{x} p^x \cdot (1-p)^{n-x}}$$

$$\Rightarrow \sup_{0<p<1} p^x(1-p)^{n-x} = \left(\frac{x}{n}\right)^x (1-x/n)^{n-x}$$

where $p = x/n$

$$\therefore \sup_{p \le p_0} p^x(1-p)^{n-x} = \begin{cases} p_0^x(1-p_0)^{n-x} & ; p_0 < \frac{x}{n} \\ (x/n)^x(1-x/n)^{n-x} & x/n < p_0 \end{cases}$$

$$\Rightarrow \lambda(x) = \begin{cases} \dfrac{p_0^x(1-p_0)^{n-x}}{(x/n)^x(1-x/n)^{n-x}} & ; p_0 < x/n \end{cases}$$

Note that,

$\lambda(x) \le 1$ or $np_0 < x$ and $\lambda(x)$

If $x < np_0$ and it is follows that $\lambda(x)$ is decreasing fn. of x. and generalized LR Testing reject H_0 is $x > c'$.

$\therefore$ LRT is given by.

$$\phi(x) = \begin{cases} 1 & \text{if } \lambda(x) < \lambda_\alpha \\ \gamma & \text{if } \lambda(x) = \lambda_\alpha \\ 0 & \text{if } \lambda(x) > \lambda_\alpha \end{cases}$$

where λ_α and γ are obtained by

$$E[\phi(x)/H_0] = \alpha.$$

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LRT for Exponential Distribution.

Let us obtain the LRT for testing $H_0: \theta \le \theta_0$ against $H_1: \theta > \theta_0$ using the random sample of 'n' independent observation from $P(\theta)$. Here

$$\Theta_0 = \{\theta / 0 < \theta \le \theta_0\}$$

$$\Theta = \{\theta / \theta > 0\} \text{ and}$$

$$L(\theta/x) = e^{-n(\theta)} \cdot \theta^{n\bar{x}}$$

when $\theta \in \Theta$, MLE of θ is $\bar{x}$

and when $\theta \in \Theta_0$, MLE of θ is $\bar{x}$ if $\bar{x} \le \theta_0$ and is θ_0 itself if $\bar{x} > \theta_0$

Then the LR is given by,

$$\lambda(x) = 1, \text{ if } \bar{x} \le \theta_0$$

$$\lambda(x) = \frac{e^{-n\theta_0} \theta_0^{n\bar{x}}}{e^{-n\bar{x}} \bar{x}^{n\bar{x}}} \quad \Big| \text{ where } \theta = \bar{x}$$

$$= e^{-n(\theta_0 - \bar{x})} \left(\frac{\theta_0}{\bar{x}}\right)^{n\bar{x}} ; \text{ if } \bar{x} > \theta_0$$

where $\bar{x} > \theta_0$

$e^{-n(\theta_0 - \bar{x})} \cdot \left(\frac{\theta_0}{\bar{x}}\right)^{n\bar{x}}$ is a strictly decreasing fn. of $\bar{x}$ and so,

$\{ \lambda(x) < \text{some constant} \} \iff \{ \bar{x} > \text{some constant} \}$

Thus, the LR Test is given by,

$$\phi(\bar{x}) = \begin{cases} 0 & \text{if } \bar{x} < C_\alpha \\ \gamma & \text{if } \bar{x} = C_\alpha \\ 1 & \text{if } \bar{x} > C_\alpha \end{cases}$$

where C_α and γ are to be selected such that

$$E[\phi(x) / \Theta = \theta_0] = \alpha$$

Since the distn of $\bar{x}$ is discrete γ is necessary to make the level of the test exactly 'α'.

LRT for the poisson distribution:

Let $x_1, x_2 \dots x_n$ be the random sample follows poisson distn with parameter λ.

Let us consider $H_0 : \lambda = \lambda_0$ v/s $H_1 : \lambda = \lambda_1$. In this case the parameter space Θ is given by

$$\Theta : \{ x : \lambda_1, \ \lambda > 0 \}$$

$$\Theta_0 : \{ x : \lambda_0, \ \lambda_0 > 0 \}$$

Let us define the likelihood fn. L is

$$L = \prod_{i=1}^{n} f(x_i, \theta)$$

The pb.m.fn of poisson distn is given by.

$$P(x:\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$L(x_i;\lambda) = e^{-n\lambda} \cdot \frac{\pi\, \lambda^{x_i}}{x!}$$

$$L(x_i;\lambda) = \frac{e^{-n\lambda}\lambda^{\Sigma x_i}}{\pi\, x_i!}$$

$$L(\hat{\theta}_0) = \frac{e^{-n\lambda} \cdot \lambda^{\Sigma x_i}}{\pi\, x_i!}$$

$$\Rightarrow \log L = -n\lambda + \Sigma x_i \log\lambda - \log \pi\, x_i$$

$$\frac{\partial \log L}{\partial \lambda} = 0 \Rightarrow -n + \frac{\Sigma x_i}{\lambda} = 0$$

$$\frac{\Sigma x_i}{\lambda} = +n$$

$$\lambda = \frac{\Sigma x_i}{n} = \bar{x}$$

$$\therefore \hat{\lambda} = \bar{x}.$$

$\therefore$ LRT is $\lambda(x)$

$$\Rightarrow \lambda(x) = \frac{e^{-n\lambda_0} \cdot \lambda_0^{\Sigma x_i}}{e^{-n\lambda} \cdot \lambda^{\Sigma x_i}}$$

$$\Sigma x_i = n\bar{x} = n\hat{\lambda}$$

$$\Rightarrow \lambda(x) = e^{-n(\lambda_0 - \lambda)} \cdot \left(\frac{\lambda_0}{\lambda}\right)^{\Sigma x_i}$$

$$= e^{-n(\lambda_0-\lambda)} \cdot \left(\frac{\lambda_0}{\lambda}\right)^{n\hat{\lambda}}$$

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where,

$$-2 \log \lambda(x) = -2\{-n(\lambda_0 - \lambda) + n\hat{\lambda}(\frac{\lambda_0}{\hat{\lambda}})$$

$$= -2n\{-(\lambda_0 - \lambda) + \hat{\lambda} \log(\frac{\lambda_0}{\hat{\lambda}})\}$$

where $\hat{\lambda}$ is estimated value, we reject the H_0 at α level if

$-2 \log \lambda(x) > \chi^2_{1,\alpha}$ with 1 d.f.

Likelihood Ratio Criterion:

The Method of Maximum likelihood gives estimate which possess some optimum properties under certain condition. A test procedure which is closely to related to this likelihood Ratio Method was introduced by Neymann-Pearson for testing a simple hypothesis or composite hypothesis.

Consider the random variable x with probability density function $f(x, \theta)$. The set Ω which is the set of all possible value of θ is called the parameter space.

For examble, if $x \sim N(\mu, \sigma^2)$ then the parameter space $\Omega = \{(\mu, \sigma^2), -\infty < \mu < \infty, 0 < \sigma < \infty\}$ consider a general family of distribution

$\{f(x; \theta_1, \theta_2, \dots \theta_k), \theta_i \in \Omega; i = 1, 2, \dots k\}$

The null hypothesis H_0 will stat that the parameter belongs to some subspace is C of the parameter space.

Let $x_1, x_2 \cdots x_n$ be a r.s of size $n(n>1)$ from a popln with density function $\{f(x; \theta_1, \theta_2, \ldots \theta_k)\}$ where Ω, the parameter space in the of all parameter that $\{\theta_1, \theta_2, \ldots \theta_k\}$ can be assumed to test the null hypothesis:

$$H_0 : (\theta_1, \theta_2 \cdots \theta_k) \in C \quad V/s$$

Alternative hypothesis:

$$H_1 : (\theta_1, \theta_2 \cdots \theta_k) \in \bar{C}$$

where $\bar{C} = \Omega - C$

The likelihood function of the sam observation is given by $L = \prod_{i=1}^{n} f(x; \theta_1, \theta_2 \cdots \theta_k)$

The criterion of the likelihood ratio of test is defined as the ratio of two maximum and is given by $\lambda = \frac{L_1}{L_0} > K$

L_0 and L_1 are the maximum likelihood function with respect to the parameters.

The quantity λ is a function of sample observations only and does not involve any parameter, thus 'λ' is a random variable. the CR for testing H_0 vs H_1 in some an initial $0 < \lambda < \lambda_0$ where λ_0 is number (<1) determined by the distribution of λ and desired Prob of Type-1 Error

λ_0 is given by the equation $P\{\lambda > \lambda_0 ; H_0\} = \alpha$.

Thus a test that has the CR defind as $0 < \lambda < \lambda_0$ and $P\{\lambda > \lambda_0 ; H_0\} = \alpha$ is a likelihood ratio test for testing H_0.

Applications of likelihood ratio test:

(8)

Test for the mean of a normal popln.

Test for the variance of a normal popln.

Test for the equality of means of two normal popln.

Test for the equality of means of several normal popln.

Test for the equality of varianc of several normal popln.

properties of likelihood ration Test:

The LRT prinicple is an important testing one if we are testing a simple null hypothesis H_0 vs simple Alternative hypothesis H_1, then the LRT Principle leads to the same test as given by Neymann pearson lemma this suggestion that likliho ratio test has some desirable properties specially large sample.

1. under certain conditions $-2 \log e^{\lambda}$ has an assumptiotic χ^2 distn.

2. under certain assumptions, likelihood Ratior test is consistance.

LRT of Normal Distn:

Let $H_0 : \mu = \mu_0$ and $H_1 : \mu = \mu_1$ ($\mu_1 > \mu_0$)

In this case $\theta = \mu$ since H_0 is fixes μ at μ_0

$\Theta_0 = \{\mu_0\}$. Also H_1 fixes μ at μ_1 and so

$\Theta_1 = \{\mu_1\}$ and $\Theta = \{\mu_0, \mu_1\}$

$\therefore$ the likelihood fn. is gn by,

$$L(\mu/x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}\right\}\left\{\Sigma(x_i-\mu)^2\right\}$$

So where $\theta \in \Theta_0$, MLE of $\mu = \hat{\mu} = \mu_0$

$$\sup L(\mu/x) = L(\mu_0/x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{\frac{1}{2\sigma^2}\right\}\left\{\Sigma(x_i-\mu)^2\right\} \to (1)$$

Let

$$L(\mu_1/x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}\right\}\left\{\Sigma(x_i-\mu)^2\right\} \longrightarrow (2)$$

Also if $\theta \in \Theta$, MLE of $\mu = \hat{\mu} = \mu_0$ or μ_1

according as $(1) \geq (2)$ or $(1) < (2)$

and

$$\sup L(\mu/x) = L(\hat{\mu}/x) = \mathrm{Max}\{(1), (2)\}$$

$\therefore$ LR is given by $\lambda(x) = 1$ if $(1) \geq (2)$ and if $(1) < (2)$

$$\lambda(x) = \frac{(1)}{(2)} \Rightarrow \exp\left[\left\{-\frac{1}{2\sigma^2}\right\}\left\{\Sigma(x_i-\mu_0)^2 - \Sigma(x_i-\mu_1)^2\right.\right.$$

$$= \exp\left[\left\{\frac{-1}{2\sigma^2}\right\}\left\{\Sigma(x_i^2 - 2x_i\mu_0 + \mu_0^2) - \Sigma(x_i^2 - 2x_i\mu_1 + \mu_1^2)\right\}\right.$$

$$= \exp\left[\left\{\frac{-1}{2\sigma^2}\right\}\left\{\Sigma\, \cancel{x_i^2} - \cancel{x_i^2} - 2x_i\mu_0 + 2x_i\mu_1 + \mu_0^2 - \mu_1^2\right.\right.$$

$$= \exp\left[\left\{\frac{-1}{2\sigma^2}\right\}\left\{-2n\bar{x}\mu_0 - 2n\bar{x}\mu_1 + \mu_0 - \mu_1\right\}\right]$$

$$= \exp\left[\left\{\frac{1}{2\sigma^2}\right\}\left\{2n\bar{x}(\mu_0-\mu_1) - n(\mu_0^2-\mu_1^2)\right\}\right]$$

$$= \exp\left[\left\{\frac{n(\mu_0-\mu_1)}{\sigma^2}\right\}\left\{\bar{x} - \frac{\mu_0+\mu_1}{2}\right\}\right]$$

Thus if $\mu_1 > \mu_0$, the LR test is given by

$\phi(x) = 0$ if ① > ② that is

if $\bar{x} - \frac{\mu_0+\mu_1}{2} < 0$ and if ① < ②

$$\therefore \phi(x) = \begin{cases} 1 & \text{if } \lambda(x) < \lambda_\alpha \iff \bar{x} > c_\alpha \\ \gamma & \text{if } \lambda(x) = \lambda_\alpha \iff \bar{x} = c_\alpha \\ 0 & \text{if } \lambda(x) > \lambda_\alpha \iff \bar{x} < c_\alpha \end{cases}$$

where γ and c_α should be choosen to satisfy the size condition

$$\therefore E[\phi(x)/H_0] = \alpha$$

Similarly proceeding, if $\mu_1 < \mu_0$ the LR test is given by

$$\phi(x) = \begin{cases} 1 & \text{if } \bar{x} \le \mu_0 - z_\alpha - \frac{\sigma}{\sqrt{n}} \\ 0 & \text{if } \bar{x} > \mu_0 - z_\alpha - \frac{\sigma}{\sqrt{n}} \end{cases}$$

~~scribble~~

(D) Theorm:

Let $H: \mu = \mu_0$ and $K: \mu > \mu_0$ As in the Previous case, H fixes μ at μ_0 and so $\Theta_0 = \{\mu_0\}$. In this case, $\Theta = \{\mu / \mu \geq \mu_0\}$ and the likelihood function is given by,

$$L(\mu/x) = \exp\left\{-\frac{1}{2\sigma^2}\right\}\left\{\Sigma(x_i - \mu)^2\right\}$$

$$= \exp\left\{-\frac{1}{2\sigma^2}\right\}\left\{\Sigma(x_i - \bar{x})^2 + n(\bar{x} - \mu)^2\right\}$$

since

$$\frac{\partial \ln L}{\partial \mu} = \left\{\frac{1}{\sigma^2}\right\} n(\bar{x} - \mu) > 0, \text{ if } \mu < \bar{x} \text{ and}$$
$$< 0, \text{ if } \mu > \bar{x}$$

$L(\mu/x)$ is strictly increasing if $\mu < \bar{x}$ and there after strictly decreasing. so, if $\bar{x} \in [\mu_0, \infty)$ then $L(\mu/x)$ takes the maximum value at $\bar{x}$ which is the MLE of μ. If $\bar{x} < \mu_0$ then $L(\mu/x)$ takes the maximum value at μ_0 and so, the MLE of μ is μ_0 itself, Thus,

$$L(\hat{\mu}/x) = \exp\left\{-\frac{1}{2\sigma^2}\right\}\left\{\Sigma(x_i - \bar{x})^2\right\}, \text{ if } \bar{x} \geq \mu_0$$

$$= \exp\left\{-\frac{1}{2\sigma^2}\right\}\left\{\Sigma(x_i - \mu_0)^2\right\}, \text{ if } \bar{x} < \mu_0$$

So, if $\bar{x} < \mu_0$

$$l(x) = 1$$

and if $\bar{x} \geq \mu_0$

$$l(x) = \exp\left[\left\{\frac{-1}{2\sigma^2}\right\}\left\{\Sigma(x_i - \mu_0)^2 - \Sigma(x_i - \bar{x})^2\right\}\right]$$

$$= \exp\left[\left\{-\frac{1}{2\sigma^2}\right\}\left\{n(\bar{x} - \mu_0)^2\right\}\right]$$

So, when $\bar{x} \geq \mu_0$

$l(x) <$ some constant $\Leftrightarrow (\bar{x} - \mu_0)^2 >$ some other constant

$\Leftrightarrow \bar{x} >$ some other constant

Thus, the LR test is given by,

$$\varphi(x) = \begin{cases} 1, & \text{if } l(x) \leq l_\alpha \text{ or equivalently if } \bar{x} \geq c_\alpha \\ 0, & \text{if } l(x) > l_\alpha \text{ or } \bar{x} < c_\alpha \end{cases}$$

where c_α should be selected, such that

$$E\{\varphi(x)/\mu_0\} = P(\bar{x} \geq c_\alpha/\mu_0\} = \alpha$$

This gives,

$$c_\alpha = \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$$

Thus, we finally get

$$\varphi(x) = \begin{cases} 1, & \text{if } \bar{x} \geq \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \\ 0, & \text{if } \bar{x} < \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \end{cases}$$

which is precisely the test φ_6 we, have already seen that this is UMP.

Similarly one can show that the test φ_7 is the LR test for testing $H: \mu = \mu_0$ against $K: \mu < \mu_0$ which is also a UMP test.

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2) Let $H: \mu \leq \mu_0$ and $K: \mu > \mu_0$ In this case,
$\Theta_0 = \{\mu / \mu \leq \mu_0\}$ and $\Theta = \{\mu / -\infty < \mu < \infty\}$.
we have shown in case 2 that $L(\mu/x)$ is strictly increasing if $\mu < \bar{x}$ and there after, strictly decreasing, so, $\hat{\mu} = \bar{x}$ if $\bar{x} \leq \mu_0$ and $\hat{\mu} = \mu_0$ if $\bar{x} > \mu_0$ Also, it is easily seen that $\hat{\mu} = \bar{x}$.

Hence if $\bar{x} \leq \mu_0$

$$l(x) = 1$$

and if $\bar{x} > \mu_0$

$$l(x) = \exp\left[\left\{-\frac{1}{2\sigma^2}\right\}\left\{\Sigma(x_i - \mu_0)^2 - \Sigma(x_i - \bar{x})^2\right\}\right]$$

$$= \exp\left[\left\{-\frac{1}{2\sigma^2}\right\}\left\{n(\bar{x} - \mu_0)^2\right\}\right]$$

So, if $\bar{x} > \mu_0$, as shown in case 2, $l(x) \leq$ or $> l_\alpha \Leftrightarrow \bar{x} \geq$ or $< c_\alpha$, Thus, the LR test is given by

$$\psi(x) = \begin{cases} 1, & \text{if } \bar{x} \geq c_\alpha \\ 0, & \text{if } \bar{x} < c_\alpha \end{cases}$$

where c_α should be selected, such that $E\{\psi(x)/\mu_0\} = P\{\bar{x} \geq c_\alpha / \mu_0\} = \alpha$, this will give $c_\alpha = \mu_0 + z_\alpha \sigma/\sqrt{n}$. this is the test obtained above in case 1 and 2. Viz, we have already seen that is ump.

large Sample properties:

Consistency of LRT was introduced by wald and Wolfowitz in 1940. this the based on the rational that any good test should be able to reject the hypothesis being tested, when it is false, with increasing certaintly as the sample size increases on this basis.

Definitions:

A test is said to be consistent if the power of the test tends to units as the sample size tends to infinity and this must be true under every distn in the alternative family, that is

$E\{\phi(x)/H_1\} \to 1$ as $n \to \infty$ for all elements of H_1

Example 1:

The LR Test is consistent under the conditions under which the MLE's are constant.

Proof:

It is known that, the MLE's are consistant, so if H_1, is true, the true value of θ is in both H_0 and $H_0 \cup H_1$ and so $\hat{\theta}$.

The MLE of θ under $H_0 \cup H_1$ and $\bar{\theta}$, the MLE θ under H_0 both ~~no~~ will converge in prob to the same quantity

Hence if H is true.

$\therefore$ The CR $\lambda(x) \to 1$ in prob as $n \to \infty$

In otherwords, the distn of $\lambda(x)$ will tend to be degenerate at unity as $n \to \infty$ under H_0, since the constant λ_α of the LRT should satisfy the condition that

$$P\{\lambda(x) \leq \lambda_\alpha / H_1\} \to 1 \text{ as } n \to \infty.$$

Which shows that the LRT is consistent.

2) Any reasonable test based on a consistent estimator will be a consistent test.

Proof:

Let us prove the theorem for the case of a single parameter θ assuming $H_0: \theta = \theta_0$ Vs $H_1: \theta \neq \theta_0$

Let α be the level of the test.

Let us select two positive numbers α_1 and α_2 such that $\alpha_1 + \alpha_2 = \alpha$.

Let T_n be any consistent estimator of θ select the two positive numbers a_n and b_n such that

$$P\{T_n \leq \theta_0 - a_n / \theta_0\} = \alpha_1; \; P\{T_n \geq \theta_0 + b_n / H_0\} = \alpha_2 \to (1)$$

So, that

$$P\{\theta_0 - a_0 < T_n < \theta_0 + b_n / \theta_0\} = 1-\alpha$$

Now, define a test function $\phi(x)$ by,

$$\phi(x) = \begin{cases} 1 & \text{if } T_n < \theta_0 - a_n \text{ (or) } T_n > \theta_0 + b_n \\ 0, & \text{otherwise} \end{cases} \longrightarrow (2)$$

It is obvious that $\phi(x)$ is a reasonable test for testing H_0 against H_1 at level α since T_n is consistent for θ, if θ is true value of θ. then to any given positive number ϵ.

We can find +ve integer no. m, such that

$$P\{|T_n - \theta_0| > \epsilon / \theta_0\} < \alpha, \text{ if } n > m$$

Combining this with (1), we get if $n > m$

$$P\{|T_n - \theta_0| > \epsilon / \theta_0\} < \alpha_2 = P\{T_n \geq \theta_0 + b_n / \theta_0\}$$
$$\leq P\{|T_n - \theta_0| \geq b_n / \theta_0\}.$$

that is

$$P\{|T_n - \theta| \leq \epsilon / \theta_0\} \geq P\{|T_n - \theta_0| < b_n / \theta_0\}$$
if $n > m$.

that is

$$(\theta_0 - \epsilon, \theta_0 + \epsilon) > (\theta_0 - b_n, \theta_0 + b_n).$$

This means $b_n \leq \epsilon$, since ϵ is arbitrary $b_n \leq \epsilon \Rightarrow b_n \to 0$ as $n \to \infty$, In a similar manner If can be shown that

$a_n \to 0$ as $n \to \infty$

at $\theta > \theta_0$, select a positive no ϵ.

Such that $\theta - \epsilon > \theta_0$ since $b_n \to 0$ as $n \to \infty$.

We can choose 'n' sufficiently large

Such that $\theta_0 + b_n < \theta - \epsilon$. Hence under the same condition.

$$\{x / T_n > \theta_0 + b_n\} > \{x / T_n > \theta - \epsilon\} > \{x / |T_n - \theta| < \epsilon\}$$

$$\therefore P\{T_n > \theta_0 + b_n\} \geq P_\theta\{|T_n - \theta| < \epsilon\}$$

If 'n' a sufficiently large But prob on the right $\to 1$, as $n \to \infty$, since T_n is a consistent estimator of θ, so the prob on the right also $\to 1$ as $n \to \infty$ for all $\theta > \theta_0$

In a similar manner,

$$P_\theta\{T_n < \theta_0 - a_n\} \to 1 \text{ as } n \to \infty \text{ for all } \theta < \theta_0$$

this is shows that the power of the test is ①.

$$E_\theta(\phi) = P_\theta\{T_n \leq \theta_0 - a_n\} + P_\theta\{T_n > \theta_0 + b_n\} \to 1$$

as $n \to \infty$

for all $\theta \neq \theta_0$ Hence the test is consistent.

The LRT of Assymptotic Distribution:

Let us define the $LR = \prod_{i=1}^{n} f(x_i, \theta)$

$$R = \frac{L(\theta_0)}{L(\theta)}$$

Since, W.R.T $-2 \log \lambda(x)$ is a decreasing fn. it follows that the critical region of the LRT can also be expressed in the term.

$$C_1 = \{ x : -2 \log \lambda(x) \geq c \}$$

$$\therefore \lambda(x) = -2 \log \lambda(x) = 2[l(\hat{\theta}; \theta) - L(\theta_0; x)]$$

The CR may be written as

$$\theta_1 = \{x : \lambda(x) \geq c\}$$

and $\lambda(x)$ is called the LRT.

We have been using the idea that values of θ class to $\hat{\theta}$ are well supported by the data. So if θ_0 is a possibile value of θ, then for large sample

$$\lambda(x) \xrightarrow{D} \chi^2_p.$$

where $p = \dim(\theta)$.

The LR statistic for Assymptotic distn:

$$l(\theta) = l(\hat{\theta}) + (\hat{\theta} - \theta_0)\, l'(\hat{\theta}) + \frac{1}{2}(\hat{\theta} - \theta_0)^2$$

and remembering that $Z''(\hat{\theta}) + \cdots$

$$Z'(\hat{\theta}) = 0$$

we have,

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$$\Lambda \simeq (\hat{\theta} - \theta_0)^2 [- l''(\hat{\theta})].$$

$$= (\hat{\theta} - \theta_0)^2 I(\hat{\theta})$$

$$= (\hat{\theta} - \theta)^2 I(\theta) \cdot \frac{I(\hat{\theta})}{I(\theta_0)}$$

But $(\hat{\theta} - \theta_0) I(\theta_0)^{1/2} \xrightarrow{D} N(0,1)$ and

$$(\hat{\theta} - \theta)^2 I(\theta) \xrightarrow{D} \chi^2, \quad \frac{I(\hat{\theta})}{I(\theta)} \xrightarrow{D} 1$$

and Slutsky's theorem gives

$$\Lambda \xrightarrow{D} \chi^2$$

Provided θ_0 is the true value of θ.

Bartlett's test for homogenety of variances:

Suppose one is interested to test the homogeneity of k popln variances. (ie) to test

$$H_0 : \sigma_1^2 = \sigma_2^2 = \dots = \sigma_k^2$$

vs H_1: at least any two of them are not goal out of various test procedures, the most accepted test procedure is one given by M.S Bartlett in 1937.

Suppose $s_1^2, s_2^2 \dots s_k^2$ are the estimate variances of $\sigma_1^2, \sigma_2^2, \dots \sigma_k^2$ based on to $(n_1-1), (n_2-1) \dots (n_k-1)$ df respectively. let s_p^2 is the pooled estimated variance where

$$s_p^2 = \frac{\sum_{i=1}^{k} (n_i - 1) s_i^2}{\sum_{i=1}^{k} (n_i - 1)}$$

To test H_0 we make use of χ^2 test where

$$\chi^2 = \log_e 10 \left[\log_{10} S_p^2 \sum_{i=1}^{k}(n_i-1) - \sum_{i=1}^{k}(n_i-1) \times \log_{10} S_i^2\right]$$

$$= 2.3026\left[\log_{10} S_p^2 \sum_{i=1}^{k}(n_i-1) - \sum_{i=1}^{k}(n_i-1) \times \log_{10} S_i^2\right]$$

χ^2 has $(k-1)$ df

it has been proved that χ^2 has an upward bias, Hence a correction c is calculated by the formula

$$C = 1 + \frac{1}{3(k-1)}\left[\sum_{i=1}^{k}\frac{1}{n_i-1} - \frac{1}{\sum_{i=1}^{k}(n_i-1)}\right]$$

Corrected value of chi-square;

$$\chi^2_c = \chi^2/c$$

Using χ^2_c distributed with $(k-1)$ df, the decision about H_0 is taken in the usual manner.