Unit - II

Uniformly Most powerful Test:

Let us take the most powerful test $H_0: \mu = \mu_0$ against $H_1: \mu = \mu_1$ (or) $(\mu_1 > \mu_0)$ using a random sample of n-observations from the distn. $N(\mu, \sigma^2)$ σ^2 known, at level α. is given by

$$\phi(x) = \begin{cases} 1 : & \text{if } \bar{x} \geq \mu_0 + Z_\alpha \frac{\alpha}{\sqrt{n}} \\ 0 : & \text{otherwise} \end{cases}$$

and if $\mu_1 < \mu_0$ the MPT of level α is given by.

$$\phi(x) = \begin{cases} 1 & \text{if } \bar{x} \leq \mu_0 - Z_\alpha \frac{\alpha}{\sqrt{n}} \\ 0, & \text{otherwise} \end{cases}$$

Definition of UMPT:

Let C_α be the collection of all level α test for testing a hypothesis H_0 against a composite alternative hypothesis H_1. A test $\phi \in C_\alpha$ is said to be UMP if $E[\phi(x)/H_1 \geq E(\phi/H_1)] \;\forall$ (for every) $\phi^* \in C_\alpha$ and for every distn in H_1.

One parameter Exponential family:

Let θ be a real parameter and let x have pdf $f(x) = c(\theta) \cdot e^{\theta T(x)} \cdot h(x)$

Null hypothesis $H_0: \theta_1 \leq \theta \leq \theta_2$

$H_1: \theta < \theta_1$ or $\theta > \theta_2$

(or)

$$P_\theta(x) = c(\theta) \, e^{Q(\theta) \cdot T(x)} \cdot S(x)$$

$$\Rightarrow P_\theta(x) = \exp[Q(\theta) + D(\theta) + h(x)].$$

where Q is strictly monotone. Then $\exists$ a UMPT ϕ for testing $H_0: \theta < \theta_0$ Vs $H_1: \theta > \theta_1$

* If Q is Increasing

$$\phi(x) = \begin{cases} 1 & \text{if } T(x) > c \\ \gamma & \text{if } T(x) = c \\ 0 & \text{if } T(x) < c \end{cases}$$

c and γ determined by $E_{\theta_0}[\phi(x)]$:

* If Q is Decreasing, then the inqualition are reversed.

$$P_\theta(x) = c(\theta) \cdot e^{Q(\theta) \cdot T(x)} \cdot S(x)$$

$c(\theta)$ is the fn. of θ alone

$Q(\theta)$ - Monotone fn. of θ

$T(x)$ - sufficient statistic

$S(x)$ - Simply fn. of x

There are four types of Testing

a)

i) $H_0 : \theta \le \theta_0$ Vs $H_1 : \theta > \theta_0$,

$$\phi(x) = \begin{cases} 1 & : T > c \\ \gamma & : T = c \\ 0 & ; T < c \end{cases}$$

where c and γ are obtaine by

$E_{\theta_0}[\phi(x)] = \alpha$.

ii) $H_0 : \theta > \theta_0$ Vs $H_1 : \theta < \theta_0$

$$\therefore \phi(x) = \begin{cases} 1 & ; T < c \\ \gamma & ; T = c \\ 0 & ; T > c \end{cases}$$

b) $H_0 : \theta \le \theta_1$ Vs $H_1 : \theta > \theta_2$, $H_1 : \theta_1 < \theta < \theta_2$

$$\therefore \phi(x) = \begin{cases} 1 & ; c_1 < T(x) < c_2 \\ \gamma_i & ; T(x) = c_i ; \ i = 1, 2 \dots \\ 0 & ; T(x) < c_1 \text{ or } > c_2 \end{cases}$$

The constant are $c_1, c_2, \gamma_1, \gamma_2$ are obtained

by $E[\phi(x)/\theta_1] = \alpha$; $E[\phi(x)/\theta_2] = \alpha$

c) $H_0 : \theta_1 \le \theta \le \theta_2$ Vs $H_1 : \theta_1 \ge \theta \ge \theta_2$

$$\therefore \phi(x) = \begin{cases} 1 & : T(x) < c_1 \text{ or } > c_2 \\ \gamma_i & : T(x) = c_i \ ; \ i = 1, 2 \\ 0 & ; c_1 < T(x) < c_2 \end{cases}$$

The constants are obtained by

$E[\phi(x)/\theta_1] = E[\phi(x)/\theta_2] = \alpha$.

d) $H_0: \theta = \theta_0$ Vs $H_1: \theta \neq \theta_0$ (or) $\theta > \theta_0$

$$\therefore \phi(x) = \begin{cases} 1 & ; \quad c_2 < T(x) < c_1 \\ \gamma & ; \quad T(x) = c_i \\ 0 & ; \quad c_1 < T(x) < c_2 \end{cases}$$

α - Similar Test:

A test ϕ is said to be a similar on a subset of Ω^* of Θ

$$\beta_\phi(\theta) = E_\theta[\phi(x)] = \alpha; \quad \theta \in \Omega^*$$

A test is said to be on a set Ω is subset of Θ, if it is α similar on Ω^* for same α; $0 < \alpha < 1$

It is clear that $\exists$ at least one similar test on every Ω^* namely $\phi(x)$ $0 \leq \alpha \leq 1$

Let $\beta_\phi(\theta)$ be the continous in θ. for any ϕ. If ϕ is an unbiased size 'α' test of $H_0: \theta \in \Omega_0$ against $H_1: \theta \in \Omega_1$ it is 'α' similar on the boundary $\Lambda = \bar{\Omega}_0 \cap \bar{\Omega}_1$

Let the power fn. of every test ϕ of $H_0: \theta \in \Omega_0$ Vs $H_1: \theta \in \Omega_1$ be continuous in θ. then a UMPT α similar test is UMP unbiased Provided that is size α for testing H_0 against H_1.

A test ϕ is UMP α similar test on the boundary $\Delta = \Omega_0 \cap \bar{\Omega}$, is said to be UMP α similar test.

Monotone Liklihood Ratio Property:

When the hypothesis tested is simple and the alternative hypothesis is composite, the best possible solution to the testing Problem is the UMP test, if it exists. But, unfortunately, UMP tests do not exists always. However in the case of a single parameter there is one family of distn. known as the family of distn possessing the monotone likelihood ratio Property. For which UMP tests exist when the hypothesis tested is one sided. This is investigated in this section.

Definition 7.12:

Let $\{P_\theta(x) \mid \theta \in \Theta\}$ be the family of joint probability densities of a random sample of observations on a R.V. $X \sim f(x, \theta)$ $\theta \in \Theta$. This family is said to passes the Monotone likelihood Ratio (MLR) Property through a statistic T if, for any two values of the parameter θ, say $\theta_1 > \theta_0$ the ratio

$$\frac{P_1}{P_0} = \frac{P_{\theta_1}(x)}{P_{\theta_0}(x)}$$

is a non-decreasing fun of t.

Many of the common distn used in applications have this Property as shown below.

Examble: 7.13:

In the case of a random sample of n observations on $X \sim B(m, \theta)$, (Binomial Distn)

$$P_\theta(x) = \prod \binom{m}{x_i} \theta^t (1-\theta)^{(mn-t)},$$

Where $t = \sum x_i$ and $0 < \theta < 1$

So, if $0 < \theta_0 < \theta_1 < 1$, then

$$\frac{P_1}{P_0} = \left(\frac{\theta_1}{\theta_0}\right)^t \left(\frac{1-\theta_1}{1-\theta_0}\right)^{mn-t}$$

$$= \left[\frac{\theta_1(1-\theta_0)}{\theta_0(1-\theta_1)}\right]^t \left(\frac{1-\theta_1}{1-\theta_0}\right)^{mn}$$

This is a strict increasing fun of t since $\{\theta_1(1-\theta_0)/\theta_0(1-\theta_1)\} > 1$. so the family $P_\theta(x)$ has MLR Property through T, the sample total or equivalently the sample mean $\bar{x}$.

Examble: 7.14: (Normal distn)

Let $X \sim N(\mu, \sigma^2)$, σ^2 known, In this case

$$P_\mu(x) = (\sigma\sqrt{2\pi})^{-n} \exp\left[\left\{\frac{-1}{(2\sigma^2)}\right\}\left\{\sum(x_i-\mu)^2\right\}\right]$$

So if $\mu_1 > \mu_0$

$$\frac{P_1}{P_0} = \exp\left[\left\{\frac{-1}{(2\sigma^2)}\right\}\left\{\Sigma(x_i-\mu_1)^2 - \Sigma(x_i-\mu_0)^2\right\}\right]$$

$$= \exp\left[\left\{\frac{n\bar{x}(\mu_1-\mu_0)}{\sigma^2}\right\}+\left\{\frac{n(\mu_0^2-\mu_1^2)}{2\sigma^2}\right\}\right]$$

Since $\mu_1 > \mu_0$ P_1/P_0 is a strict increasing function of $\bar{x}$, so the family $\{P_\mu(x) \mid \mu \in R\}$ has MLR Property through the sample Mean $\bar{x}$.

Example 7.15: (uniforly distr). In this case

Let $x \sim U(0,\theta)$,

$$P_\theta(x) = \left(\frac{1}{\theta^n}\right) c(\theta - x_{(n)}), \text{ where}$$

$\theta > 0$ and $0 < x_{(n)} < \theta$

where $c(u) = 0$ or 1 according as

$x \leq 0$ or > 0.

So if $\theta_1 > \theta_0 > 0$, then

$$\frac{P_1}{P_0} = \begin{cases} \left(\frac{\theta_0}{\theta_1}\right)^n, & \text{if } x_{(n)} < \theta_0 \\ \infty, & \text{if } x_{(n)} \geq \theta_0 \end{cases}$$

Thus as $x_{(n)}$ goes from 0 to ∞, P_1/P_0 jumps from $(\theta_0/\theta_1)^n$ to ∞ (takes only two values). So in this case also the family $\{P_\theta(x) \mid \theta > 0\}$ has the MLR Property through the n^{th} order statistic $X_{(n)}$.

Example 7.16

Let $X \sim C(\theta, 1)$, viz. cauchy distn with location parameter θ and scale parameter unity. In this case.

$$P_\theta(x) = \left(\frac{1}{\pi}\right)^n \left[\pi\left(1+(x_i-\theta)^2\right)^{-1}\right]$$

so if $\theta_1 > \theta$

$$\frac{P_1}{P_0} = \pi\left\{\frac{1+(x_i-\theta_0)^2}{1+(x_i-\theta_1)^2}\right\}$$

This cannot be expressed as an increasing function of any sample statistic so in the case of the cauchy family of distributions, the joint PDF does not possess the MLR property.

Example 7.17:

Let the PDF of the R.V X belong to the one parameter exponential family (3.28) The joint PDF of n independent random observations on X is given by.

$$P_\theta(x) = \exp\left\{\alpha(\theta)\sum a(x_i) + \sum b(x_i) + n\beta(\theta)\right\}$$

Where $\alpha(\theta)$ is a strict monotonic fun of θ. Without loss of generality, $\alpha(\theta)$ can be assumed to be increasing and so $\alpha(\theta)$ and θ determine each other uniquely

So if $\theta_1 > \theta_0$

$$\frac{P_1}{P_0} = \exp\left\{t\left[\alpha(\theta_1) - \alpha(\theta_0)\right] + n\left[\beta(\theta_1) - \beta(\theta_0)\right]\right\}$$

Where $t = \sum a(x_i)$, since $\alpha(\theta)$, is strictly increasing and so $\alpha(\theta_1) - \alpha(\theta_0) > 0$, P_1/P_0 is a strict increasing function of t.
Thus we see that, in that, in the case of the one parameter exponential family. the joint PDF of the sample observations has the MLR Property through the statistic, $T = \sum a(x_i)$. In, Example 7.15 we showed that the ~~go~~ joint PDF of the sample data, in the case of $U(0, \theta)$, has the MLR Property. We also know that this distn is not a member of the one parameter exponential family of distribution.

<u>Remarks</u>
Thus, the Exponential family of distn is a proper subset of the family of distn having the MLE property.

Theorem:
Let $P_\theta(x)$, the joint PDF of a sample of observations on a random variable $X \sim f(x, \theta)$, $\theta \in \Theta$ have the MLR Property through a statistic T. then

a) For testing $H: \theta \leq \theta_0$ against $K: \theta > \theta_0$, $(\theta_0 \in \Theta)$, at level α, there exists given by

$$\psi(T) = \begin{cases} 1, & \text{if } T > c \\ \gamma, & \text{if } T = c \\ 0, & \text{if } T < c \end{cases}$$

where c and γ are to be selected such

unbiased Test:

unbiased test is a test in which the Probability of rejecting H_0, when H_0 is false greater than or equal to the Probability of rejecting H_0 when H_0 is true that is $1-\beta \geq \alpha$.

To Test $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$ the critical Region ω and its test is said to be unbiased test

If the power of the test exceed the size ω of critical Region.

Power of the test > size of the critical Region.

$$\therefore 1-\beta > \alpha.$$

$$\Rightarrow P_{\theta_1}(x) > P_{\theta_0}(x) \text{ (or) } P_{\theta_1}(\omega) > P_{\theta_0}(\omega)$$

$$\Rightarrow P[x \in \omega / H_1] > P[x \in \omega / H_0]$$

$$\Rightarrow \int_\omega L_1 dx > \int_\omega L_0 dx$$

Theorem:

If ω is the Most powerful or uniformly most powerful critical region then its necessary unbiased.

Proof:

Since ω is the MPCR of size α for testing $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$ by Neymann Pearson lemma, we have for $\forall\ k > 0$.

$W = \{x : L(x, \theta) \geq k L(x, \theta_0)\}$

$= \{x : L_1 \geq k L_0\}$ and $W' = \{x : L(x, \theta_1) < k L(x, \theta_0)\} = \{x : L_1 < k L_0\}$ where k is determind so that the size of the test is α (ie),

$$P_{\theta_0}(\omega) = P(x \in \omega / H_0) = \int_{\omega} L_0 \, dx = \alpha \longrightarrow (1)$$

To Prove that W is unbiased, we have to show that power of $W \geq \alpha$ (ie) $P_{\theta}(\omega) \geq \alpha \longrightarrow (2)$.

We have $P_{\theta_1}(\omega) = \int_{\omega} L_1 \, dx \geq k \int_{\omega} L_0 \, dx = k\alpha$

[$\because$ On W, $L_1 \geq k L_0$ and using (1)]

(ie) $P_{\theta_1}(\omega) \geq k\alpha$, $\forall$ $k > 0 \longrightarrow (3)$

Also.

$1 - P_{\theta_1}(\omega) = 1 - P(x \in \omega / H_1) = P(x \in \omega' / H_1) = \int_{\omega'} L_1 \, dx$

$< k \int_{\omega} L_0 \, dx = k P(X : x \in \omega' / H_0)$

[$\because$ on w', $L_1 < k L_0$]

Case (i): If $k \geq 1$ the from (3), we get

$$P_{\theta_1}(\omega) \geq k\alpha > \alpha$$

$\Rightarrow$ W is unbiased CR.

Case (ii): $0 < k < 1$. if $0 < k < 1$ then from (4), we get $1 - P_{\theta_1}(\omega) < 1 - \alpha$

$\Rightarrow P_{\theta_1}(\omega) > \alpha \Rightarrow W$ is unbiased CR Hence MP Critical region is unbiased

This test is called an unbiased test.

UMPU Tests: (uniformly Most powerful unbiased test):

As usual, let x stand for the vector of observations on a random variable X with pdf $f(x, \theta)$, $\theta \in \Theta$, the Parameter space. Also, let $P_\theta(x)$ stand for the joint pdf of the sample observations and $P = \{P_\theta(x) | \theta \in \Theta\}$. When θ is a vector, i.e. the number of Parameters involved in the PDF is more than one and we are concerned with testing a hypothesis relating to one but not all of them, the other parameter left unspecified by the hypothesis are called nuisance parameters. Their presence cannot be avoided. but they create Problems. So, such parameters are called nuisance parameter. In the case of testing for the mean of a normal population, the unknown variance is a nuisance parameter. One method of overcoming the Problems of nuisance parameters. in hypothesis testing is to use similar tests explained earlier and ~~use~~ for this purpose. We need to introduce a new concept, viz., tests with Neymann structure, which is defined as follows.

Definition (Tests with Neymann structure)

Let T be sufficient for the family

$$P^* = \{P_\theta(x) / \theta \in \Theta^*\}$$

Also let,

$$P_t^* = \{P_\theta(t) / \theta \in \Theta^*\}$$

Viz., the family of distributions of the sufficient statistic T as θ varies in Θ^*. Then, any test function φ satisfying the condition.

$$E_\theta\{\varphi(x) / T = t\} = \alpha, \text{ for all } t \text{ and for}$$

all $\theta \in \Theta^*$

is said to have Neymann structure with respect to T on P_t^*. It means that on each of the surfaces $T=t$.

$$E_\theta\{\varphi(x)\} = \alpha, \text{ for all } \theta \in \Theta^*.$$

Remarks:

one can easily see that if a test has Neymann structure with respect to T on P_t^*, then it is similar with respect to P^*. This is because.

$$E_\theta\{\varphi(x)\} = E_\theta[E\{\varphi(x) / T = t\}] = E_\theta(\alpha) = \alpha.$$

$\forall \; \theta \in \Theta.$

Prove that for one parameter exponential family there is no UMP test:

Proof:

$$f(x) = \theta e^{-\theta x} \qquad x > 0$$

$$H_0 : \theta = \theta_0 \quad \text{Vs} \quad H_1 : \theta \neq \theta_0$$

$$L = \prod_{i=1}^{n} f(x_i, \theta) = \theta^n \exp\left[-\theta \sum_{i=1}^{n} x_i\right]$$

Consider $H_1 : \theta = \theta_1 \quad (\theta_1 \neq \theta_0)$

The best critical region, using Neymann Pearson lemma is given by

$$(\theta_0 - \theta_1) \sum x_i \geq \log\left\{k \cdot \left(\frac{\theta_0}{\theta_1}\right)^n\right\} = k_1 \text{ (say)} \to (1)$$

The constants λ_1 and λ_2 are so determine that

$$\left.\begin{array}{l} P\left[\sum x_i : \leq \lambda_1 \leq \lambda_1 / H_0\right] = \alpha \\ \Rightarrow P\left[2\theta \sum x_i \leq 2\theta \lambda_1 / H_0\right] = \alpha \end{array}\right\} \to (2)$$

$$\left.\begin{array}{l} P\left[\sum x_i \geq \lambda_2 / H_0\right] = \alpha \\ \Rightarrow P\left[2\theta \sum x_i \geq 2\theta \lambda_2 / H_0\right] = \alpha \end{array}\right\} \to (3)$$

But in random sampling from the given exponential distribution.

$$M_{\sum x}(t) = M_{x_i}(t) = \prod_{i=1}^{n} M_{x_i}(t) = \left\{M_{x_i}(t)\right\}^n$$

$$= \left(1 - \frac{t}{\theta}\right)^{-n}$$

$M_{2\theta \sum x_i}(t) = M_{\sum x}(2t\theta) = (1-2t)$

which is the mgf of a χ^2 - variate with $2n$ df Here by uniqueness theorem of m.g.f's $2\theta \sum_{i=1}^{n} x_i \sim \chi^2(2n)$

using this result in (2)

$P[2\theta_0 \sum x_i \leq \mu_1] = P[\chi^2(2n) \leq \mu_1] = \alpha$

$\Rightarrow \mu_1 = \chi_1^2 - \alpha . 2n$

where $\chi^2_{\alpha, n}$ is the upper 'α' point of χ^2 distrn with n df given by

$P(\chi^2 > \chi^2_{\alpha . n}) = \alpha \longrightarrow (4)$

Hence, BCR for testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_1$ $(>\theta_1)$ is given.

$w_0 = \{x : 2\theta_0 \sum x \leq \chi^2 - \alpha, 2n\} =$

$\{x : \sum x_i < \frac{1}{2\theta_0} \chi_1^2 - \alpha, 2n\}.$

and since it is independent of θ_1 w_0 is UMPCR for $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_1$ $(>\theta_0)$ 11ly from (3) and (4), we get

$w_1 = \{x : 2\theta_0 \sum x_i \geq \chi^2 \alpha, 2n\} = \{x \sum x_i \geq \frac{1}{2\theta_0} \alpha 2n\}$

since the two critical regions w_0 and w_1 are different there exits no critical region of size α which is UMP for $H_0 \theta = \theta_0$ against the two tailed Alternative hypothesis $H_1 : \theta \neq \theta_1$.

Similar Test:

1) Explain similar Critical region with examples

1. Consider the null hypothesis H_0 for testing $H_0: \theta \in \Theta_0$ where Θ_0 consists of many points. (ie) The null hypothesis is composite null hypothesis.

2. To test composite null hypothesis H_0 $\theta \in \Theta_0$.. we have to search for rejection region in the sample space which is similar for all θ.

3. If ω be the similar critical region to test the null hypothesis $H_0: \theta \in \Theta_0$ then we should have $\alpha = P_{H_0}(\omega)$ $\forall \theta \in \Theta_0$

4. The test based on similar critical region is called as similar test..

5. The necessary and sufficient condition for similar critical region is

$$E(I_\omega / t) = \alpha \quad \forall \theta \in \Theta_0$$

where

$$I_\omega = \begin{cases} 1 & \text{if } x \in \omega \\ 0 & \text{otherwise} \end{cases}$$

t is sufficient statistics

$$\Rightarrow E_\theta(I_\omega) = P_\theta(\omega) = \alpha \quad \forall \theta \in \Theta_0$$

2) Explain Neymann structure with examples?

1. A test with critical region ω is said to be a Neymann's structure with respective to the sufficient statistic 'T'. If

$E(I_{\omega}/t)$ is the same almost every

where $\omega \in \Theta_0$

2. The characteristic of Neymann's structure is the condition probability of rejecting H_0 when it is true is equal to α on each surfaces 'T' t.

(ie) $P_{\alpha}[\{\omega / H_0 : \theta \in \Theta_0\}] = \alpha$

3. Every that based on similar region will have Neymann structure with sufficient statistic 'T'.

Pblm:

1) Find the most powerful similar CR of size α for testing $H_0: \mu = \mu_0$ vs $H_1: \mu = \mu_1$ from a normal popln.

Sol:

$H_0: \mu = \mu_0$ vs $H_1: \mu = \mu_1$

The joint pdf of $x_1, x_2 \dots x_n$ under H_0, where σ^2 is regared as the parameter is,

$$f(x / \mu_0, \sigma^2) = \frac{1}{(\sigma\sqrt{2\pi})^n} \exp\left[-\sum (x_i - \mu_0)^2 / 2\sigma^2\right]$$

This shows that.

$V = \sum_1 (x_i - \mu_0)^2$ is sufficient for σ^2.

Consider now a particular simple hypothesis $h_0 \in H_0$ and simple alternative hypothesis $h_1 \in H_1$ viz.

$$h_0 : \mu = \mu_0 , \sigma^2 = \sigma_0^2$$

$$h_1 : \mu = \mu_1 : \sigma^2 = \sigma_1^2$$ and the most powerful similar region of size α for testing h_0 vs h_1 is

$$w_0 : \{x \mid f(x, \mu_1 \sigma_1^2) > k(v) f(x \mid \mu_0, \sigma_0^2)\}$$

where $k(v)$ is such that the condition size of w_0 given $V = v$, is α now, if we take logarithm on both sides, we see that

$$f(x \mid \mu_1 \sigma^2) > k(v) f(x \mid \mu_0, \sigma^2)$$

$$(\mu_1 - \mu_0)(\bar{x} - \mu_0) > k_1(v) \quad \ldots (say)$$

where $k_1(v)$ is related to $k(v)$

case (i) $\mu_1 > \mu_0$

Here the condition is equavalent to

$$(\bar{x} - \mu_0) > k_2(v)$$

$$\sqrt{n}(\bar{x} - \mu_0)/\sqrt{v} \Rightarrow k_3(v) \text{ say}$$

As such

we may write

$$w_0 = \{x / \sqrt{n}(\bar{x} - \mu_0)/\sqrt{v} > k_3(v)\}$$

$k_3(v)$ is to determined such that

$$P_{\theta_0}(w_0/v) = \alpha.$$

Such that $P_{\theta_0}[\sqrt{n}(\bar{x} - \mu_0)/\sqrt{v} > k_3] = \alpha$

we also note that,

$$\frac{\sqrt{n}\,(\bar{x}-\mu_0)}{\sqrt{v}} = \frac{\sqrt{n}\,(\bar{x}-\mu_0)}{\{n(\bar{x}-\mu_0)^2 + \Sigma(x-\bar{x})^2\}^{1/2}}$$

$$= \frac{1}{\sqrt{t^2+n-1}}$$

where $t = \sqrt{n}\,(\bar{x}-\mu_0)/s$ is student's t statistic with $n-1$ degrees of freedom, since

$\sqrt{n}\,(\bar{x}-\mu_0)/\sqrt{v} > k_3$

iff $t > k_4$ (say)

we may also write

$$w_0 = \{x \,/\, \sqrt{n}\,(\bar{x}-\mu_0)/\sqrt{v} > k_3\}$$

$$= \{x \,/\, t > k_4\}$$

where k_4 is such that

$$P_{\theta_0}[t > k_4] = \alpha$$

This shows that k_4 is the upper α-point of the t-distribution with $n-1$ degrees of freedom. Denote this by $t_{\alpha, n-1}$ we have, finally

$$w_0 = \{x/\sqrt{n}(\bar{x}-\mu_0) > t_{\alpha, n-1}\}$$

Since this is independent of σ_0^2 and σ_1^2, it is the MP similar region of size α for testing H_0 vs H_1.

case (ii) $\mu_1 < \mu_0$:

Arguing in a similar fashion the similar critical region is

$$w_0 = \{x \,/\, \sqrt{n}\,(\bar{x}-\mu_0)/\sqrt{s} > -t_{\alpha, n-1}\}.$$

2) Pblm:

Test the null hypothesis

$H_0: \sigma^2 = \sigma_0^2$ V/s

$H_1: \sigma^2 = \sigma_1^2$ (composite)

From a normal popln with mean μ and variance σ^2. Determine the most powerful similar CR.

Sol:

$H_0: \sigma^2 = \sigma_0^2$ V/s

$H_1: \sigma^2 = \sigma_1^2$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$L = \prod_{i=1}^{n} f(x_i, \mu)$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \frac{\Sigma(x_i - \mu)}{\sigma}$$

Consider a simple hypotheses h_0 to H_0 and an alternative simple hypothesis h_1 i.e. H_1 such that

h_0 $\mu = \mu_0$, $\sigma^2 = \sigma_0^2$

h_1 $\mu = \mu_1$ $\sigma^2 = \sigma_1^2$

and

we get the MP similar region of size α for testing H_0 VS H_1 as

$$W_0 = \{x \mid f(x \mid \mu, \sigma_1^2) > k(\bar{x}) f(x \mid \mu_0, \sigma_0^2)\}$$

where $k(\bar{x})$ is such that the conditional size of W_0, given $\bar{x} = \bar{x}$ is α. Now the condition

$$f(x \mid \mu_1, \sigma_1^2) > k(\bar{x}) f(x \mid \mu_0, \sigma_0^2)$$

reduces, if we take logarithms of both sides we get.

$(\sigma_1^2 - \sigma_0^2) \cdot \Sigma_i (x_i - \bar{x})^2 > k(\bar{x})$

say,

Case 1: $\sigma_1^2 > \sigma_0^2$

Here condition is equivalent to

$\Sigma_i (x_i - \bar{x})^2 > k_2(\bar{x})$

$\Sigma_i (x_i - x)^2 / \sigma_0^2 > k_3(\bar{x})$

Say,

we may therefore we write.

$w_a = \{x \mid \Sigma (x_i - \bar{x})^2 / \sigma_0^2 > k_3(\bar{x})\}$

where $k_3(\bar{x})$ is to be so determined

that

$P_{\theta_0}(|w_0|\bar{x}) = \alpha$

we know that $\Sigma_i (x_i - \bar{x})^2 / \sigma_0^2$ and

independently distributed.

Hence conditional distribution of

$\Sigma_i (x_i - \bar{x}_i)^2 / \sigma_0^2$ given $x = \bar{x}$ is the same as its.

marginal distribution.

implying that $k_3(\bar{x})$ is independent of

$\bar{x}$ writing k_3 for this constant we note that

it is to be so chosen that.

$P_{\theta_0}[\Sigma_i (x_i - \bar{x}) / \sigma_0^2 > k_3] = \alpha.$

since $\Sigma (x_i - \bar{x})^2 / \sigma_0^2$ has under H_0 χ^2

distribution with $n-1$ degrees of freedom k_3 must

be the upper α points of this distribution which.

may be denoted by $\chi^2_{\alpha, n-1}$.

as such.

$w_0 = \{x \mid \Sigma_i (x_i - \bar{x})^2 / \sigma_0^2 > \chi^2_{\alpha, n-1}\}$

since this is an independent of μ_0 and μ_1

it is the MP similar region of size α for testing H_0 vs H_1.

UMP - unbiased test for one parameter exponential family:

Null hypothesis $H_0 : \theta_1 \le \theta \le \theta_2$

$H_1 : \theta < \theta_1$ or $\theta > \theta_2$

Consider the general pdf.

$$f(x) = C(\theta)\, e^{\theta T(x)}\, h(x) \text{ --- } (1).$$

The critical region is obtained and are as follows.

$$\phi(x) = \begin{cases} 1 & T(x) < c_1 \text{ or } T(x) > c_2 \\ \gamma & T(x) = c_i \quad i = 1,2 \\ 0 & c_1 < T(x) < c_2 \end{cases} \text{ --- } (2)$$

Such that.

$$E_{\theta_1} \phi(x) = \alpha \text{ --- } (3)$$

$$E_{\theta_2} \phi(x) = \alpha \text{ --- } (4)$$

From e.q. (3)

$E_{\theta_1} \phi(x) = \alpha$ multiplying $t(x)$ on both sides and Taking expectation we get.

$$E(T(x)\, \phi(x)) = E(T(x))\, \alpha \text{ --- } (5)$$

Now we have to maximize $E[\phi(x)]$ for any value at $\theta = \theta_1$ By maximizing (3), (4) and (5) the critical region is obtained and is given by,

$$C(\theta_0)\,(k_1 + k_2 t)\, e^{\theta_0 t} < C(\theta')\, e^{\theta' t}$$

The constants k_1, k_2 can be determine by satisfying e.q (3) Hence UMPU test exsist for one parameter exponential family.

Start with a group G of appropriate transformations 'g' which map the sample space $\mathcal{X}$' onto itself on a one to one basis and the associated group '$\bar{G}$' of transformations $\bar{g}$ which map the parameter space Θ onto itself on a one to one basis.

Definition of Invariant Test 1:

A hypotheses testing problem for testing $H_0: \theta \in \Theta_0$ against $H_1: \theta \in \Theta_1$ is said to be invariant under a group 'G' of transformation 'g' if the associated group of transformation '$\bar{G}$' of $\bar{g}$ preserves both Θ_0 and Θ_1. That is $\bar{g}(\Theta_0) = \Theta_0$ and $\bar{g}(\Theta_1) = \Theta_1$ for all elements $\bar{g}$ of $\bar{G}$.

After identifying an appropriate group 'G' of transformations on x and the association group '$\bar{G}$' of transformation $\bar{G}$ on Θ with the above stated properties.

2. A test fn $\phi(x)$ is said to be invariant under a group G of transformations is $\phi[g(x)] = \phi(x)$ for $x \in \mathcal{X}$ and for all $g \in G$

3. Two points x and y belonging to $\mathcal{X}$ are said to be equivalent ph under the group 'G' if there exists a $g \in G$ such that $y = g(x)$.

4. For any $x \in \mathcal{X}$ the set $[g(x) | g \in G]$ viz the set of all those points of $\mathcal{X}$ obtained as the various transformations in G are applied to this pt x is called the orbit generated by x. So all points on an orbit are equivalent pts.

Maximal Invariants:

Which is very helpful to identify the invariant test fn.

Definition of Maximal Invariants:

A fn $\phi(x)$ defined on $\mathcal{X}$ is said to be a Maximal Invariant

If i) it is invariant that is

$$\phi[g(x)] = \phi(x) \text{ for all } x \in \mathcal{X} \text{ and } g \in G$$

ii) $\phi(x) = \phi(y) \Rightarrow y = g(x)$, for some $g \in G$

This definition implies that a maximal invariant fn takes the same value at every point in an orbit but takes different values at pts in the different orbits.

Theorem

Let $T(x)$ be a maximal invariant with respect to the group of transformations G. Then a necessary and sufficient condition for a test function ϕ to be invariant with respect to G is that ϕ depends on x only through $T(x)$. This means that there is a function h such that

$\phi(x) = h\{T(x)\}$: for all x in $\mathcal{X}$.

Proof :

Sufficiency suppose that there exists a function h such that $\phi(x) = h\{T(x)\}$ for all x in $\mathcal{X}$. Then for any $g \in G$

$$\phi\{g(x)\} = h[T\{g(x)\}] = h\{T(x)\} = \phi(x)$$

This means ϕ is invariant with respect to G

Necessity Let ϕ be invariant with respect to G

Also since $T(x)$ is a maximal invariant, for any $x, y \in \mathcal{X}$, if $T(x) = T(y)$ then there exists a $g \in G$ such that $y = g(x)$ then.

$$\phi(y) = \phi\{g(x)\} = \phi(x)$$

since ϕ is invariant ; so for any $x, y \in \mathcal{X}$.

$$T(x) = T(y) \Rightarrow \phi(x) = \phi(y)$$

This is possible only if $\phi(x)$ is a function of $T(x)$

Example 10.6 :

Recalling that x stands for $(x_1, x_2 \dots x_n)$ we define

$$g_c(x) = (x_1 + c, x_2 + c, \dots x_n + c)$$

Let $G = \{g_c(x) \mid c \in R\}$ under G, one invariant function of x is

$$T(x) = y = (y_1, y_2 \dots y_{n-1})$$

where $y_i = x_i - x_n$, $i = 1, 2 \dots n-1$, since $y = g_c(x)$ for any c. Further, if x and x' are two points in $\mathcal{X}$ with $T(x) = T(x')$, thus,

$T(x) = T(x')$

$y_i = y_i'$, $\quad i = 1, 2, \dots n-1$

$x_i - x_n = x_i' - x_n'$, $\quad i = 1, 2, \dots n-1$

$\Rightarrow x_i' = x_i + c$, $i = 1, 2 \dots n-1$, where

$c = x_n' - x_n$

$x' = g_c(x)$

Thus, $T(x)$ is a maximal invariant.

Note that maximal invariant need not be unique. Another maximal invariant in this example is $M(x) = (x_1 - \bar{x}, x_2 - \bar{x}, \dots x_n - \bar{x})$

Example 10.7:

Let $G = \{g_c'(x)\}$ where $g_c'(x) = (cx_1, cx_2, \dots cx_n)$, $c \in R$ and $c > 0$. Under this G, an invariant function of x is

$$M(x) = y = 0 \quad \text{if each } x_i = 0$$
$$= (y_1, y_2, \dots y_n) \quad \text{otherwise.}$$

where,

$$y_i = \frac{x_i}{(\sum x_i^2)^{1/2}}, \quad i = 1, 2 \dots n$$

for,

$$M\{g_c'(x)\} = M\{(cx_1, cx_2 \dots cx_n)\} = M(x)$$

where

$$c = (\sum x_i^2)^{-1/2}$$

Also note that

$M(x) = M(x') \Rightarrow$ either $M(x) = M(x') = 0$

$\Rightarrow$ each x_i = each $x_i' = 0$

or

$$\frac{x_i}{c} = \frac{x_i'}{c'}, \quad \text{for} \quad i = 1, 2, \ldots n$$

where

$$c' = \left(\Sigma x_i'^2\right)^{-1/2}$$

This implies

$$x_i' = \left(\frac{c'}{c}\right) x_i, \quad i = 1, 2, \ldots n$$

That is

$$x_i' = k x_i, \quad i = 1, 2, \ldots n$$

where $k = \frac{c'}{c}$

So; $x' = g_k'(x)$ in all cases.

Hence; $M(x)$ is a maximal invariant.

Example 10.8:

Let $G = \{ g_{a,b}(x) \mid a > 0, \; a, b, \in R \}$

where

$$g_{a,b}(x) = (a x_1 + b, \; a x_2 + b, \ldots \; a x_n + b)$$

Proceeding as in the above examples, one can show that

$$M(x) = 0, \quad \text{if} \quad V = 0.$$

where

$$V = \Sigma \left\{ \frac{(x_i - \bar{x})^2}{n} \right\}^{1/2}$$

$$M(x) = \frac{(x_1 - \bar{x})}{\gamma}, \frac{(x_2 - \bar{x})}{\gamma}, \ldots \frac{x_n - \bar{x}}{\gamma}$$

If $\gamma \neq 0$.

UMP Invariant Tests:

Now, we proceed to obtain MPT having the invariance property making use of the maximal invariants defined in the preceding section.

If invariance is a required quality of a hypothesis testing procedure then we 1st identify a group G of transformation with reference to which we seek invariance. Then we check whether the hypothesis testing problem is invariant and then consider test functions which are invariant ~~and then~~ with ~~consider~~ respect to G. Since any invariant function should be a function of the maximal invariant we first identify a maximal invariant and ~~then~~ ~~and~~ consider only those test functions which are functions of the maximal invariant and than select one among them which has the main desirable quality of maximum power. Such a test will have maximum power among all invariant test and so is the most desirable test. It is called the most powerful invariant test.