Unit V	Topic
	5.1 Renewal process 5.2 Renewal function and its properties 5.3 Elementary renewal theorem 5.4 Strict and wide sense stationary processes with Examples 5.5 Basic ideas of Time series 5.6 Auto-regressive and moving average processes.

5.1 Renewal process

Renewal process is a generalization of Poisson process. Renewal theory has assumed great importance because of its theoretical structure as well as for its application various situations, such as demography, manpower studies, reliability, replacement and maintenance, inventory control, queueing, simulation and so on.

A renewal (counting) process $\{N(t),\ t \geq 0\}$ is a non-negative integer-valued stochastic process that registers the successive occurrences of an event during the time interval (0, t], where time durations between consecutive 'events' are positive, independent, identically distributed, random variables (i.i.d.r.v).

Consider the following examples,

a) The number of bulbs replaced in an industrial campus in an interval of time (0, t) whenever the bulbs fail and denote it by N(t).

b) No. of defectives components of a machine being renewed in the interval (0, t) and denote it by N(t).

In the above examples N(t), represents the occurrence of a particular event on the interval (0, t]. Since N(t) counts the no. of renewals in some interval (0, t), the collection {N(t)} may be called as a counting process. The event of interest under renewal is called Renewal event. For ex. in the above cases fused bulbs, defective components are the events of interest and are called Renewal events.

The inter occurence time between the renewals are all i.i.d r.v's. Let $x_1, x_2, \ldots x_n$ be the time between the renewals where $x_n = t_{n+1} - t_n$, the time between $(n+1)^{th}$ and n^{th} renewals, S_n being the sum of 'n' i.i.d r.v's is also a r.v and the process $\{S_n\}$ is called the Renewal Process.

A renewal process is a stochastic process $\{N(t), t \geq 0\}$ where N(t) registers the no. of renewals happened in the interval (0, t) with the property that the inter occurrence time between the renewals denoted by $x_1, x_2 \cdots x_n$ where x_n is the time between $(n-1)$ and n^{th} renewals are all positive and i.i.d r.v's.

Suppose $P[x_n \leq x] = F(x)$; $n = 1, 2 \cdots$ where it is stipulated for $F(0) = 0$, $F(x) > 0$.

Let $S_n = \sum_{i=1}^{n} x_i$, the time for 'n' renewals. Then by convolution $P[S_n \leq x] = F_n(x)$ where by convention $S_0 = 0$. With the aid of the prescribed function $F_n(x)$ and the link between the process $\{N(t)\}$ & $\{S_n\}$, various r.v's related with them may be established.

5.2 Renewal function:

For the Renewal Process $\{N(t)\}$ where $N(t)$ represents no. of renewals in $(0, t)$ with the renewal time S_n, $n = 1, 2, \cdots$, the distribution function of S_n is

$$P[S_n \leq x] = F_n(x)$$

The expected no. of renewals in $(0, t)$ is given by $E[N(t)] = M(t)$ is called the Renewal function.

An expression for the renewal function may be obtained by making use of the link between $N(t)$ and S_n. They are linked by the relation $\{N(t) \geq n\}$, and $\{S_n \leq t\}$. There are equivalent events and we have equal Probabilities. Therefore,

$$P[N(t) \geq n] = P(S_n \leq t) = F_n(t)$$

Consider $P[N(t)=n] = P[N(t) \geq n] - P[N(t) \geq n+1]$
$= F_n(t) - F_{n+1}(t)$

$M(t) = E[N(t)] = \sum_n n \cdot P[N(t)=n]$
$= \sum_{n=1}^{\infty} n \, [F_n(t) - F_{n+1}(t)]$

$= [F_1(t) - F_2(t)] + 2[F_2(t) - F_3(t)] + 3[F_3(t) - F_4(t)] + \dots$

$= \sum_{n=1}^{\infty} F_n(t)$

which is the expression for renewal function.

Integral equation satisfied by the renewal function:

Statement:

The renewal function

$M(t) = E[N(t)]$
$= \sum_{n=1}^{\infty} F_n(t)$

satisfies an integral of the equation of the form.

$M(t) = F(t) + \int_0^t m(t-x)\, dF(x)$

$= F(t) + F * m(t)$

Proof:

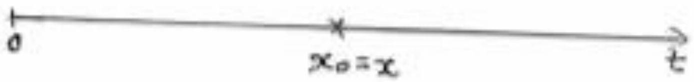
To prove this we use the renewal argument which is renewal invoking a condition on the first renewal x_0 and considering the conditional time expectation under this condition. Thus, we improve.

the condition $x_0 = x$, $x > 0$, then we have

$$E\left[N(t)/_{x_0=x}\right] = \begin{cases} 0 & \text{if } x > t \\ 1+M(t-x) & \text{if } x < t \end{cases} \quad \text{—①}$$

It follows from the argument that

If the $x > t$ makes that no. renewal is in the interval $(0,t)$ and therefore, the expected no. of renewal under this condition is zero.



If $x < t$, it means that there is one renewal in the interval of time 'x' and in the remaining interval of length $(t-x)$, the expected no/s of renewals is $m(t-x)$ and hence the equation ① using the fact that

$$E(x) = E\left[E(x/y)\right]$$

$$M(t) = E\left(N(t)\right)$$

$$= E\left[N(t)/x_0 = x\right]$$

$$= \int_0^t E\left\{N(t)/_{x_0=x}\right\} dF(x)$$

$$= \int_0^t \left\{1+m(t-x)\right\} dF(x)$$

$\cdot F(t) + F * m(t)$

5.3: Elementary Renewal Theorem:

For a renewal process $N(t)$ with renewal function

$$m(t) = E[N(t)] = \sum_{k=1}^{\infty} F_k(t)$$

$$\lim_{t\to\infty} \frac{M(t)}{t} = \frac{1}{\mu} \quad \text{where } \mu = E(X_1)$$

Proof:

Since $S_{N(t)} + 1 = X_1 + X_2 + \dots + X_{N(t)} + 1$

It clearly follows that

$$t \le S_{N(t)} + 1$$
$$\le E[S_{N(t)} + 1]$$
$$\le E(X_1)[1 + m(t)]$$
$$\le \mu[1 + M(t)]$$
$$1 + M(t) \ge t/\mu$$
$$M(t) \ge \frac{t}{\mu} - 1 \Rightarrow \frac{M(t)}{t} \ge \frac{1}{\mu} - \frac{1}{t}$$

$$\lim_{t\to\infty} \frac{M(t)}{t} \ge \frac{1}{\mu} \longrightarrow (1)$$

To establish the other inequality let for any $c > 0$

$$X_i^c = \begin{cases} X_i & c > X_i \\ c & c < X_i \end{cases}$$

Also let the process response to the renewal time X_i^c is $N^c(t)$

The renewal function $M(t)$ and the time for $N(t)+1$ renewals in $S^c_{N(t)}+1$ and it follows that $t+c \geq S^c_{N(t)}+1$.

Taking expectation.

$$t+c \geq E\left(S^c_{N(t)}+1\right)$$

$$\geq \mu^c \left[1+M^c(t)\right]$$

$$1+M^c(t) \leq \frac{t}{\mu^c}+\frac{c}{\mu^c}$$

$$\frac{M^c(t)}{t} \leq \frac{1}{\mu^c}+\frac{c}{t\mu^c}-\frac{1}{t}.$$

$$\lim_{t\to\infty} \frac{M(t)}{t} \leq \frac{1}{\mu} \longrightarrow ③$$ comparing ① & ③

$$\lim_{t\to\infty} \frac{M(t)}{t} = \frac{1}{\mu} \quad \text{where } \mu = E(X_1)$$

5.4
Stationary process:

If certain prob. distns do not depend on 't', then the r.p $\{X(t)\}$ is called stationary process.

Strict sense and stationary process (SSS) (or) strongly stationary process:

Definition:

A r.p is called SSS process if all its finite dimensional distns are invariant under transition of the time parameter that is If the joint probability density of $X(t_1), X(t_2), \ldots, X(t_n)$ is the same as that of $X(t_1+h), X(t_2+h), \ldots, X(t_n+h)$ for all $t_1, t_2 \ldots t_n$ and $h>0$ and $\forall n \geq 1$.

If the above condition holds good for a particular 'n', the process is called stationary of order 'n'. If the process is stationary of order 'n' for any integer 'n', then it is strongly stationary.

1st order stationary:

A r.p. $\{x(t)\}$ is said to be 1st order stationary process if $E\{x(t)\} = \mu$ is a constant.

2nd order stationary.

A r.p. $\{x(t)\}$ is said to be 2nd order stationary. If $f(x_1, x_2, t_1, t_2) = f(x_1, x_2, t_1+h, t_2+h)$ for any h (ie) the 2nd order density must be invariant under translation of time. (ie) the joint density of $x(t)$ and $x(t+\tau)$ depends only on time difference 'τ' and not on 't'. $\tau = t_2 - t_1$.

Wide Sence Stationary process: (WSS) (or) Weekly stationary process.

Definition:

A r.p. $\{x(t)\}$ is called WSS process, if its mean is constant and its auto-correlation function depends only on the time difference, (τ).

Conditions:-

i) $E\{x(t)\}$ is always const

ii) $E(x(t_1)) \cdot E(x(t_2)) = R_{xx}(\tau)$ where $\tau = t_2 - t_1$.

(or) $E[x(t)] \cdot E[x(t+\tau)] = R_{xx}(\tau)$

Note:

A r.p. $\{x(t)\}$ which is not stationary any sence is called an evalutionary process.

Problem-1:

Show that the r.p $X(t) = A \cdot \cos(\omega t + \theta)$ is not a stantionary where 'A' and 'ω' are constants and θ is uniformaly distibute r.v in $(0, \pi)$.

Sol:

Mean: $E[X(t)]$

$$= \int_0^{\pi} X(t) \cdot f(\theta)\, d\theta.$$

$$= \int_0^{\pi} A \cos(\omega t + \theta) \frac{1}{\pi}\, d\theta$$

$$= \frac{A}{\pi} \left[\sin(\omega t + \theta)\right]_{\theta=0}^{\pi}$$

$$= \frac{A}{\pi} \left[\left(\sin \omega t + \pi\right) - \sin(\omega t)\right]$$

$$\frac{1}{\pi} \left[-\sin(\omega t) - \sin(\omega t)\right]$$

$$= \frac{-2 \sin(\omega t)}{\pi}$$

which is not a constant.

The given random process is not stationary.

Problem-2:

Show that the r.p $x(t) = A \cdot \sin(\omega t + \theta)$ is WSS where A and ω are constants and θ is uniformly distributed r.v. in $(0, 2\pi)$

$x(t) = A \sin(\omega t + \theta)$

Since θ is uniformly distributed in $(0, 2\pi)$ the density of θ is given by

$$f_x(x) = f(\theta) = \frac{1}{2\pi} \quad 0 < \theta < 2\pi$$

$$\text{Mean} = \int_0^{2\pi} x(t) f(\theta) d\theta$$

$$= \int_0^{2\pi} A \sin(\omega t + \theta) \frac{1}{2\pi} d\theta$$

$$= \frac{A}{2\pi} \left[-\cos(\omega t + \theta)\right]_0^{2\pi}$$

$$= \frac{A}{2\pi} - \left[\cos(\omega t + 2\pi) - \cos(\omega t)\right]$$

$$= \frac{A}{2\pi} \left[\cos(\omega t) - \cos(\omega t)\right]$$

$$= 0$$

which is constant.

Auto correlation $= E[x(t) \cdot x(t + \tau)]$

$= E[A \sin(\omega t + \theta) \cdot A \sin(\omega(t + \tau) + \theta)]$

$= A^2 E[\sin(\omega t + \theta) \sin(\omega t + \omega\tau + \theta)]$

We know that
$$\sin A \cdot \sin B = \frac{\cos(A+B) - \cos(A-B)}{2}$$

$$\Rightarrow \sin(\omega t+\theta)\sin(\omega t+\omega\tau+\theta) = \frac{\cos(2\omega t+2\theta+\omega\tau)}{2} - \left(\frac{\cos(\omega\tau)}{2}\right)$$

$$\text{Auto correlation} = \frac{A^2}{2}\left[\int_0^{2\pi} \frac{\cos(2\omega t+2\theta+\omega\tau) - \cos(\omega\tau)}{2}\right]$$

$$= \frac{A^2}{2}\left[\frac{\sin(2\omega t+2\theta+\omega\tau)}{2\pi\cdot 2}\right]_0^{2\pi} - \frac{A^2}{2}\cos(\omega\tau)$$

$$= \frac{A^2}{2}\left[\frac{\sin(2\omega t+2(2\pi)+\omega\tau)}{2\cdot 2\pi} - \frac{\sin(2\omega t+\omega\tau)}{2\cdot 2\pi}\right]$$

$$- \frac{A^2}{2}\cos(\omega\tau)$$

$$= \frac{A^2}{2}\left[\frac{\sin(\omega t+\omega\tau) - \sin(2\omega t+\omega\tau)}{2\cdot 2\pi}\right]$$

$$- \frac{A^2}{2}\cos(\omega\tau)$$

$$= 0 - A^2/2 \cos(\omega\tau)$$

$$\text{Auto correlation} = -A^2/2 \cos(\omega\tau)$$

which dependents on τ.

Since, mean of the process is constant and its Auto-correlation depends on τ only the given random process is a WSS process

Reference:

Medhi. J: Stochastic Processes, Wiley Eastern limited, New Delhi, Second Edition, 1994.

Renewal Function and Renewal Density

The function $M(t) = E\{N(t)\}$ is called the *renewal function* of the process with distribution F. It is clear that

$$\{N(t) \geq n\} \Leftrightarrow \{S_n \leq t\} \tag{2}$$

or $\{N(t) \geq n\}$ if and only if $\{S_n \leq t\}$.

Equivalently $\{N(t) < n\} \Leftrightarrow \{S_n > t\}$.

Theorem The distribution of $N(t)$ is given by

$$p_n(t) = \Pr\{N(t) = n\} = F_n(t) - F_{n+1}(t) \tag{3}$$

and the expected number of renewals by

$$M(t) = \sum_{n=1}^{\infty} F_n(t) \tag{4}$$

Proof: We have

$$\begin{aligned}\Pr\{N(t) = n\} &= \Pr\{N(t) \geq n\} - \Pr\{N(t) \geq n+1\} \\ &= \Pr\{S_n \leq t\} - \Pr\{S_{n+1} \leq t\} \\ &= F_n(t) - F_{n+1}(t).\end{aligned}$$

Again,

$$\begin{aligned}M(t) = E\{N(t)\} &= \sum_{n=0}^{\infty} n\, p_n(t) \\ &= \sum_{n=0}^{\infty} n\{F_n(t) - F_{n+1}(t)\} \\ &= \sum_{n=1}^{\infty} F_n(t) \\ &= \sum_{n=1}^{\infty} \Pr\{S_n \leq t\}. \text{ Hence proved.}\end{aligned}$$

The relation (4) can be put in terms of Laplace transform as follows:

Let $F'(x) = f(x)$ be the density function of (p.d.f.) of K_n and $g^*(s)$ denote the Laplace transform of a function $g(t)$. Then taking Laplace transform of both sides of (4), we get

$$M^*(s) = \sum_{n=1}^{\infty} F_n^*(s) = \frac{1}{s}\sum_{n=1}^{\infty} f_n^*(s)$$

$$= \frac{1}{s}\sum_{n=1}^{\infty}\left[f^*(s)\right]^n = \frac{f^*(s)}{s\left[1-f^*(s)\right]}. \qquad (5)$$

This is equivalent to

$$f^*(s) = \frac{sM^*(s)}{1+sM^*(s)}. \qquad (.6)$$

These show that $M(t)$ and $F(x)$ can be determined *uniquely* one from the other.

Note that $M(t) = E\{N(t)\}$ *is a sure* function and *not* a random function or stochastic process.

Renewal Density

The derivative $m(t)$ of $M(t)$ (*i.e.* $M'(t) = m(t)$) is called the *renewal density*. We have

$$m(t) = \lim_{\Delta t \to 0} \frac{\Pr\{\text{one or more renewals in } (t, t+\Delta t)\}}{\Delta t}$$

$$= \sum_{n=1}^{\infty} \lim_{\Delta t \to 0} \frac{\Pr\{n \text{ the renewal occurs in } (t, t+\Delta t)\}}{\Delta t}$$

$$= \sum_{n=1}^{\infty} \lim_{\Delta t \to 0} \frac{f_n(t)\Delta t + o(\Delta t)}{\Delta t}$$

(assuming that $F(x)$ is absolutely continuous and $F_n'(t) = f_n(t)$)

$$= \sum_{n=1}^{\infty} f_n(t) = \sum_{n=1}^{\infty} F_n'(t)$$

$$= M'(t).$$

The function $m(t)$ specifies the mean number of renewals to be expected in a narrow interval near t.

Note that $m(t)$ is *not* a probability density function. As $m^*(s) = L.T.\ \{m(t)\} = s\,M^*(s)$, it follows from (5) that

$$m^*(s) = \frac{f^*(s)}{1-f^*(s)}.$$

Example . Let X_n have gamma distribution having density

$$f(x) = \frac{a^k x^{k-1} e^{-ax}}{(k-1)!}, \quad x \geq 0.$$
$$= 0, \text{ elsewhere.}$$

Then,

$$f^*(s) = \left(\frac{a}{s+a}\right)^k$$

and the density $F_n'(x)$ of

$$S_n = X_1 + X_2 + \cdots + X_n$$

has the L.T.

$$\left(\frac{a}{s+a}\right)^{nk}$$

Thus

$$F_n'(x) = \frac{a^{nk} x^{nk-1} e^{-ax}}{(nk-1)!}$$

and hence

$$F_n(x) = \int_0^x F_n'(y)\,dy$$

$$= 1 - e^{-ax} \sum_{r=0}^{nk-1} \frac{(ax)^r}{r!}, \qquad n \geq 1.$$

Thus

$$p_n(t) = F_n(t) - F_{n+1}(t)$$

$$= e^{-at} \sum_{r=nk}^{(n+1)k-1} \frac{(ax)^r}{r!},$$

and using (.5), we get

$$M^*(s) = \frac{a^k}{s\left[(s+a)^k - a^k\right]}. \qquad (\ 7)$$

Example *Hyper-exponential distribution.* Let X_n have density

$$f(t) = pae^{-at} + (1-p)be^{-bt},\ 0 \le p \le 1,$$
$$a > b > 0. \qquad (.8)$$

Such a model may be used to describe a system which has two kinds of components – a proportion 'p' of components having, say, a high failure rate a and the remaining proportion $(1-p)$ of components having a different, say, a lower failure rate b. We have

$$f^*(s) = \frac{pa}{s+a} + \frac{(1-p)b}{s+b} \qquad (.9)$$

so that

$$M^*(s) = \frac{ab + s\left[pa + (1-p)b\right]}{s^2\left[s + (1-p)a + pb\right]}$$

Writing $A = pa + (1-p)b$ and $B = (1-p)a + pb$, we get

$$M^*(s) = \frac{As + ab}{s^2(s+B)}$$

$$= \frac{A}{s(s+b)} + \frac{ab}{s^2(s+B)}$$

$$= \frac{A}{B}\left\{\frac{1}{s} - \frac{1}{s+B}\right\} + \frac{ab}{B}\left[\frac{1}{s^2} - \left\{\frac{1}{s} - \frac{1}{s+B}\right\}\frac{1}{B}\right].$$

Inverting the L.T., we get

$$M(t) = \frac{A}{B}\left(1 - e^{-Bt}\right) + \frac{ab}{B}\left\{t - \frac{1-e^{-Bt}}{B}\right\}$$

$$=\frac{abt}{B}+C\left(1-e^{-Bt}\right), \tag{. 10}$$

where

$$C=\left(\frac{A}{B}-\frac{ab}{B^2}\right)$$

$$=\frac{p(1-p)(a-b)^2}{B^2}\geq 0.$$

Markovian case: When p = 1 (or p = 0) the distribution of X_n reduces to negative exponential and then $C = 0$, *i.e.* the second term of (10) vanishes, so that we get $M(t) = at$ (or bt).

RENEWAL EQUATION

An integral equation can be obtained for the renewal function

$$M(t) = E\{N(t)\},$$

which gives the expected number of renewals in $[0, t]$.

Theorem The renewal function M satisfies the equation

$$M(t)=F(t)+\int_0^t M(t-x)\,dF(x). \tag{ 1}$$

Proof: By conditioning on the duration of the first renewal X_1, we get

$$M(t)=E\{N(t)\}=\int_0^\infty E\{N(t)\mid X_1=x\}\,dF(x).$$

Consider $x > t$, given that $X_1 = x > t$, no renewal occurs in $[0, t]$, so that

$$E\{N(t)\mid X_1=x\}=0.$$

Consider $0 \leq x \leq t$; given that the first renewal occurs at x ($\leq t$), then the process starts again at epoch x, and the expected number of renewals in the remaining interval of length $(t-x)$ is $E[N(t-x)]$, so that

$$E\{N(t)\mid X_1=x\}=1+E\{N(t-x)\}$$

$$=1+M(t-x).$$

Thus, considering the above two equations, we get

$$M(t)=\int_0^t \{1+M(t-x)\}\,dF(x)$$

$$=F(t)+\int_0^t M(t-x)\,dF(x).$$

We have thus established (1).

The equation (. 1) is called the *integral equation of renewal theory* (or simply *renewal equation*) and the argument used to derive it is known as *'renewal argument'*. The renewal equation is also expressed as

$$M = F + M^{*} f.$$

The equation (. 1) can also be established as given below.

We have
$$M(t) = \sum_{n=1}^{\infty} F_n(t) = F_1(t) + \sum_{n=1}^{\infty} F_{n+1}(t)$$

$$= F(t) + \sum_{n=1}^{\infty} \left\{ \int_0^t F_n(t-x)\, dF(x) \right\},$$

F_{n+1}, being the convolution of F_n and $F_1 = F$. Thus, assuming the validity of the change of order of integration and summation, we get

$$M(t) = F(t) + \int_0^t \left\{ \sum_{n=1}^{\infty} F_n(t-x) \right\} dF(x);$$

or,
$$M(t) = F(t) + \int_0^t M(t-x)\, dF(x).$$

It follows that $M(t) = \sum_{n=1}^{\infty} F_n(t)$ satisfies the integral equation (.1). The renewal equation (. 1) can be generalised as follows

$$v(t) = g(t) + \int_0^t v(t-x)\, dF(x),\ t \geq 0, \qquad (.\ 2)$$

where g and F are known and v is unknown. The equation (. 2) is called *a renewal type equation*.

5.5 Basic ideas of Time Series

We shall discuss here only the models, where the random part involves stationary stochastic processes (real-valued). Here, we shall also confine mostly to processes in discrete time. A time series $\{x_i, t = 1,..., m\}$ may be considered as a (single) particular realisation of the stationary stochastic process $\{X_t, t \in I\}$. We shall be concerned here with the stationary process $\{X_t, t \in I\}$ rather than with the analysis of the time series $\{x_t, t = 1,..., m\}$.

We consider here a stochastic process $\{X_t, t \in I\}$ which is wide sense or covariance stationary; then (*i*) the mean value function $E\{X_t\} = m$ is independent of t, and (*ii*) the covariance function $E[(X_t - m)(X_{t+k} - m)] = C_k$ is a function of the difference $(t + k) - t = k$. The correlation function is $\rho_k = C_k/C_0$, where $C_0 = \text{var}(X_t)$. The graph of the correlation function ρ_k, $k = 0, 1, 2,...$ is called the *correlogram* of the stochastic process. Considered below are some stationary processes which are used as models for the generation of the random part of a time series.

MODELS OF 'IME SERIES

1 Purely Ra dom Process (or White Noise Process)

A completely rand m process $\{X_t\}$ has

$$E\{X_t\} = m, \qquad (\ 1)$$

which may be ass ned to be 0, for simplicity, and

$$E\{X_t X_{t+k}\} = \begin{cases} \sigma^2, & k = 0 \\ 0, & k \neq 0 \end{cases} \qquad (\ 2)$$

The process is covariance stationary.

2 First Order Markov Process

$\{X_t\}$ has the structure

$$X_t + \alpha_1 X_{t-1} = e_t, \quad |\alpha_1| < 1, \qquad (\ 3)$$

where $\{e_t\}$ is purely random process, say, with $m = 0$, $\sigma = 1$. Further e_{t+1}, is not involved in $X_t, X_{t-1},...,$ *i.e.*, X_t depends on $e_1, e_2,...e_t$, but is independent of $e_{t+1}, e_{t+2},...$

Multiplying both sides of (3) by X_{t-1} and taking expectation, we get

$$C_1 + \alpha_1 C_0 = 0,$$

or, $$\rho_1 + \alpha_1 = 0, \text{ i.e., } \rho_1 = -\alpha_1 = a \text{ (say)}, \ |a| < 1.$$

Proceeding in the same way, we get

$$\rho_2 + \alpha_1 \rho_1 = 0, \quad \text{i.e., } \rho_2 = (-\alpha_1)^2 = a^2$$
$$\cdots \qquad \cdots \qquad \cdots$$
$$\rho_n + \alpha_1 \rho_{n-1} = 0, \quad \text{i.e., } \rho_n = (-\alpha_1)^n = a^n$$
$$\cdots \qquad \cdots \qquad \cdots$$

The process is covariance stationary. The representation (. 3) is a linear stochastic difference equation; X_t can be expressed as a formal solution of the difference equation as follows.

Denote the backward shift operator by B, i.e. $BX_t = X_{t-1}$.

Then $$B^2 X_t = X_{t-2} \text{ and } B^k X_t = X_{t-k}, \ k = 1, 2, 3, \ldots$$

Thus the difference equation can be written as

$$(1 + \alpha_1 B) X_t = e_t$$

or $$X_t = (1 + \alpha_1 B)^{-1} e_t = \sum_{k=0}^{\infty} \left\{ (-\alpha_1)^k B^k \right\} e_t$$

$$= \sum_{k=0}^{\infty} \rho_1^k e_{t-k}. \qquad (. 4)$$

3 Moving Average (MA) Process

Here $\{X_t\}$ is represented as

$$X_t = a_0 e_t + a_1 e_{t-1} + \cdots + a_h e_{t-h}, \qquad (. 5)$$

where the a's are real constants and $\{e_t\}$ is a purely random process with mean 0 and variane σ^2. X_t can also be represented as

$$X_t = f(B) e_t, \quad \text{where } f(B) = \sum_{r=0}^{h} a_r B^r;$$

when $a_h \neq 0$, $\{X_t, t \in T\}$ is called an MA process of order h.

4 Autoregressive Process (AR Process)

The process $\{X_t\}$ given by

$$X_t + b_1 X_{t-1} + b_2 X_{t-2} + \cdots + b_h X_{t-h} = e_t, \quad b_h \neq 0, \tag{ .8}$$

where $\{e_t\}$ is a purely random process, with mean 0, is called an *autoregressive process of order h.* X_t can be obtained as a solution of the linear stochastic difference equation

$$g(B)X_t = e_t, \tag{ 9}$$

where
$$g(B) = \sum_{r=0}^{h} b_r B^r,\ b_0 = 1.$$

Suppose that $g(B) = \Pi(1 - z_i B)$, $z_i \neq z_j$, *i.e.*, $z_1^{-1}, \ldots, z_h^{-1}$ are the distinct roots of the equation $g(z) = 0$. Further suppose that $|z_i| < 1$ for all i, *i.e.*, all the roots of $g(z) = 0$ lie outside the unit circle; the roots z_i of the characteristic equation $f(z) \equiv \sum_{r=0}^{h} b_r z^{h-r} = 0$ (where $f(z) = z^{-h} g(z^{-1})$) all lie within the unit circle. The complete solution of (2.9) can be written as

$$X_t = \sum_{r=1}^{h} A_r z_r^t + \frac{1}{g(B)} e_t,$$

where A_r's are constants. Now

$$\frac{1}{g(B)} e_t = \prod_{i=1}^{h} (1 - z_i B)^{-1} e_t$$

5 Autoregressive Process of Order Two (Yule Process)

This is given by

$$X_t + b_1 X_{t-1} + b_2 X_{t-2} = e_t \tag{ 15}$$

where $\{e_t\}$ is a purely random process with mean 0.

Assume that the roots z_1, z_2 of the characteristic equation

$$z^2 + b_1 z + b_2 = 0$$

are distinct, and that both the roots lie within the unit circle, *i.e.* $b_2 < 1$, and $4b_2 > b_1^2$.

5.6 Auto-regressive and moving average processes.

6 Autoregressive Moving Average Process (ARMA Process)

Consider a process having the representation

$$\sum_{r=0}^{p} b_r X_{t-r} = \sum_{s=0}^{q} a_s e_{t-s} \quad (b_0 = 1), \qquad (21)$$

where $\{e_t\}$ is a purely random process with mean 0, and the roots of $f(z) = \sum_{r=0}^{p} b_r z^{p-r} = 0$ all lie within the unit circle. Such a process is called an *auto-regressive moving average process* and is denoted by ARMA (p, q). We can write (21) as $g(B)\, X_t = h(B)\, e_t$, where $g(B)$ and $h(B)$ are polynomials in B of degrees p and q respectively. Then X_t can be represented as an infinite moving average $X_t = \sum v_r e_{t-r}$ as follows:

Put $u_t = \sum_{s=0}^{q} a_s e_{t-s}$ and let the roots z_i, $i = 1, 2, \ldots, p$ of the equation $f(z) = 0$ be distinct.

Then

$$X_t = \sum_{r=1}^{p} A_r z_r^t + \prod_{i=1}^{q} (1 - z_i B)^{-1} u_t.$$

Now as $|z_i| < 1$, we have for large t (neglecting the terms $\sum A_r z_r^t$)

$$X_t = \prod_{i=1}^{p} (1 - z_1 B)^{-1} u_t = \sum_{r=0}^{\infty} b_r' u_{t-r} = \sum_{r=0}^{\infty} b_r' \left(\sum_{r=0}^{q} a_s e_{t-r-s} \right)$$

$$= \sum_{r=0}^{\infty} v_r e_{t-r} \ (\text{say}). \qquad (22)$$

The v_r's, $r = 0, 1, 2, \ldots$ in (22) can be determined by substituting the above expression for X_t in the l.h.s. of (21) and then comparing the coefficients of e_{t-r} from both sides.

We have $E\{X_t\} = 0$. Multiplying (21) X_{t-k} $(k \geq 0)$ and taking expectations, we get

$$g(B)\rho_k = \sum_{r=0}^{p} b_r \rho_{k-r} = 0, \quad k > q$$

whence

$$\rho_k = \sum_{r=1}^{p} \alpha_r z_r^k, \qquad k > q, \qquad (23)$$

α_r being constants.