Unit IV	Topic
	4.1 Branching Process (one dimension only) 4.2 Generating Functions 4.3 Properties of Generating Functions 4.4 Concept of Weiner Process 4.5 Weiner Process as a limit of random walk.

4.1 Branching Process (one dimension only)

In real life situations, it sometimes happens that although a suitable Markov Chain can be formed, it may be difficult to define transition probabilities.

Branching Processes are example of Markov chains. For example, consider the organisms that produce offspring. Suppose an organism at the end of its lifetime produces a random number ' ξ ' of offspring with probability distribution,

$$Pr\{\xi = k\} = p_k,\ k = 0, 1, 2, \ldots \qquad \ldots\ldots\ldots\ldots(1)$$

where $p_k \geq 0$ and $\sum_{k=0}^{\infty} p_k = 1$.

Let us assume that all the offspring act independently of each other and at the end of their lifetime, individually have progeny in accordance with the probability distribution given in (1).

The process $\{X_n\}$ where X_n is the population size at the n^{th} generation, is a Markov Chain. Then the process $\{X_n\}$ is called **Discrete Time Branching Process.**

The only knowledge regarding the distribution of $X_{n1}, X_{n2}, \ldots\ldots X_{nr}, \ldots\ldots X_n$, $n_1 < n_2 < n_3 \ldots\ldots < n_r$, is the last known population count, since the number of the offspring is a function merely of the present population size. The transition matrix is given by

$$P_{ij} = Pr\{ X_{n+1} = j / X_n = i \}$$
$$= Pr\{ \xi_1 + \xi_2 + \ldots\ldots + \xi_i = j \}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where ξ's follow independent probability distribution given in (1).

In the n^{th} generation, the 'i' individuals independently give rise to number of offspring $\{\xi_k ; k = 1, 2, \ldots.\ i\}$ and hence the cumulative number produced is $\xi_1 + \xi_2 + \ldots\ldots + \xi_i$.

Then the Generating Function (GF) of $\xi_1 + \xi_2 + \ldots\ldots + \xi_i$ is $[\varphi(s)]^i$. Hence, P_{ij} is the j^{th} coefficient in the power series expansion of $[\varphi(s)]^i$.

Consider another example of Electron Multipliers. An electronic multiplier is a device amplifies a weak current of electron. A series of plates are set in the path of electrons by source. Each electron, as it strikes the 1st plate, generates a random number of new electrons, which in turn strike the next plate and produce more electrons etc., Let X_0 be the number electrons initially emitted, X_1 be the number electrons emitted on the 1st plate by the impact due to the X_0 initial electrons; in general X_n be the number electrons emitted on the nth plate due to the electrons emanating from the *(n-1)th* plate.

The sequence of random variables $\{X_i, i = 0, 1, 2, ..., n\}$ constitutes a branching process.

4.2 Generating Function

The Generating Functions are extremely useful in the study of branching processes.

We will develop some relations for the probability generating functions of the X_n. Assume first that the initial population consists of one individual i.e., assume $X_0 = 1$. Clearly we can write for every $n = 0, 1, 2, \ldots$

$$X_{n+1} = \sum_{r=1}^{X_n} \xi_r,$$

where ξ_r $(r \geq 1)$ are independently identically distributed random variables with distribution

$$\Pr\{\xi_r = k\} = p_k, \qquad k = 0, 1, 2, \ldots, \qquad \sum_{k=0}^{\infty} p_k = 1.$$

We introduce the probability generating function

$$\varphi(s) = \sum_{k=0}^{\infty} p_k s^k$$

and

$$\varphi_n(s) = \sum_{k=0}^{\infty} \Pr\{X_n = k\} s^k, \qquad \text{for} \quad n = 0, 1, 2, \ldots.$$

Manifestly,

$$\varphi_0(s) \equiv s \qquad \text{and} \qquad \varphi_1(s) = \varphi(s).$$

Further,

$$\varphi_{n+1}(s) = \sum_{k=0}^{\infty} \Pr\{X_{n+1} = k\} s^k$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \Pr\{X_{n+1} = k | X_n = j\} \Pr\{X_n = j\} s^k$$

$$= \sum_{k=0}^{\infty} s^k \sum_{j=0}^{\infty} \Pr\{X_n = j\} \cdot \Pr\{\xi_1 + \cdots + \xi_j = k\}$$

$$= \sum_{j=0}^{\infty} \Pr\{X_n = j\} \cdot \sum_{k=0}^{\infty} \Pr\{\xi_1 + \cdots + \xi_j = k\} s^k. \qquad (2.1)$$

Since ξ_r $(r = 1, 2, \ldots, j)$ are independent, identically distributed random variables with common probability generating function $\varphi(s)$, the sum $\xi_1 + \cdots + \xi_j$ has the probability generating function $[\varphi(s)]^j$. Thus,

$$\varphi_{n+1}(s) = \sum_{j=0}^{\infty} \Pr\{X_n = j\} [\varphi(s)]^j.$$

But the right-hand side is just the generating function $\varphi_n(\cdot)$ evaluated at $\varphi(s)$. Thus,

$$\varphi_{n+1}(s) = \varphi_n(\varphi(s)). \tag{2.2}$$

Iterating this relation we obtain

$$\varphi_{n+1}(s) = \varphi_n(\varphi(s)) = \varphi_{n-1}(\varphi(\varphi(s))) = \varphi_{n-1}(\varphi_2(s)) = \varphi_{n-2}(\varphi_2(\varphi(s))) = \varphi_{n-2}(\varphi_3(s)).$$

It follows, by induction, that for any $k = 0, 1, \ldots, n$

$$\varphi_{n+1}(s) = \varphi_{n-k}(\varphi_{k+1}(s)).$$

In particular, with $k = n - 1$,

$$\varphi_{n+1}(s) = \varphi(\varphi_n(s)). \tag{2.3}$$

If instead of $X_0 = 1$ we assume $X_0 = i_0$ (constant), then

$$\varphi_0(s) \equiv s^{i_0} \quad \text{and} \quad \varphi_1(s) = [\varphi(s)]^{i_0}$$

because

$$X_1 = \sum_{j=1}^{i_0} \xi_j.$$

We still have

$$\varphi_{n+1}(s) = \varphi_n(\varphi(s))$$

Let us consider the branching process $\{X_n\}$ where X_n is the population size at the n^{th} generation with random number ξ of offspring which are all independent with probability distribution,

$\Pr\{\xi = k\} = p_k$, $k = 0, 1, 2, \ldots\ldots$ ....(1)

where $p_k \geq 0$ and $p_k = 1$. Also assume that $X_0 = 1$.

It is known that $$\varphi_n(s) = \sum_{k=0}^{\infty} \Pr\{X_n = k\} s^k, \quad \text{for} \quad n = 0, 1, 2, \ldots.$$

and the generating function such that

$\varphi_1(s) = \varphi(s)$ ....(1)

$\varphi_{n+1}(s) = \varphi[\varphi_n(s)]$

$\varphi_{n+1}(s) = \varphi_n[\varphi(s)]$ ....(2)

Let $m = EX_1$ and $\sigma^2 = \operatorname{Var} X_1 = E(X_1^2) - [E(X_1)]^2$ exist.

(ie) m = the expected no of off springs per individual.

To find $E(x_n)$, first we note that

$E(x_n) = Q_n(1)$

$[Q_n(s) = \sum_k Pr[x_n = k] s^k]$

$E(x_1) = Q_1'(1) = Q'(1) = m$

Differentiating (5) w.r to 's' and taking s=1, we have,

$Q_{n+1}'(s) = Q_n'[Q(s)] \cdot Q'(s)$ s=1

$Q_{n+1}'(1) = Q_n'(Q(1)) \cdot [Q'(1)]$

$= Q_n'(1) m$

$= m \cdot Q_n'(1)$

which is in the form a recurrence relation. By repeatedly substituting we have,

$Q_{n+1}'(1) = m[m\, Q_{n-1}'(Q(1))]$

$= m^2 [Q_{n-1}'(1)]$

$= m^n Q'(1)$

$= m^{n+1}$

(ie) $E[x_{n+1}] = m^{n+1}$

$\therefore E[x_n] = m^n$

Variance:

$V(x_{n+1}) = E(x_{n+1}^2) - [E(x_{n+1})]^2$

$= E(x_{n+1}^2) - (m^{n+1})^2$

Consider $E(X_{n+1}^2)$:

$Q_n(s) = \sum_k P[X_n = k] \cdot s^k$

$Q_n'(s) = \sum_k [P[X_n = k] \cdot k \cdot s^{k-1}]_{s=1}$

$= \sum_k k \cdot P[X_n = k]$

$= E(X_n)$

$Q_n''(s) = \sum_k [P[X_n = k] \cdot k(k-1) s^{k-2}]_{s=1}$

$= \sum k^2 P(X_n = k) - \sum k \cdot P(X_n = k)$

$= E(X_n^2) - E(X_n)$

$\therefore E(X_n^2) = Q_n''(1) + E(X_n)$

$\Rightarrow E(X_{n+1}^2) = Q_{n+1}''(1) + m^{n+1}$

$[\because E(X_{n+1}) = m^{n+1}]$

$V(X_{n+1}) = Q_{n+1}''(1) + m^{n+1} - (m^{n+1})^2$

Now, to find $Q_{n+1}''(1)$, differentiate eqn (2) twice with respect to 's' and taking $s=1$.

$Q_{n+1}'(s) = Q_n'(Q(s)) \cdot Q'(s)$

$Q_{n+1}''(s) = Q_n'[Q(s)]\, Q'(s)$

$Q_{n+1}''(s) = Q_n''\{Q(s)\}[Q'(s)]^2 + Q_n'[Q(s)], Q''(s)]_{s=1}$

$Q_{n+1}''(s) = Q_n''[Q_1(s)]^2 + Q'(n) \ldots [Q''(1)]$.

$Q_{n+1}''(s) = Q_n''(1) m^2 + Q_n'(1) Q''(1)$ ——— (3)

consider, $Q(s) = \sum P_k s^k$

$\Rightarrow Q'(s) = [\sum P_k \cdot k s^{k-1}]_{s=1}$ ——— (4)

$\Rightarrow Q'(1) = \sum k \cdot P_k = m$

Eqn (4) $\rightarrow$ $Q''(s) = [\sum_k P_k \cdot k(k-1) s^{k-2}]_{s=1}$.

$$\Rightarrow Q''(1) = \sum k^2 P_k - \sum k P_k$$

$$= E(X_1^2) - m$$

$$V(X_1) = E(X_1^2) - [E(X_1)]^2$$

$$E(X_1^2) = V(X_1) + [E(X_1)]^2$$

$$= \sigma^2 + m^2$$

$$\Rightarrow Q''(1) = \sigma^2 + m^2 - m = M \text{ (say)}$$

Eqn (3) $\Rightarrow$

$$Q''_{n+1}(1) = Q''_n(1)\, m^2 + m^n(\sigma^2 + m^2 - m) \qquad \left[\because Q'_n(s) = E(X_n) = m^n\right]$$

$$= M m^n + m^2 Q''_n(1)$$

$$= Q''_n(1) m^2 + M m^n$$

Repeadtlly substituting for $Q'_n(1)$ we have

$$= M(m^n + m^{n+1} + \cdots + m^{2n})$$

$$V(X_{n+1}) = M(m^n + m^{n+1} + \cdots + m^{2n}) + m^{n+1} - m^{2(n+1)}$$

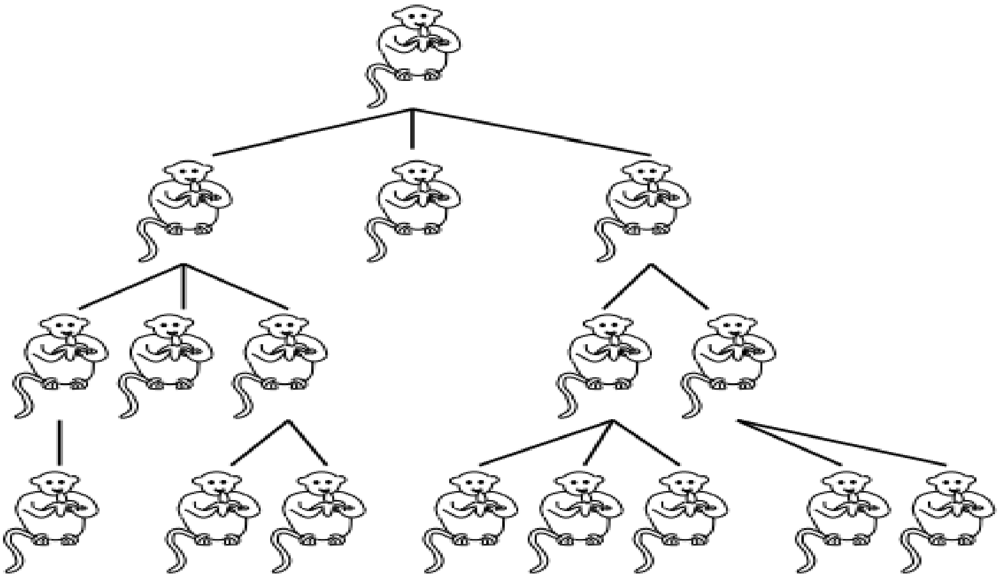
$$= \sigma^2 m^n (1 + m + m^2 + \cdots + m^n)$$

$$= \sigma^2 (n+1); \quad \text{when } m = 1$$

$$= \sigma^2 m^n \left(\frac{1 - m^{n+1}}{1 - m}\right); \quad \text{when } m \neq 1$$

Variance: $V(X_n) = \sigma^2 m^{n-1}\left(\frac{1 - m^n}{1 - m}\right); \quad \text{when } m \neq 1$

$$= \sigma^2 n; \quad \text{when } m = 1.$$



4.3 Properties of Generating Functions

1) There is a one-to-one relation between the pmf and the gf:

$$p_k = \frac{1}{k!} \left. \frac{d^k \phi(s)}{ds^k} \right|_{s=0}$$

2) If $\xi_1, \ldots, \xi_n$ are independent random variables with generating functions $\phi_1(s), \ldots, \phi_n(s)$, and we let $X = \xi_1 + \cdots + \xi_n$, then

$$\phi_X(s) = \phi_1(s) \cdot \phi_2(s) \cdots \phi_n(s).$$

3) The moments of ξ can be computed by differentiating the g.f., for example the first moment is given by

$$\mathrm{E}[\xi] = p_1 + 2p_2 + 3p_3 + \cdots = \left. \frac{d\phi(s)}{ds} \right|_{s=1}$$

4) The p.g.f. φ(s) is continuous, non-decreasing and convex on [0,1].

5) $\varphi^{'}(1) = \left[\frac{d\varphi(s)}{ds}\right]_{s=1} = E(X).$

6) $\varphi^{''}(1) = \left[\frac{d^2\varphi(s)}{ds^2}\right]_{s=1} = Var(X).$

Another definition for Branching process is as follows:

A Galton-Watson process is a Markov chain $\{X_n, n = 0, 1, 2, \ldots\}$ having state space N (the set of non-negative integers), such that

$$X_{n+1} = \sum_{r=1}^{X_n} \zeta_r, \tag{1}$$

where ζ_r are i.i.d. random variables with distribution $\{p_k\}$.

Let

$$P(s) = \sum_k \Pr\{\zeta_r = k\} s^k = \sum_k p_k s^k \tag{2}$$

be the p.g.f. of $\{\zeta_r\}$ and let

$$P_n(s) = \sum_k \Pr\{X_n = k\} s^k, \quad n = 0, 1, 2, \ldots \tag{3}$$

be the p.g.f. of $\{X_n\}$.

We assume that $X_0 = 1$; clearly $P_0(s) = s$ and $P_1(s) = P(s)$. The r.v.'s X_1 and ζ_r both give (the same) offspring distribution.

Theorem 1. We have

$$P_n(s) = P_{n-1}(P(s)) \tag{4}$$

and

$$P_n(s) = P(P_{n-1}(s)). \tag{5}$$

Proof: We have, for $n = 1, 2, \ldots$

$$\Pr\{X_n = k\} = \sum_{j=0}^{\infty} \Pr\{X_n = k \mid X_{n-1} = j\} \cdot \Pr\{X_{n-1} = j\}$$

$$= \sum_{j=0}^{\infty} \Pr\left\{\sum_{r=1}^{j} \zeta_r = k\right\} \cdot \Pr\{X_{n-1} = j\}$$

so that,

$$P_n(s) = \sum_{k=0}^{\infty} \Pr\{X_n = k\} s^k$$

$$= \sum_{k=0}^{\infty} s^k \left[\sum_{j=0}^{\infty} \Pr\left\{ \sum_{r=1}^{j} \zeta_r = k \right\} \Pr\{X_{n-1} = j\} \right]$$

$$= \sum_{j=0}^{\infty} \Pr\{X_{n-1} = j\} \left[\sum_{k=0}^{\infty} \Pr\{\zeta_1 + \zeta_2 + \cdots + \zeta_j = k\} s^k \right].$$

The expression within the square brackets, being the p.g.f. of the sum $\zeta_1 + \cdots + \zeta_j$ of j i.i.d. random variables each with p.g.f $P(s)$, equals $[P(s)]^j$. Thus

$$P_n(s) = \sum_{j=0}^{\infty} \Pr\{X_{n-1} = j\} [P(s)]^j$$

$$= P_{n-1}(P(s)).$$

Thus we get (4). Putting $n = 2, 3, 4, \ldots$, we get, when $X_0 = 1$,

$$P_2(s) = P_1(P(s)) = P(P(s)),\ P_3(s) = P_2(P(s)),\ P_4(s) = P_3(P(s))$$

and so on. Iterating (4) we get

$$P_n(s) = P_{n-1}(P(s)) = P_{n-2}(P(P(s))) \qquad (6)$$

$$= P_{n-2}(P_2(s)).$$

For $n = 3$, $\quad P_3(s) = P_1(P_2(s)) = P(P_2(s)).$

Again iterating (6), we get

$$P_n(s) = P_{n-3}(P(P_2(s))) = P_{n-3}(P_3(s)),$$

and for $n = 4$, $\quad P_4(s) = P_1(P_3(s)) = P(P_3(s)).$

Thus $\quad P_n(s) = P_{n-k}(P_k(s)),\ k = 0, 1, 2, \ldots, n.$

and for $k = n - 1$

$$P_n(s) = P_1(P_{n-1}(s)) = P(P_{n-1}(s)).$$

Thus we get (5).

Note that even when $X_0 = i \neq 1$, the relation (5) holds but (4) does not hold.

Moments of X_n

Theorem 1 could be used to find the moments of X_n. We have

$$P'(1) = E(\zeta_r) = E(X_1) = m \text{ (say)}.$$

Theorem .2. If $m = E(X_1) = \sum_{k=0}^{\infty} kp_k$, and $\sigma^2 = \text{var}(X_1)$ then

$$E\{X_n\} = m^n \quad (.7)$$

and
$$\text{var}(X_n) = \frac{m^{n-1}(m^n - 1)}{m-1}\sigma^2, \qquad \text{if } m \neq 1 \quad (.8)$$

$$= n\sigma^2, \qquad \text{if } m = 1.$$

Proof: Differentiating (4), we get

$$P_n'(s) = P_{n-1}'(P(s))P'(s)$$

whence
$$P_n'(1) = P_{n-1}'(1)P'(1) = mP_{n-1}'(1)$$

and on iterating

$$P_n'(1) = m^2 P_{n-2}'(1)$$

$$\cdots$$

$$\cdots$$

$$= m^{n-1}P'(1) = m^n.$$

Thus
$$E(X_n) = P_n'(1) = m^n.$$

Differentiating (5) twice and proceeding in a similar fashion, one can find the second moment $P_n''(1)$, and thus the variance of X_n in the form (.8).

One can likewise proceed to get higher moments of X_n.

Examples:

Example 3(a). Let p_k, $k = 0, 1, 2$ be the probability that an individual in a generation generates k offsprings. Then $P(s) = p_0 + p_1 s + p_2 s^2$, and $P_2(s)$, $P_3(s)$ can be calculated by simple algebra. The probability of extinction is one if $m \le 1$; if $m > 1$, it is given by the root less than 1 of $s = P(s)$. Suppose that $p_0 = 2/3$, $p_1 = 1/6$, and $p_2 = 1/6$; then $m = 1/2 < 1$. The equation $s = P(s)$ becomes $s^2 - 5s + 4 = 0$ with roots 1 and 4; the probability of extinction is 1. Suppose that $p_0 = 1/4$, $p_1 = 1/4, P_2 = 1/2$; then $m = 5/4 > 1$; the equation $s = P(s)$ has the roots 1/2 and 1. The root 1/2 (smaller than unity) gives the probability of extinction. Note that the probability of extinction is p_0/p_2 or 1 according as $p_0 < p_2$ or $p_0 \geq p_2$ and also that $p_0 <$ (or $\geq$) p_2 *iff* $m >$ (or $\leq$) 1.

Example 3(b). Let the probability distribution of the number of offsprings generated by an individual in a generation be Poisson with mena λ *i.e.* $P(s) = e^{\lambda(s-1)}$. It can be easily seen that the graph of $P(s)$ in $0 \leq s \leq 1$ (*i.e.*, between the points $(0, e^{-\lambda})$ and (1, 1)) is convex, and that the curve of $y = P(s)$ always lies above $y = s$ when $\lambda \leq 1$, there being no other root of $s = P(s)$ except unity in (0, 1); the probability of extinction is then 1. When $\lambda > 1$, the curve $y = P(s)$ intersects $y = s$ in another point whose s-coordinate has a value < 1 and the probability of extinction will be this value of s. For example, if $\lambda = 2$, it can be seen that $s = e^{2(s-1)}$ has a root approximately equal to 0.2 which is smaller than 1, and the probability of extinction is $q = 0.2$.

Example 3(c). Let the distribution of the number of offsprings be geometric with $p_k = b(1-b)^k$, $k = 0, 1, 2, \ldots$ $(0 < b < 1)$.Then $m = (1-b)/b$ and $P(s) = b/(1-s(1-b))$. The equation $s = P(s)$ has the roots 1 and $b/(1-b)$. If $m \leq 1$, then the probability of extinction is 1; if $m > 1$, the root $b/(1-b) < 1$, and the probability of extinction is equal to the root $b/(1-b)$.

Example 3(d). Let $p_k = bc^{k-1}$, $k = 1, 2, \ldots, 0 < b, c, b+c < 1$ and $p_0 = 1 - \sum_{k=1}^{\infty} p_k$. Then $m = b/(1-c)^2$.

We have

$$P(s) = 1 - \frac{b}{1-c} + \frac{bs}{1-cs}. \tag{3.7}$$

The quadratic equation $s = P(s)$ has the roots

$$1 \text{ and } \frac{1-(b+c)}{c(1-c)} = s_0 \text{ (say)}.$$

If $m = 1$, then $s_0 = 1$ and the probability of extinction is 1; if $m > 1$, $s_0 < 1$, and the probability of extinction is $q = s_0$ (< 1).

This model was applied in a series of interesting papers by Lotka to find out the probability of extinction for American male lines of descent. The values estimated by him (in 1939) from census figures of 1920 give $b = 0.2126$, $c = 0.5893$ ($m \approx 1.25 > 1$) and the probability of extinction $q = s_0 = 0.819$.

Note: It is not always possible to put the generating functions $P_n(s)$ in closed form. The generating functions $P(s)$ obtained in Examples 3(c) and 3(d) are of interesting forms: they may be considered as particular cases of the more general fractional linear form (or general bilinear form)

$$P(s) = \frac{\alpha + \beta s}{\gamma + \delta s}, \quad \alpha\delta - \beta\gamma \neq 0.$$

When $P(s)$ is the above form, $P_n(s)$ is also of the same form

$$P_n(s) = \frac{\alpha_n + \beta_n s}{\gamma_n + \delta_n s}$$

where $\alpha_n, \beta_n, \gamma_n, \delta_n$ are functions of $\alpha, \beta, \gamma, \delta$.

Further, it may be noted that the equation $s = P(s)$ (where $P(s)$ is of fractional linear form) has two finite solutions 1 and s_0 and that $s_0 <, =, >$ or 1 according as $m = P'(1) >, =,$ or < 1.

Example 3(e). $P_n(s)$ for Lotka's model considered in Example 3(*d*) above. For any two points u, v, we get

$$\frac{P(s)-P(u)}{P(s)-P(v)} = \frac{s-u}{s-v} \cdot \frac{1-cv}{1-cu}.$$

Put $u = s_0$, $v = 1$,then $P(s_0) = s_0$, $P(v) = 1$, so that

$$\frac{P(s)-s_0}{P(s)-1} = \frac{s-s_0}{s-1} \cdot \frac{1-c}{1-cs_0}$$

whence

$$\frac{1-c}{1-cs_0} = \left\{\frac{P(s)-s_0}{s-s_0}\right\} \Big/ \left\{\frac{P(s)-1}{(s-1)}\right\}.$$

Let $m \neq 1$, then taking limits of the right hand side as $s \to 1$, we get

$$\frac{1-c}{1-cs_0} = \frac{1}{m}.$$

Hence

$$\frac{P(s)-s_0}{P(s)-1} = \left(\frac{1}{m}\right)\frac{s-s_0}{s-1}.$$

Thus

$$\frac{P_2(s)-s_0}{P_2(s)-1} = \frac{P(P(s))-s_0}{P(P(s))-1} = \left(\frac{1}{m}\right)\frac{P(s)-s_0}{P(s)-1}$$

$$= \left(\frac{1}{m^2}\right)\frac{s-s_0}{s-1}$$

and on iteration

$$\frac{P_n(s)-s_0}{P_n(s)-1} = \left(\frac{1}{m^n}\right)\frac{s-s_0}{s-1}, \quad n = 1, 2, \ldots$$

Solving for $P_n(s)$, we get

$$P_n(s) = 1 - m^n\left(\frac{1-s_0}{m^n - s_0}\right) + \frac{m^n\left(\frac{1-s_0}{m^n-s_0}\right)^2 s}{1-\left(\frac{m^n-1}{m^n-s_0}\right)s}, \quad m \neq 1. \tag{3.8}$$

If $m=1$, then $s_0=1$ and $P(s)=\dfrac{c+(1-2c)s}{1-cs}$

and
$$P_n(s)=\frac{nc+(1-c-ns)s}{(1-c+nc)-ncs}.$$

Limiting Results

Suppose that $m=1$, then $\sigma^2=2c/(1-c)$,
$$n\Pr\{X_n>0\}=n\{1-P_n(0)\}=\frac{n(1-c)}{1-c+nc}$$

and
$$\lim_{n\to\infty} n\Pr\{X_n>0\}=\frac{1-c}{2}=\frac{2}{\sigma^2} \quad \text{(see Theorem 9.8}(a)).$$

Suppose that $m<1$, then $s_0>1$ and
$$\lim_{n\to\infty} m^{-n}\Pr\{X_n>0\}=\lim_{n\to\infty} m^{-n}\{1-P_n(0)\}$$
$$=\lim_{n\to\infty}\frac{1-s_0}{m^n-s_0}=\frac{s_0-1}{s_0}.$$

Again
$$\sum_k \Pr\{X_n=k \mid X_n>0\}s^k=\frac{P_n(s)-P_n(0)}{1-P_n(0)}$$
$$=1-\frac{1-P_n(s)}{1-P_n(0)}$$
$$=\frac{\left(\dfrac{1-s_0}{m^n-s_0}\right)s}{1-\left(\dfrac{m^n-1}{m^n-s_0}\right)s}.$$

Thus
$$\lim_{n\to\infty}\sum_k \Pr\{X_n=k \mid X_n>0\}s^k=s\left(1-\frac{1}{s_0}\right)\left(1-\frac{s}{s_0}\right)^{-1}$$

and
$$\lim_{n\to\infty}\Pr\{X_n=k \mid X_n>0\}=\left(1-\frac{1}{s_0}\right)\left(\frac{1}{s_0}\right)^{k-1}, \quad k=1,2,\ldots$$

In other words, for large n, the distribution of $\{X_n\}$, given $X_n>0$, is geometric with mean $s_0/(s_0-1)$ and p.g.f.

$$b(s)=\frac{(1-1/s_0)s}{1-s/s_0}.$$

It can be easily verified that $b(s)$ satisfies the equation
$$b(P(s))=mb(s)+1-m$$

4.4 Concept of Weiner Process (Brownian Motion Process)

R. Brown (1972) observed that small particles immersed in a liquid exhibit ceaseless irregular motion. The Brownian Motion Process arose an early attempt to explain this phenomenon. Today, the Brownian motion process and its many generations and extensions arise in numerous areas such as economics, communication theory, biology, management science and mathematical statistics.

The **Wiener process** is a real valued continuous-time stochastic **process** named in honor of American mathematician Norbert **Wiener** for his investigations on the mathematical properties of the one-dimensional Brownian motion.

Definition of Wiener Process:

The stochastic process $\{X(t)\}$ is called a ***Wiener Process (or Wiener-Einstein Process or Brownian Motion Process)*** with mean μ and variance σ^2, *if*

i) $X(t)$ has independent increments, ie., for every pair of disjoint intervals of time (s,t) and (u,v), where $s \leq t \leq u \leq v$, the r.v.s $\{X(t) - X(s)\}$ and $\{X(v) - X(u)\}$ are independent.

ii) Every increment $\{X(t) - X(s)\}$ is normally distributed with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$.

Note:

1) $i) \Longrightarrow$ Wiener Process is a Markov Process with independent increments.

2) $ii) \Longrightarrow$ Wiener Process is a Gaussian.

3) A Wiener Process $\{X(t), t \geq 0\}$ with X(0) = 0, $\mu = 0$ and $\sigma = 1$, is called a ***Standard Wiener Process.***

4.5 Weiner Process as a limit of random walk

Consider that a Brownian particle performs a **random walk** such that in a small interval of time of duration Δt, the displacement of the particle to the right or to the left is also of small magnitude Δx, the total displacement X(t) of the particle in time 't' being 'x'.

Suppose that a random variable Z_i denotes the length of the i^{th} step taken by the particle in a small interval of time Δt and that

$$Pr\{Z_i = \Delta x\} = p \text{ and } Pr\{Z_i = -\Delta x\} = q,$$

$p + q = 1, 0 < p < 1$, where p is independent of *'x' and 't'*.

Suppose that the interval of length *'t'* is divided into *'n'* equal subintervals of *length* Δt and that the displacement Z_i , $i=1, 2, ..., n$ in the *'n'* steps are mutually independent random variables. Then $n.\Delta t = t$ and the total displacement $X(t)$ is the sum of *'n'* *i.i.d.* random variables Z_i, $i=1, 2, ..., n$.

$$ie., X(t) = \sum_{i=1}^{n(t)} Zi, n \equiv n(t) = {}^{t}/_{\Delta t}.$$

We know $E\{ Zi \} = (p-q)\, \Delta x$ and $V\{ Zi \}= 4pq\, (\Delta x)^2$.

Hence, $E\{ X(t) \} = n. E\{ Zi \} = t. (p-q) \, {}^{\Delta x}/_{\Delta t}$ ....(1)

and $V\{ X(t) \}= n. V\{ Zi \} = 4pqt \, {}^{(\Delta x)^2}/_{\Delta t}$.

As $\Delta x \to 0, \Delta t \to 0$, we must have

$${}^{(\Delta x)^2}/_{\Delta t} \to a \ limit, \ (p-q) \to a \ multiple \ of \ \Delta x. \quad ....(2)$$

Particularly in an interval of length *'t'*, $X(t)$ has mean-value function $=\mu t$ and variance function $= \sigma^2 t$. In other words, as $\Delta x \to 0, \Delta t \to 0$, in such a way that the equation (2) is satisfied and per unit of time,

$$E\{X(t)\} \to \mu \text{ and } V\{X(t)\} \to \sigma^2. \quad ....(3)$$

From (1) for t =1 and from (3) we have

$$\frac{(p-q)(\Delta x)}{\Delta t} \to \mu \quad and \quad \frac{4pq(\Delta x)^2}{\Delta t} \to \sigma^2. \quad ....(4)$$

The relation (2) and (4) will be satisfied, when

$$\Delta x = \sigma(\Delta t)^{1/2},$$

$$p = \frac{1}{2}[1 + \frac{\mu}{\sigma}.(\Delta t)^{1/2}] \ and \ q = \frac{1}{2}[1 - \frac{\mu}{\sigma}.(\Delta t)^{1/2}]$$

Since, Z_i *'s* are *i.i.d.* random variables, the sum $\sum_{i=1}^{n(t)} Zi = X(t)$ for large $n(t)$ $(=n)$, is asymptotically normal with mean μt and variance $\sigma^2 t$. Here, 't' is the length of the interval of time during which the displacement takes place is $X(t) - X(0)$. Thus, for $0 < s < t$, $\{X(t) - X(s)\}$ is normally distributed with mean $\mu(t\text{-}s)$ and variance $\sigma^2(t\text{-}s)$. Further, the increments $\{X(s) - X(0)\}$ and $\{X(t) - X(s)\}$ are mutually independent.

$\Longrightarrow \{X(t)\}$ is a Markov Process.

Since, $\{X(t) - X(0)\}$ is normally distributed with mean μt and variance $\sigma^2 t$, the transition probability density function of a Wiener Process is given by

$$p(x_0, x; t)dx = Pr\{x \le X(t) < x+dx \,/\, X(0) = x_0\}$$

$$= = \frac{1}{\sigma\sqrt{2\pi t}} . e^{-\frac{(x-x_0-\mu t)^2}{2\sigma^2 t}} dx.$$

Brownian Motion as a Limit of Random Walks. One of the many reasons that Brownian motion is important in probability theory is that it is, in a certain sense, a limit of rescaled simple random walks. Let $\xi_1, \xi_2, \ldots$ be a sequence of independent, identically distributed random variables with mean 0 and variance 1. For each $n \geq 1$ define a continuous–time stochastic process $\{W_n(t)\}_{t\geq 0}$ by

(1) $$W_n(t) = \frac{1}{\sqrt{n}} \sum_{1\leq j\leq \lfloor nt \rfloor} \xi_j$$

1

This is a random step function with jumps of size $\pm 1/\sqrt{n}$ at times k/n, where $k \in \mathbb{Z}_+$. Since the random variables ξ_j are independent, the increments of $W_n(t)$ are independent. Moreover, for large n the distribution of $W_n(t+s) - W_n(s)$ is close to the NORMAL$(0,t)$ distribution, by the Central Limit theorem. Thus, it requires only a small leap of faith to believe that, as $n \to \infty$, the distribution of the random function $W_n(t)$ approaches (in a sense made precise below) that of a standard Brownian motion.

Why is this important? First, it explains, at least in part, why the Wiener process arises so commonly in nature. Many stochastic processes behave, at least for long stretches of time, like random walks with small but frequent jumps. The argument above suggests that such processes will look, at least approximately, and on the appropriate time scale, like Brownian motion.

Second, it suggests that many important "statistics" of the random walk will have limiting distributions, and that the limiting distributions will be the distributions of the corresponding statistics of Brownian motion. The simplest instance of this principle is the central limit theorem: the distribution of $W_n(1)$ is, for large n close to that of $W(1)$ (the gaussian distribution with mean 0 and variance 1). Other important instances do not follow so easily from the central limit theorem. For example, the distribution of

(2) $$M_n(t) := \max_{0\leq s\leq t} W_n(t) = \max_{0\leq k\leq nt} \frac{1}{\sqrt{n}} \sum_{1\leq j\leq k} \xi_j$$

converges, as $n \to \infty$, to that of

(3) $$M(t) := \max_{0\leq s\leq t} W(t).$$

The distribution of $M(t)$ will be calculated explicitly below, along with the distributions of several related random variables connected with the Brownian path.

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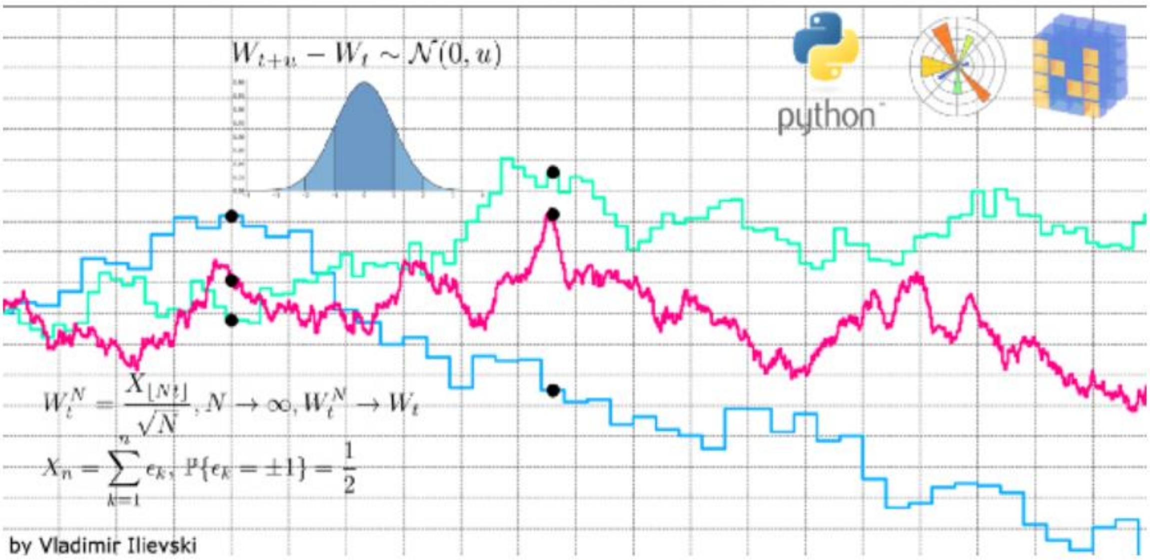

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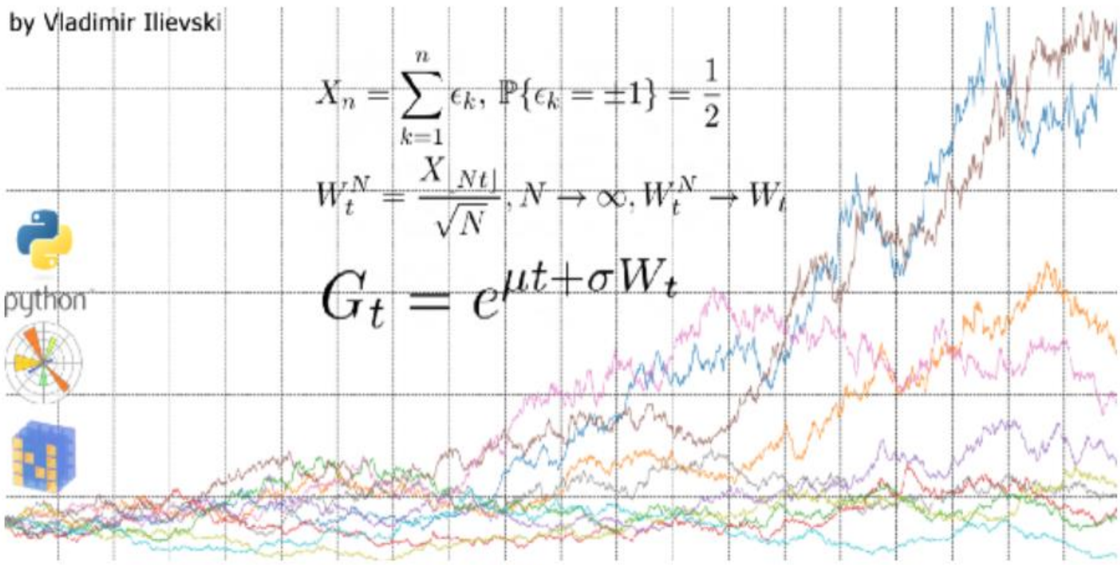

Geometric Brownian Motion

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