Unit III	Topic
	3.1 Continuous time Markov Chain 3.2 Kolmogrov's differential equations 3.3 Poisson Process 3.4 Pure Birth process 3.5 Birth and Death Process 3.6 Simple applications.

3.1 Continuous Time Markov Chains

A **Continuous Time Markov Chain** $\{X(t);\ t > 0\}$ *(or)* $\{X(t);\ t \epsilon T\}$ is a Markov Process satisfying the Markovian dependent condition, defined over a finite state space S, the index set being continuous, $T = \{t;\ t > 0\}$ because of the continuity nature of the index set 'T".

The r.v X_t *or* $X(t)$ represents the state of an event in the interval of length 't'. Consequently, we have the process having started from state 'i' reaches the state 'j' in the interval 't'.

The transition probabilities can also be represented as a TPM,

$$P = \left[P_{ij}^{(t)}\right].$$

Postulator:

For a continuous time Markov Chain {X(t); t > 0}, the transition probability function satisfying the following conditions is said to be **Standard Postulator**:

i) $P_{ij}^{(t)} \geq 0;$ for all $i, j \epsilon S$. *(non-negativity)*

ii) $P_{ij}^{(t+s)} = Pr\,[\,X(t+s) = j\ /\ X(t) = i\,];\ t \geq 0;\ \ i, j\ =\ 0,1,2, \ldots ..$

This probability is independent of S.

iii) Chapman Kolmogorov equation $\quad P_{ij}^{(t+s)} = \sum_{k\epsilon S} P_{ik}^{(t)}\, P_{kj}^{(s)}$

iv) For any $P_{ij}^{(t)} \geq 0, \quad such\ that\ \lim_{t\to 0} P_{ij}^{(t)}\ =\ 1\ if\ \ i = j,$

$$= 0\ if\ \ i \neq j$$

The main role is taken by the n^{th} step transition probabilities, in case of continuous time Markov chain, which are infinite decimal matrix.

i) $\lim\limits_{n\to 0}\frac{1-p_{ij}^{(t)}}{t} = P_{ij}^{!\,(t)}(0) = q_i$

ii) $\lim\limits_{n\to 0}\frac{p_{ij}^{(t)}}{t} = P_{ij}^{!}(0) = q_{ij}$

The above quantities form a matrix called infinite decimal matrix.

$A = [a_{ij}]$

where $a_{ij} = q_i$ if $i = j$,
$= q_{ij}$ if $i \neq j$

3.2 Kolmogorov differential equations

For a Continuous Time Markov Chain, it is known that

$$\lim_{n\to 0}\frac{1-p_{ij}^{(t)}}{t} = P_{ij}^{!\,(t)}(0) = q_i$$

$$\lim_{n\to 0}\frac{p_{ij}^{(t)}}{t} = P_{ij}^{!}(0) = q_{ij}$$

If $\sum_{i\neq j} q_{ij} = q_i$ for all $'i'$, then the process is said to be **Conservative.** The significance of the conservative is that TPF of a process satisfies the system of differential equations. By solving these differential equations by Ordinary Differential Equations (ODE) method, the TPF may be obtained and hence the complete determination of the process.

The set of differential equations satisfied by the TPF, $P_{ij}^{(t)}$, are called Kolmogorov Backward and Forward differential equations. To obtain the above equations in the interval *(0, t+h) for h>0, 'h'* is a small interval, and is split up, in two ways, into two intervals viz.,

i) *(0, h) & (h, t+h)* which will lead to Kolmogorov **Backward** differential equation (KBE),

$$P_{ij}^{!\,(t)} = \sum_{k\neq i} q_{ik}^{(h)} P_{kj}^{(t)} - P_{ij}^{(t)}.q_i$$

ii) *(0, t) & (t, t+h)* which will lead to Kolmogorov **Forward** differential equation (KFE).

$$P_{ij}^{!(t)} = \sum_{k \neq i} q_{ik}^{(t)} P_{kj}^{(h)} - P_{ij}^{(t)} . q_i$$

To derive Kolmogorov Backward differential equation (KBE):

Consider the first split up *(0, h) & (h, t+h),*

$$P_{ij}^{(t+h)} = \sum_{k \epsilon S} P_{ik}^{(h)} P_{kj}^{(t)}$$

On both sides, subtracting $P_{ij}^{(t)}$, dividing by *'h'* and taking limit *h→0,* we have

$$\lim_{h\to 0} \frac{P_{ij}^{(t+h)} - P_{ij}^{(t)}}{h} = \lim_{h\to 0} \frac{1}{h} \left[\sum_{k \epsilon S} P_{ik}^{(h)} P_{kj}^{(t)} - P_{ij}^{(t)} \right]$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\sum_{k\neq i} P_{ik}^{(h)} P_{kj}^{(t)} - P_{ii}^{(h)} . P_{ij}^{(t)} - P_{ij}^{(t)} \right]$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\sum_{k\neq i} P_{ik}^{(h)} P_{kj}^{(t)} - P_{ij}^{(t)} \left\{ 1 - P_{ii}^{(h)} \right\} \right]$$

$$\Rightarrow P_{ij}^{!(t)} = \sum_{k\neq i} \lim_{h\to 0} \frac{P_{ik}^{(h)}}{h} P_{kj}^{(t)} - P_{ij}^{(t)} . \lim_{h\to 0} \left\{ \frac{1 - P_{ii}^{(h)}}{h} \right\}$$

$$\Rightarrow P_{ij}^{!(t)} = \sum_{k\neq i} \lim_{h\to 0} \frac{P_{ik}^{(h)}}{h} P_{kj}^{(t)} - P_{ij}^{(t)} . q_i \quad .... (1)$$

$$\left[\text{Since, by definition, } \lim_{h\to 0} \left\{ \frac{1 - P_{ii}^{(h)}}{h} \right\} = q_i \right]$$

Consider the 1st term of the equation (1),

$$\sum_{k\neq i} \lim_{h\to 0} \frac{P_{ik}^{(h)}}{h} P_{kj}^{(t)} \geq \sum_{k \neq i}^{N} \lim_{h\to 0} \frac{P_{ik}^{(h)}}{h} P_{kj}^{(t)}$$

Taking limit N→∞,

$$\sum_{k\neq i} \lim_{h\to 0} \frac{P_{ik}^{(h)}}{h} P_{kj}^{(t)} \geq \sum_{k \neq i} q_{ik} P_{kj}^{(t)}$$

.... (2)

Consider the LHS of the equation (2),

$$\sum_{k\neq i} \lim_{h\to 0} \frac{P_{ik}^{(h)}}{h} P_{kj}^{(t)} = \lim_{h\to 0} \frac{1}{h}\left[\sum_{k\neq i}^{N} P_{ik}^{(h)} P_{kj}^{(t)} + \sum_{k=N+1}^{\infty} P_{ik}^{(h)} P_{kj}^{(t)}\right]$$

$$\leq \lim_{h\to 0} \frac{1}{h}\left[\sum_{k\neq i}^{N} P_{ik}^{(h)} P_{kj}^{(t)} + \sum_{k=N+1}^{\infty} P_{ik}^{(h)}\right]$$

$$\leq \lim_{h\to 0} \frac{1}{h}\left[\sum_{k\neq i}^{N} P_{ik}^{(h)} P_{kj}^{(t)} + 1 - \sum_{k=1}^{N} P_{ik}^{(h)}\right]$$

$$\leq \lim_{h\to 0} \frac{1}{h}\left[\sum_{k\neq i}^{N} P_{ik}^{(h)} P_{kj}^{(t)} + \{1 - P_{ii}^{(h)}\} - \sum_{k\neq i}^{N} P_{ik}^{(h)}\right]$$

$$\leq \sum_{k\neq i}^{N} \lim_{h\to 0} \frac{P_{ik}^{(h)}}{h} P_{kj}^{(t)} + \lim_{h\to 0}\left\{\frac{1 - P_{ii}^{(h)}}{h}\right\} - \lim_{h\to 0}\left\{\frac{\sum_{k\neq i}^{N} P_{ik}^{(h)}}{h}\right\}$$

$$\Rightarrow \sum_{k\neq i} \lim_{h\to 0} \frac{P_{ik}^{(h)}}{h} P_{kj}^{(t)} \leq \sum_{k\neq i}^{N} q_{ik} P_{kj}^{(t)} + q_i - \sum_{k\neq i}^{N} q_{ik}$$

Taking limit N→∞,

$$\sum_{k\neq i} \lim_{h\to 0} \frac{P_{ik}^{(h)}}{h} P_{kj}^{(t)} \leq \sum_{k\neq i}^{N} q_{ik} P_{kj}^{(t)} + q_i - q_i$$

$$\leq \sum_{k\neq i}^{N} q_{ik} P_{kj}^{(t)}$$

.... (3)

From (2) and (3),

$$\sum_{k\neq i} \lim_{h\to 0} \frac{P_{ik}^{(h)}}{h} P_{kj}^{(t)} = \sum_{k\neq i}^{N} q_{ik} P_{kj}^{(t)} \quad ....(4)$$

Substituting (4) *in* (1), *we get,*

Hence, $\boldsymbol{P_{ij}^{!\,(t)}} = \sum_{k\neq i} \boldsymbol{q_{ik}^{(h)} P_{kj}^{(t)}} - \boldsymbol{q_i \,.\, P_{ij}^{(t)}}$ which is the required **KBE**.

The differential equation may be written in the matrix form as

$P^! = A.P$

where $P = \left[P_{ij}^{(t)}\right]$

$$A = [a_{ij}]; \quad a_{ij} = q_i \quad if\ i=j$$
$$= q_{ij} \quad if\ i \neq j$$

3.3 Poisson Process

The Poisson process is an example for continuous time Markov Chain *{X(t); t > 0}* where *X(t)* represents no. of occurrences of a particular event in the interval *(0, t)*. The name of the Poisson process is due to the concept of **rare events.**

Examples:

Consider the random process *X(t)* which represents

i) No. of accidents occurred in a locality in the interval *(0, t)*.

ii) No. of break downs of a machine in the interval *(0, t)*.

iii) No. of customers arriving at a service station in the interval *(0, t)*.

iv) No. of bulbs fused in an industry in the interval *(0, t)*.

In all these examples, the occurrence of the events is rare and hence the random process, *X(t)* can be related to the Poisson Process.

Definition:

If *X(t)* represents the no. of occurrences of a certain event in the interval *(0, t)*, then the discrete random process *{X(t)}* is called the Poisson process provided the following postulates are satisfied:

a) The increments are all independent. ie., for any $t_1, t_2, \ldots\ldots t_n$ such that $t_1 \leq t_2 \leq \ldots\ldots \leq t_n$, the increments, $X(t_2) - X(t_1), X(t_3) - X(t_2), \ldots\ldots X(t_{i+1}) - X(t_i)$ .... are all independent.

b) The transition probabilities satisfy the stationary property that the distribution of r.v. $X(t) - X(s)$ for any set depends only on the time difference $(t-s)$, not on the time *'s'* when it starts.

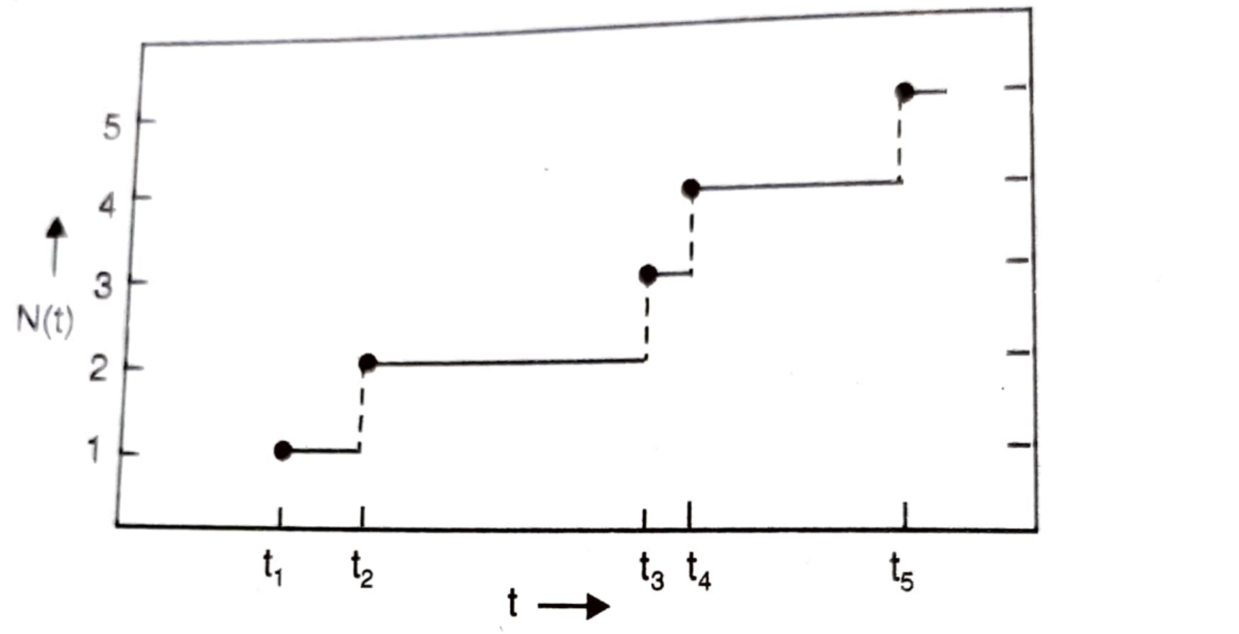
c) The probability that **atleast one event** happens in a small interval of time $h > 0$,

P(h) = f(h) + o(h) where $h > 0$ is the time parameter and o(h) is defined such that $\{o(h) / h\} \rightarrow 0$ as $h \rightarrow 0$.

d) The probability that **more than one event** happens in a small interval of time $h > 0$, implies that only one event happens when $h \rightarrow 0$.

e) The initial value of the state of the process is 'zero'. ie., $X_0 = 0$.

Let us consider the total number $N(t)$ of occurrences of an even E in an interval $(0, t)$ of duration *'t'*, ie., if we start from an initial time point (or instant) $t=0$, $N(t)$ denotes the number of occurrences upto the time point 't'. For example, if an event actually occurs at the time points $t_1, t_2, t_3, ...$, then $N(t)$ jumps abruptly from 0 to 1 at $t = t_1$, from 1 to 2 at $t = t_2$ and so on; the situation is represented in the figure given below.



Number $N(t)$ of occurrences of an event in an interval of duration t

Derivation of Poisson Process:

Theorem: Under the postulates the process *X(t)* follows Poisson distribution with mean λt., ie., $P_n(t)$ is given by the Poisson law:

$$P_n(t) = Pr[X(t)=n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, n = 0, 1, 2, ...$$

Proof:

Let 'λ' be the number of occurrences of the event in a unit time. Also let $P_n(t) = Pr[X(t)=n]$ be the probability of '*n*' number of occurrences in the interval *(0, t)*.

$$Pr[(t+\Delta t)] = Pr[(X(t+\Delta t)=n] \quad .... (1).$$

Consider $P_n(t+\Delta t)$ for n ≥ 0, the probability of '*n*' number of occurrences in the interval *(0, t+Δt)***:**

The 'n' events in interval of duration 't+ Δt' can happen in the following mutually exclusive ways with their respective probabilities:

a) ***'n'*** *events in (0, t) and* ***no*** *event in (t, t+Δt)* with the probability

$$P_n(t+\Delta t) = P_n(t) . P_0(\Delta t) = P_n(t) \ (1-\lambda \Delta t) + P_{n-1}(t) . \lambda \Delta t + O(\Delta t) ....(2)$$

b) ***'n-1'*** *events in (0, t) and* ***1*** *event in (t, t+Δt)* with the probability

$$P_{n-1}(t+\Delta t) = P_{n-1}(t) . P_1(\Delta t) = P_{n-1}(t) \ (\lambda \Delta t) + O(\Delta t)$$

c) ***'n-2'*** *events in (0, t) and* ***2*** *events in (t, t+Δt)* with the probability

$$P_{n-2}(t+\Delta t) = P_{n-2}(t) . P_2(\Delta t) \leq P_2(\Delta t)$$

and so on for $P_{n-3}(t+\Delta t), P_{n-4}(t+\Delta t)$..........

Consider the equation (2),

$$P_n(t+\Delta t) = P_{n-1}(t).\lambda \Delta t + P_n(t) \ (1-\lambda \Delta t) + O(\Delta t)$$

$$= P_{n-1}(t).\lambda \Delta t + P_n(t) - P_n(t).\lambda \Delta t \quad \textit{[by neglecting } O(\Delta t) \textit{ terms]}$$

$$\Rightarrow \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \lambda [P_{n-1}(t) - P_n(t)]$$

Taking limit, as $\Delta t \to 0$, we get the **differential equation** as

$$P_n^{!}(t) = \lambda [P_{n-1}(t) - P_n(t)], \ n \geq 1 \quad ....(3)$$

Consider the eqn.(2), $P_n(t+\Delta t) = P_n(t) \,.\, P_0(\Delta t) = P_n(t)\ (1-\lambda\Delta t) + O(\Delta t)$

For n=0, we get

$$P_0(t+\Delta t) = P_0(t)\,.\,P_0(h) = P_0(t)\ (1-\lambda\Delta t) + O(\Delta t)$$

$$\Rightarrow \frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda\,.\,P_0(t) + \frac{O(\Delta t)}{\Delta t}$$

As $\Delta t \to 0$, we get the **second differential** equation as

$$P_0^!\ (t) = -\lambda\,.\,P_0(t) \qquad \ldots\ldots\ldots\ldots\ldots(4)$$

Initial condition: Suppose the process starts from scratch at time '0', so that

X(t) = 0, ie., $P_0\ (0) = 1\ ;$ $P_n\ (0) = 0\ \ for\ n \neq 0.$ $\ldots\ldots\ldots\ldots\ldots(5)$

The differential equations (3) and (4) together with the initial conditions of the equation (5) completely specify the **system of differential equations**. The solution of the system of differential equations gives the probability distribution $\{P_n(t)\}$ of X(t).

Let the solution of the differential equation (3) be

$$P_n(t) = \frac{(\lambda t)^n}{n!}\,.\,f(t), \qquad \ldots\ldots\ldots\ldots(6)$$

Differentiating eqn.(5) w.r.to 't', we get

$$P_n^!\ (t) = \frac{(\lambda)^n}{n!}\ [n.\,t^{n-1}.\,f(t) + t^n.\,f^!(t)] \qquad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Substituting the equation (6) in (3), we can have,

$$P_n^!\ (t) = \lambda\left[\frac{(\lambda t)^{n-1}}{(n-1)!}.\,f(t) - \frac{(\lambda t)^n}{n!}.\,f(t)\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Since, LHSs of (7) & (8) are equal, their RHSs are also equal.

$$\Rightarrow \frac{(\lambda)^n}{n!}\ [n.\,t^{n-1}.\,f(t) + t^n.\,f^!(t)] = \lambda\left[\frac{(\lambda t)^{n-1}}{(n-1)!}.\,f(t) - \frac{(\lambda t)^n}{n!}.\,f(t)\right]$$

$$\Rightarrow \frac{\lambda^n}{n!}\ n.\,t^{n-1}.\,f(t) + \frac{\lambda^n}{n!}\ t^n.\,f^!(t)] = \left[\frac{\lambda^n t^{n-1}}{(n-1)!}.\,f(t) - \frac{\lambda^{n+1}t^n}{n!}.\,f(t)\right]$$

$$\Rightarrow \frac{\lambda^n t^{n-1}}{(n-1)!}.\,f(t) + \frac{\lambda^n}{n!}\ t^n.\,f^!(t)] = \left[\frac{\lambda^n t^{n-1}}{(n-1)!}.\,f(t) - \frac{\lambda^{n+1}t^n}{n!}.\,f(t)\right]$$

$$\Rightarrow \frac{\lambda^n}{n!}\ t^n.\,f^!(t) = \left[-\frac{\lambda^{n+1}t^n}{n!}.\,f(t)\right]$$

$$\Rightarrow \quad f^{!}(t) = \; [\, -\lambda \; . f(t)]$$

$$\Rightarrow \quad \lambda = -\frac{f^{!}(t)}{f(t)}$$

$$\Rightarrow \frac{f^{!}(t)}{f(t)} = -\lambda$$

Integrating on both sides we get

$$Log\ f(t) = -\lambda t + c$$

$$\Rightarrow f(t) = e^{-\lambda t}.e^{c}$$

$$= k.e^{-\lambda t} \text{ where } k = e^{c} \qquad \text{...... ....(9)}$$

From equation (6),

$$P_n(t) = \frac{(\lambda t)^n}{n!}.f(t) \qquad \text{.....................(10)}$$

When $n=0$, $P_0(t) = \frac{(\lambda t)^0}{0!}.f(t) = f(t)$

When $t=0$, $P_0(0) = \frac{(\lambda t)^0}{0!}.f(0) = f(0) = k.e^{\,0} = k$ .... (11) [using (9)]

According to the initial conditions given in the equation (5), $P_0(0)$ =1.

Therefore, from the equation (11),

$$P_0(0) = k = 1.$$

Substituting the above result ($k=1$) in (9) and hence $f(t) = e^{-\lambda t}$ in (10), we get the solution of the system of differential equations as

$$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n = 0, 1, 2, \ldots.$$ which is the required Poisson process.

Mean and variance of the Poisson Process: E{X(t)} = λt; Var{X(t)} = λt.

Properties of Poisson Process:

i) Additive Property: Sum of two independent Poisson Processes is also a Poisson Process. Let $X_1(t)$ & $X_2(t)$ be two Poisson Processes with parameters λ_1 & λ_2 respectively. Then the process $X(t) = X_1(t) + X_2(t)$ is also a Poisson Process with parameter $\lambda_1 + \lambda_2$.

ii) The probability difference of two independent Poisson Processes, $X(t) = X_1(t) - X_2(t)$ is given by

$$P[X(t)=n] = \left\{\frac{\lambda_1}{\lambda_2}\right\}^{n/2} e^{-(\lambda_1+\lambda_2)t} \, I_n(x)\left(2t\sqrt{\lambda_1.\lambda_2}\right), \quad n = 0, \pm 1, \pm 2, ...$$

$$where \quad I_n(x) = \sum_{r=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2+r\,n} e^{-\lambda t}}{r!\,\Gamma(r+n-1)}$$

iii) Decomposition of a Poisson Process: A random selection from a Poisson Process yields a Poisson Process.

iv) By a random selection, a Poisson Process $\{X(t);\ t \geq 0\}$ of parameter 'λ' *is* decomposed into two independent Poisson Processes $\{X_1(t);\ t \geq 0\}$ and $\{X_2(t);\ t \geq 0\}$ with parameters λp *and* $(1 - \lambda p)$ respectively.

v) If $X(t)$ is a Poisson Process and $s < t$, then

$$Pr\,[X(s) = k \,/\, X(t) = n] = \binom{n}{k}\left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}$$

vi) If $X(t)$ is a Poisson Process, then the auto-correlation between

$X(t)$ *and* $X(t+s)$ *is* $\left(\frac{t}{t+s}\right)^{\frac{1}{2}}$.

....

3.4 Pure Birth Process

In the classical Poisson process we assume that the conditional probabilities are constant. Here, the probability that k events occur between t and $t+h$ given that n events occurred by epoch t is given by

$$\begin{aligned} p_k(h) = \Pr\{N(t+h) - N(t) = k \mid N(t) = n\} &= \lambda h + o(h), & k = 1 \\ &= o(h), & k \geq 2 \\ &= 1 - \lambda h + o(h), & k = 0. \end{aligned}$$

$p_k(h)$ is independent of n as well as t. We can generalise the process by considering that λ is not constant but is a function of n or t or both; the resulting processes will still be Markovian in character.

Here we consider that λ is a function of n, the population size at the instant. We assume that

$$p_k(h) = \Pr\{N(t+h) - N(t) = k \mid N(t) = n\} = \lambda_n h + o(h), \quad k = 1$$
$$= o(h), \quad k \geq 2 \quad \text{—(1)}$$
$$= 1 - \lambda_n h + o(h), \quad k = 0.$$

we shall have the following equation

$$p_n(t+h) = p_n(t)(1 - \lambda_n h) + p_{n-1}(t)\lambda_{n-1}h + o(h), \; n \geq 1.$$

Proceeding as before, we get

$$p_n'(t) = -\lambda_n p_n(t) + \lambda_{n-1}p_{n-1}(t), \quad n \geq 1 \quad \text{—(2)}$$

$$p_0'(t) = -\lambda_0 p_0(t) \quad \text{—(3)}$$

For given initial conditions, explicit expressions for $p_n(t)$ can be obtained from the above equations.

We shall consider here only a particular case of interest, the case $\lambda_n = n\lambda$ and describe a situation where this can happen.

While the above process is called *pure birth process* the process corresponding to $\lambda_n = n\lambda$ is called *Yule-Furry process*.

Yule-Furry Process

Consider a population whose members are (2) either physical or biological entities. Suppose that members can give birth (by splitting or otherwise) to new members (who are exact replicas of themselves) but cannot die. We assume that in an interval of length h each member has a ~~proba~~(4) $_n + o(h)$ of giving birth to a new member. Then, if n individuals are present at time t, the probability (5) there will be one birth between t and $t + h$ is $n\lambda_n + o(h)$. If $N(t)$ denotes the total number of members by epoch t and $p_n(t) = \Pr\{N(t) = n\}$, then by putting $\lambda_n = n\lambda$ in and (3) we obtain the following equations for $p_n(t)$:

$$p_n'(t) = -n\lambda p_n(t) + (n-1)\lambda p_{n-1}(t), \quad n \geq 1 \qquad (3.10)$$

$$p_0'(t) = 0 \qquad (3.11)$$

If the initial conditions are given, explicit expressions for $p_n(t)$ can be obtained.

Suppose that the initial condition is $p_1(0) = 1$, $p_i(0) = 0$ for $i \neq 1$, *i.e.* the process started with only one member at time $t = 0$. The solution can be obtained by the method of induction as follows:

For $n = 1$, we have

$$p_1'(t) = -\lambda p_1(t)$$

whose solution is

$$p_1(t) = c_1 e^{-\lambda t};$$

and putting $p_1(0) = 1$, we have $c_1 = 1$, so that, $p_1(t) = e^{-\lambda t}$.

For $n = 2$, we have

$$p_2'(t) = -2\lambda p_2(t) + \lambda p_1(t)$$

or
$$p_2'(t)+2\lambda p_2(t)=\lambda p_1(t)=\lambda e^{-\lambda t}.$$

This linear equation has the integrating factor $e^{2\lambda t}$ and therefore
$$e^{2\lambda t}p_2(t)=\int \lambda e^{2\lambda t-\lambda t}dt=e^{\lambda t}+c_2;$$

since $p_2(0)=0$, we have $c_2=-1$

and
$$p_2(t)=e^{-2\lambda t}\left(e^{\lambda t}-1\right)=e^{-\lambda t}\left(1-e^{-\lambda t}\right).$$

Proceeding in this way it can be shown that

(4)
$$p_n(t)=e^{-\lambda t}\left(1-e^{-\lambda t}\right)^{n-1},\ n\geq 1.$$

Solving (3.11) and noting that $p_0(0)=0$, we get
$$p_0(t)=0.$$

The distribution is of the geometric form (see Exercise 1.2). Its p.g.f. is given by
$$P(s,t)=\sum_{n=1}^{\infty}\left\{e^{-\lambda t}\left(1-e^{-\lambda t}\right)^{n-1}\right\}s^n$$
$$=\frac{se^{-\lambda t}}{1-s\left(1-e^{-\lambda t}\right)}.$$

The mean of the process is given by
$$E\{N(t)\}=e^{\lambda t};$$

Further, $\operatorname{var}\{N(t)\}=e^{\lambda t}\left(e^{\lambda t}-1\right).$

3.5 Birth and Death Process

Let us ... considered a pure birth process, where

Pr {Number of births between t and $t + h$ is k, given that the number of individuals at epoch t is n}

is given by

$$p(k, h | n, t) = \begin{cases} \lambda_n h + o(h), & k = 1 \\ o(h), & k \geq 2 \\ 1 - \lambda_n h + o(h), & k = 0. \end{cases} \quad (4.1)$$

The above holds for all $n \geq 0$; λ_0 may or may not be equal to zero. Here k is a non-negative integer which implies that there can only be an increase by k, *i.e.* only births are considered possible. Now we suppose that there could also be a decrease by k, *i.e.* death(s) is also considered possible. In this case we shall further assume that

Pr {Number of deaths between t and $t + h$ is k, given that the number of individuals at epoch t is n} is given by

$$q(k, h | n, t) = \begin{cases} \mu_n + o(h), & k = 1 \\ o(h), & k \geq 2 \\ 1 - \mu_n + o(h), & k = 0. \end{cases} \quad (4.2)$$

The above holds for $n \geq k$; further $\mu_0 = 0$. With (4.1) and (4.2) we have, what is known as a *birth and death process*. Through a birth there is an increase by one and through a death, there is a decrease by one in the number of "individuals". The probability of more than one birth or more than one death in an interval of length h is $o(h)$. Let $N(t)$ denotes the total number of individuals at epoch t starting from $t = 0$ and let $p_n(t) = \Pr\{N(t) = n\}$. Consider the interval between 0 and $t + h$; suppose that it is split into two periods $(0, t)$ and $[t, t + h]$. The event $\{N(t + h) = n, n \geq 1\}$, (having probability $p_n(t + h)$) can occur in a number of mutually exclusive ways.

These would include events involving more than one birth and/or more than one death between t and $t + h$. By our assumption, the probability of such an event is $o(h)$. There will remain four other events to be considered:

A_{ij} : $(n - i + j)$ individuals by epoch t, i birth and j death between t and $t + h$, $i, j = 0, 1$.

We have

$$\Pr(A_{00}) = p_n(t)\{1 - \lambda_n h + o(h)\}\{1 - \mu_n h + o(h)\}$$

$$= p_n(t)\{1 - (\lambda_n + \mu_n)h + o(h)\};$$

$$\Pr(A_{10}) = p_{n-1}(t)\{\lambda_{n-1}h + o(h)\}\{1 - \mu_{n-1}h + o(h)\}$$

$$= p_{n-1}(t)\lambda_{n-1}h + o(h);$$

$$\Pr(A_{01}) = p_{n-1}(t)\{1 - \lambda_{n+1}h + o(h)\}\{\mu_{n+1}h + o(h)\}$$

$$= p_{n+1}(t)\mu_{n+1}h + o(h);$$

and

$$\Pr(A_{11}) = p_n(t)\{\lambda_n h + o(h)\}\{\mu_n h + o(h)\}$$

$$= o(h).$$

Hence we have, for $n \geq 1$

$$p_n(t+h) = p_n(t)\{1 - (\lambda_n + \mu_n)h\} + p_{n-1}(t)\lambda_{n-1}h + p_{n+1}(t)\mu_{n+1}h + o(h) \tag{4.3}$$

or

$$\frac{p_n(t+h) - p_n(t)}{h} = -(\lambda_n + \mu_n)p_n(t) + \lambda_{n-1}p_{n-1}(t) + \mu_{n+1}p_{n+1}(t) + \frac{o(h)}{h},$$

and taking limits, as $h \to 0$, we have

$$p_n'(t) = -(\lambda_n + \mu_n)p_n(t) + \lambda_{n-1}p_{n-1}(t) + \mu_{n+1}p_{n+1}(t), \ n \geq 1. \tag{4.4}$$

For $n = 0$, we have

$$p_0(t+h) = p_0(t)\{1 - \lambda_0 h + o(h)\} + p_1(t)\{1 - \lambda_0 h + o(h)\}\{\mu_1 h + o(h)\} \tag{4.5}$$

$$= p_0(t) - \lambda_0 h p_0(t) + \mu_1 h p_1(t)$$

or

$$\frac{p_0(t+h) - p_0(t)}{h} = -\lambda_0 p_0(t) + \mu_1 p_1(t) + \frac{o(h)}{h}$$

whence we get

$$p_0'(t) = -\lambda_0 p_0(t) + \mu_1 p_1(t) \tag{4.6}$$

If at epoch $t = 0$, there were i (≥ 0) individuals, then the initial condition is

$$p_n(0) = 0, \quad n \neq i, \quad p_i(0) = 1. \tag{4.7}$$

The equations (4.4) and (4.6) are the equations of the birth and death process. The birth and death processes play an important role in queueing theory. They also have interesting applications in diverse other fields such as economics, biology, ecology, reliability theory etc.

Note: The result about existence of solutions of (4.4) and (4.6) is stated below without proof.

For arbitrary $\lambda_n \geq 0$, $\mu_n \geq 0$, there always exists a solution $p_n(t)$ (≥ 0) such that $\sum p_n(t) \leq 1$. If λ_n, μ_n are bounded, the solution is unique and satisfies $\sum p_n(t) = 1$.

Example 1 Suppose that customers arrive at a Bank according to a Poisson process with a mean rate of a per minute. Then the number of customers $N(t)$ arriving in an interval of duration t minutes follows Poisson distribution with mean at. If the rate of arrival is 3 per minute, then in an arrival of 2 minutes, the probability that the number of customers arriving is:

(*i*) exactly 4 is

$$\frac{e^{-6}(6)^4}{4!} = 0.133,$$

(*ii*) greater than 4 is

$$\sum_{k=5}^{\infty} \frac{e^{-6}(6)^k}{k!} = 0.714,$$

(*iii*) less than 4 is

$$\sum_{k=0}^{3} \frac{e^{-6}(6)^k}{k!} = 0.152,$$

Example 2 A machine goes out of order whenever a component part fails. The failure of this part is in accordance with a Poisson process with mean rate of 1 per week.

Then the probability that two weeks have elapsed since the last failure is $e^{-2} = 0.135$, being the probability that in time $t = 2$ weeks, the number of occurrences (or failures) is 0.

Suppose that there are 5 spare parts of the component in an inventory and that the next supply is not due in 10 weeks. The probability that the machine will not be out of order in the next 10 weeks is given by

$$\sum_{k=0}^{5} \frac{e^{-10}(10)^k}{k!} = 0.068,$$

being the probability that the number of failures in $t = 10$ weeks will be less than or equal to 5.

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