t	Topic
II	2.1 Classification of states of a Markov Chain 2.2 Recurrent and Transient states 2.3 Criteria for classification of the states 2.4 Random walk with absorbing and reflecting barriers 2.5 Probability of absorption 2.6 Duration of Random Walk 2.7 Gambler's ruin problem.

Classification of states of a Markov chain:

Consider a discrete Markov chain $\{X_n, n=0,1,2,\dots\}$ define over a state space $S=\{0,1,\dots\}$, consisting of integers.

The states of a Markov chain can be in general classified depending upon the nature of moments of the states.

i) Accessibility
ii) Communication
iii) Essentiality

Accessability: Suppose that the state j has the property that it can be reached from any state i. i.e., it is said to be accessable from i, if there exist the probability that $P_{ij} > 0$, for $n>0$. Thus, $i \to j$ iff $P_{ij}^{(n)} > 0$, $n>0$. If $P_{ij}=0$ then $i \not\to j$: $n>0$.

$\Rightarrow i \to j$ (j is accessable from i)
$\Rightarrow i \not\to j$ (j is not accessable from i)

Communication:

If two states are accessable from each other, then they are said to be communicating with each other. That is $i \leftrightarrow j$ that is (j is accessable from i), (i is accessable from j). Thus, 'i' and 'j' are communicating with each other iff $P_{ij}^{(n)} > 0$ for $n>0$ and $P_{ji}^{(m)} > 0$ for $m>0$

Essentiality.

A state which has the property that it will communicate with any state, from which it is accessable, is said to be Essential, otherwise, Inessential.

$\Rightarrow$ The Essential state j, has the Property that it is accessable from k. that is $k \to j$

Properties of communicating states:

The communicating states satisfy the following three properties:

1) Reflexivity
2) Symmetry
3) Transitivity

Reflexivity.

Reflexivity is the property that a state will communicate with itself, that is $i \leftrightarrow i$

$P_{ij}^{(n)} > 0 ; n \geq 0$

Symmetry

Let i be the communicating state and it communicate with j which is also a communicating state with i, that is if $i \leftrightarrow j$, then $j \leftrightarrow i$.

Transitivity:

If $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$

Recurrence, Mean Recurrence and transiant

We know that $f_{ij}^{(n)}, n \geq 1$, gives the Probability that the process return to the state 'i' from 'i', for the first kind at the 'nth' step.

Clearly, the probability of "Ultimate" return is given by $f_{ii}^{*} = \sum_{n=1} f_{ii}^{(n)}$.

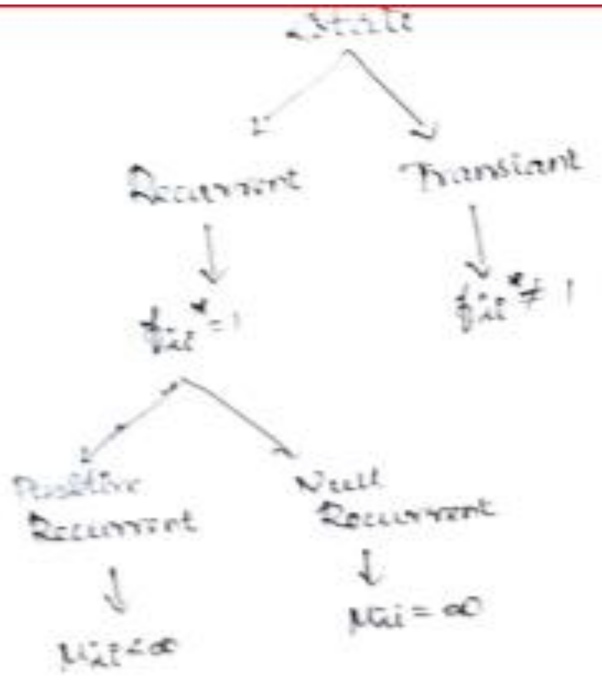
If $f_{ii}^{*} = 1$, then the state 'i' is said to be Recurrence or persistent. If $f_{ii}^{*} < 1$, then the state 'i' is said to be transcant or non-Recurrent.

The time required for the first return to the state 'i' is called recurrence Time 'i' of the process.

∴ The expected value of the time

$\sum_{n=1} n f_{ii}^{(n)} = \mu_{ii}$ is called Mean recurrence Time.

If $\mu_{ii} < \infty$, then the state 'i' is said to be positive Recurrent. If $\mu_{ii} = \infty$ then the state 'i' is said be null recurrent



Ir-reducible Markov chain:

The communicating state of Markov chain, ... the property of reflexivity satisfying this property. A class of states ... is called as equalent class.

The communicating state of a MC, constitute several equalent classes. Thus, the state space of a MC may consist a number of equalence classes.

That is, the state space can be written as $S = \{c_1, c_2, \ldots\}$ where $c_1, c_2 \ldots$ are the different equalence class constituted by communicating states.

If called every i, j we can find some 'n' such that $P_{ij}^{(n)} > 0$, then every state can be reached from other state. And the MC is said to be Ir-reduciable. otherwise the MC is reduciable.

In general, if the state space of a MC

consists only one equalence class, the chain is said to be Ir-reducible. Thus in an Ir-reducible MC, the states are communicating with each other.

Absorbing state:

When $P_{ii} = 1$, the state i is said to be Absorbing state, that is once the process enters the state 'i', gets 'Absorbed' that is, it Remains there forever and will not moved out.

Return state:

State i of a MC is called Return state, if $P_{ij}^{(n)} > 0$; $n \geq 1$.

Periodic state:

The Period d_i of a Return state i is the greatest common devisor (GCD) of all 'm' such that $P_{ii} > 0$

Thus $d_i = \text{GCD}\{m : P_{ij}^{(m)} > 0\}$;

The state 'i' is aperiodic if $d_i = 1$
and periodic if $d_i > 1$

First Return time probability:

The Probability that a Markov chain Returns to state 'i' having started from state 'i', for the First time, at the n^{th} time point is denoted by $f_{ii}^{(n)}$ and is called first return time probability

Theorem: 1

A State 'j' is persistent iff $\sum_{n=1}^{\infty} p_{jj}(n) = \infty$.
i.e., A state 'j' is persistent iff $\sum_{n=1}^{\infty} p_{jj}^{(n)}$ is divergent.

Proof:

Let $P_{jj}(s) = \sum_{n=0}^{\infty} p_{jj}^{(n)} s^n = 1 + \sum p_{jj}^{(n)} s^n, \ |s|<1$.

and $F_{jj}(s) = \sum_{n=0}^{\infty} f_{jj}^{(n)} s^n = \sum_{n=1}^{\infty} f_{jj}^{(n)} s^n, \ |s|<1$.

be the generating functions of the sequences $\{p_{jj}^{(n)}\}$ and $\{f_{jj}^{(n)}\}$ respectively.

We know that $p_{jk}^{(n)} = \sum f_{jk}^{(r)} P_{kk}^{(n-r)}, \ n \geq 1$

for any states $j \neq k$, —— ①

Multiplying both sides of ① by s^n, and adding for $n \geq 1$, we get,

$$P_{jj}(s) - 1 = F_{jj}(s) P_{jj}(s). \qquad \text{—— ②}$$

The RHS of ② is obtained, by considering RHS of ①, and is a convolution of $\{f_{jj}\}$ and $\{p_{jj}\}$ and that the generating function of the convolution is the product of the two generating functions.

Thus,
$$P_{jj}(s) = \frac{1}{1-F_{jj}(s)}, \ |s|<1. \qquad \text{—— ③}$$

Let us assume that state 'j' is persistent (Recurrence)

$\Rightarrow F_{jj} = 1$.

Using Abel's Lemma that if the series $\sum_{k=0}^{\infty} a_k$ converges then $\lim_{s \to 1^-} \sum_{k=0}^{\infty} a_k s^k = \sum_{k=0}^{\infty} a_k = a$,

We get $\lim_{s \to 1} F_{jj}(s) = 1$ and from ③, $\lim P_{jj}(s) < \infty$.

Since, the Coefficients $P_{ij} \geq 0$, we get using Abel's lemma, $\sum_n P_{jj}^{(n)} = \infty$. (divergent)

Conversely, if state 'j' is transient (non-recurrent), then by Abel's lemma, we get $\lim_{s \to 1} F_{jj}(s) < 1$

and from ③, $\lim_{s \to 1} P_{jj}(s) < \infty$. (Convergent).

Theorem (2):

Statement:

Two communicating states will be either recurrent or transient, that is, $i \leftrightarrow j$ and if 'i' is recurrent then 'j' is also recurrent.

Proof:

Given $i \leftrightarrow j \Rightarrow P_{ij}^{(n)} > 0$ and $P_{ji}^{(m)} > 0$. Let the state 'i' is recurrent, consider

$$P_{je}^{r+s+t} = P_{ji}^{(r)} \cdot P_{ii}^{(s)} \cdot P_{ij}^{(t)}$$

Taking $\sum$ over 's' on both sides

$$\sum_{s=0} P_{ji}^{r+s+t} > P_{ji}^{(r)} \cdot P_{ij}^{(t)} \cdot \sum_{s=0} P_{ii}^{(s)}$$

$$\therefore \sum_{s=0} P_{ji}^{r+s+t} > \infty$$

$\Rightarrow$ j is also recurrent if 'i' is recurrent. $\therefore \sum_{s=0} P_{ii}^{(s)} < \infty$

$$\Rightarrow \sum_{s=0} P_{ji}^{r+s+t} < \infty$$

$\Rightarrow$ j is also recurrent, but not transient. $\therefore$ two communicating states either transient (or) recurrent.

Theorem (3):

Statement:

The state is either recurrent or transient according as $Q_{ii} = 1$ or $Q_{ii} = 0$

Proof: consider $Q_{ii}^{(n)} = \sum_{r=0} f_{ii}^{(r)} Q_{ii}^{(n-1)}$

$$= \sum_{r=0} (f_{ii})^{r} Q_{ii}^{(n-1)}$$

$$= \sum_{r=0} (f_{ii}^{*})^2 \, Q_{ii}^{(n-2)}$$

$$= (f_{ii}^{*})^n$$

consider the $\lim_{n\to\infty}$

$$\lim_{n\to\infty} Q_{ii}^{(n)} = \lim_{n\to\infty} (f_{ii}^{*})^n = 1$$

$$\therefore \quad Q_{ii} = 1 \qquad (\because f_{ii}^{(n)} = 1)$$

If state 'i' is not recurrent then $f_{ii}^{*} < 1$

$$\Rightarrow Q_{ii} = 0$$

Thus the state 'i' is recurrent or transient according as $Q_{ii} = 1$ or $Q_{ii} = 0$

Theorem (4)

Statement:

If $i \leftrightarrow j$, the class is recurrent, then $Q_{ii} = 1$.

Proof:

Given that, $i \leftrightarrow j$ and the class is recurrent,

consider,

$$Q_{ij} = \sum f_{ii}^{(r)} Q_{ij}$$

W.K.T

$$Q_{ii} = 1 \text{ and } Q_{jj} = 1$$

$$\because f_{ij}^{*} = 1$$

$$\Rightarrow Q_{ii} = 1$$

Periodicity of a Markov chain:

(Q.10)

Given the state of a Markov chain, the process may return to the same state at any stage. suppose having started from 'i', the process returns to the state 'i' at $t_1, t_2 \dots$ steps. Then the greatest common devisor of all the steps can be defined as the period of the state 'i'. The period of the state 'i' is denoted by $d_i = t$. Generaly the period of state 'i' can be defind as follows:

The Period of the state 'i' is the grestest common devisors of all integers $n \geq 0$ with $P_{ij}^{(n)} > 0$

Ergodic Markov chain (EMC) (and) Positive Recurrent

A state 'i' whose period = 1, is said to be aperiodic state. for the state of convenance, We may call a positive Recurrent aperiodic state as Ergodic state and corresponding Markov chain is said to be EMC.

(2.11)

Problem ①

Let $\{X_n ; n = 1, 2, \cdots\}$ be a Markov chain with states space $S = \{0, 1, 2\}$ and one step transition probability Matrix.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1 & 0 \end{bmatrix}$$

i) Show that the chain is irreducible

ii) Find the period

Sol:

Given one step transition probability

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1 & 0 \end{bmatrix}$$

Now, To find $P^2 = P \times P$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1/4+0 & 0+1/2+0 & 0+1/4+0 \\ 0+1/8+0 & 1/4+1/4+1/4 & 0+1/8+0 \\ 0+1/4+0 & 0+1/2+0 & 0+1/4+0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/8 & 3/4 & 1/8 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

$P^3 = P^2 \times P$

$$= \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/8 & 3/4 & 1/8 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1/8+0 & 1/4+1/4+1/4 & 0+1/8+0 \\ 0+3/16+0 & 1/8+3/8+1/8 & 0+3/16+0 \\ 0+1/8+0 & 1/4+1/4+1/4 & 0+1/8+0 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1/8 & 3/4 & 1/8 \\ 3/16 & 5/8 & 3/16 \\ 1/8 & 3/4 & 1/8 \end{bmatrix}$$

$\therefore$ 1st state '0'

$p_{11}^{(2)} > 0$, $p_{11}^{(3)} > 0$

Period $\Rightarrow$ GCD(2, 3 ...) = 1

2nd state '1' is aperiod

$\Rightarrow p_{22}^{(2)} > 0$, $p_{22}^{(3)}$
$\Rightarrow$ state '1' is aperiod
Period $\Rightarrow$ GCD (2, 3 ...) = 1

3rd state '2'

$p_{33}^{(2)}$, $p_{33}^{(3)}$

Period $\Rightarrow$ GCD (2, 3 ...) = 1

state 2 is aperiod.

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Hence

$P_{11}^{(2)} > 0 \quad P_{12}^{(1)} > 0 \quad P_{13}^{(2)} > 0$

$P_{21}^{(1)} > 0 \quad P_{22}^{(1)} > 0 \quad P_{23}^{(1)} > 0$

$P_{31}^{(2)} > 0 \quad P_{32}^{(1)} > 0 \quad P_{33}^{(2)} > 0$

Problem (2)

Let $\{X_n : n = 1, 2 \ldots\}$ be a Markov chain on the states space $S = \{0, 1, 2, 3\}$ with one step probability Matrix.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

1) Is the chain is irreducible. Find the period.

Given one step Probability

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P^2 = P \times P$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 + 1/2 + 0 & 0 + 0 + 0 & 0 + 1/2 + 0 \\ 0 + 0 + 0 & 1/2 + 0 + 0 & 0 + 0 + 0 \\ 0 + 0 + 0 & 0 + 0 + 0 & 0 + 0 + 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$P^2 \times P$

$$P^3 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0 & 1/2+0+0 & 0+0+0 \\ 0+1/4+0 & 0+0+0 & 0+1/4+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/2 & 0 \\ 1/4 & 0 & 1/4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P^4 = P^3 \times P$$

$$= \begin{bmatrix} 0 & 1/2 & 0 \\ 1/4 & 0 & 1/4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1/4+0 & 0+0+0 & 0+1/4+0 \\ 0+0+0 & 1/4+0+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 1/4 & 0 & 1/4 \\ 0 & 1/4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem (3) Let $\{X_n : n=1,2\ldots\}$ be a Markov chain on the states space $S=\{0,1,2,3\}$ with one step probability Matrix

$$P=\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}$$

i) Is the chain is irreduceable,
ii) find the period.

Given one step Probability

$$P=\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}$$

$P^2 = P \times P$

$$=\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P^2=\begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^2 \times P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 0 & 1/2 & 1/4 \\ 2/4 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

State 1 $P_{11} \Rightarrow GCD(2,3,4\ldots) = 1$

$\Rightarrow$ The state '1' is aperiod

state 2 $P_{22} \Rightarrow GCD(2,3,4) = 1$

The state 2 is aperiod

state 3 $P_{33} \Rightarrow GCD(2,3,4) = 1$

The state 3 is aperiod.

$$\begin{matrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{matrix}$$

Proble (4): Let $\{X_n ; n = 1,2,3\ldots\}$ be a Markov chain on the states space $S = \{0,1,2,3\}$ with transition probability matrix

$$P = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/4 & 0 & 1/3 \\ 1 & 0 & 0 \end{bmatrix}$$

1) verify whether chain is irreduceable given comments
2) find the period.

Given one step probability

$$P = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/4 & 0 & 1/3 \\ 1 & 0 & 0 \end{bmatrix}$$

$P \times P = P^2$

$$P^2 = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/4 & 0 & 1/3 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/2 & 0 \\ 1/4 & 0 & 1/3 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 + 1/8 + 0 & 0+0+0 & 0 + 1/6 + 0 \\ 0+0+1/3 & 1/8+0+0 & 0+0+0 \\ 0+0+0 & 1/2+0+0 & 0+0+0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} \frac{1}{8} & 0 & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{8} & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

$$P^2 \times P = \begin{bmatrix} \frac{1}{8} & 0 & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{8} & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{3} \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+\frac{1}{6} & \frac{1}{16}+0+0 & 0+0+0 \\ 0+\frac{1}{32}+0 & \frac{1}{6}+0+0 & 0+\frac{1}{24}+0 \\ 0+\frac{1}{8}+0 & 0+0+0 & 0+\frac{1}{6}+0 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} \frac{1}{6} & \frac{1}{16} & 0 \\ \frac{1}{32} & \frac{1}{6} & \frac{1}{24} \\ \frac{1}{8} & 0 & \frac{1}{6} \end{bmatrix}$$

$$P^3 \times P = \begin{bmatrix} \frac{1}{6} & \frac{1}{16} & 0 \\ \frac{1}{32} & \frac{1}{6} & \frac{1}{24} \\ \frac{1}{8} & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{3} \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+\frac{1}{64}+0 & \frac{1}{12}+0+0 & 0+\frac{1}{48}+0 \\ 0+\frac{1}{64}+\frac{1}{24} & \frac{1}{64}+0+0 & 0+\frac{1}{18}+0 \\ 0+\ 0+\frac{1}{6} & \frac{1}{16}+0+0 & 0+0+0 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} \frac{1}{64} & \frac{1}{12} & \frac{1}{48} \\ \frac{1}{64}\frac{1}{12} & \frac{1}{64} & \frac{1}{18} \\ \frac{1}{6} & \frac{1}{6} & 0 \end{bmatrix}$$

2.4 Random Walk with Absorbing and Reflecting Barriers

Definition:

An one dimensional random walk is defined as a **Discrete Markov Chain {X_n}** over the state space S = [0, 1,2,3,.....} consisting of integers and the index set T={0, 1,2, ...} consisting of integers, satisfying the property that the process at any time **moves** either to the **next state** or to the **previous state** or **remains** in the **same state** in a **single** transition. ie., in a **single step** the process makes a shift to the **nearest** neighboring state or **remains** in the **same state** itself.

Notations & Transition Matrix:

Suppose a random process, at the **time point 'n'**, is in the **state 'i'**. ie., X_n=i, i∊S. Then in the next step, it moves either to **'i+1' or 'i-1' or remains** in the same **state 'i'** itself. Therefore, the transition probabilities are given by

$$
\begin{aligned}
P_{ij} &= Pr[X_{n+1} = j / X_n = i] \\
&= P_{i,i+1} \quad \text{for } j = i+1 \\
&= P_{i,i} \quad \text{for } j = i \\
&= P_{i,i+1} \quad \text{for } j = i+1.
\end{aligned}
$$

For our convenience, let $p_i = P_{i,\, i+1}$

$q_i = P_{i,\, i-1}$

$r_i = P_{i,\, i}$

where $p_0 \geq 0,\ q_0 \geq 0$ and $p_0 + q_0 = 1$

$p_i > 0,\ q_i > 0,\ p_i + q_i + r_i = 1$ for $i = 1,2,3,....$

The transition probability matrix of a random walk is of the form

$$
\mathbf{P} = \left\| \begin{matrix}
r_0 & p_0 & 0 & 0 & \cdots & & \\
q_1 & r_1 & p_1 & 0 & \cdots & & \\
0 & q_2 & r_2 & p_2 & \cdots & & \\
 & \ddots & & & & & \\
 & & 0 & q_i & r_i & p_i & 0 \\
 & & & \ddots & & & \ddots
\end{matrix} \right\|,
$$

<u>When **Player A** with **fortune k** plays a game against an **infinitely rich adversary:**</u>

The fortune of a player engaged in a series of contests is often depicted by a random walk process. Specifcally, suppose **Player A** with **fortune k** plays a game against an **infinitely rich adversary** and has the **probability** $\mathbf{p_k}$ of **winning one unit** and with probability $\mathbf{q_k = 1- pk\ (k \geq 1)}$ of **losing one unit** in each contest (the choice of the contest at each stae may depend on his fortune), and $\mathbf{r_0 = 1}$.

The process $\mathbf{\{X_n\}}$ where X_n represents his **fortune after n contests**, is clearly **a random walk**. Once the state '0' is reached (ie., the player A is wiped out), the process remains in that state. This process is commonly known as the **Gambler's Ruin.**

The random walk corresponding to $\mathbf{p_k} = p$, $\mathbf{q_k = 1- p = q}$ **for all** $\mathbf{(k \geq 1)}$ **and** $\mathbf{r_0 = 1}$ **with** $\mathbf{p > q}$ describes the situation of identical contests with a definite advantage to the Player A in each indiidual trail.

We know that with **probability** $\mathbf{(q/p)^{x_0}}$ where x_0 represents his fortune at time '0', the **player A** is ultimately ruined (his entire fortune is lost), while with probability $\mathbf{1- (q/p)^{x_0}}$, his forune increases, in the long run, without the limit.

If $\mathbf{p < q}$, then the advantage is decided in favour of the adversary (oponent) and with certaininty (probability 1) the **player A** is untimately ruined if he persists in playing as long as he is able to. The certainity of ultimate ruin is true even if the indidvdual games are **fair**, ie., $\mathbf{p_k} = \mathbf{q_k = ½}$ **.**

<u>When **BOTH the Players A & B play** with limited **fortunes:**</u>

If the adversary , the **player B**, also starts with a limited **fortune 'y'** and the player A has an initial fortune 'x' (let x + y = a), then we may again consider the Markove Chain process $\mathbf{\{X_n\}}$ representing the players A's forutne. The states of the process are now restricted to the values 0,1, 2,..... a. At any trail, $a - X_n$ is interpested as the player B's fortune.

If we allow possibility of neither player winning in a contest, the transition probability matrix takes the form

$$\mathbf{P} = \begin{Vmatrix} 1 & 0 & 0 & 0 & \cdots & & \\ q_1 & r_1 & p_1 & 0 & \cdots & & \\ 0 & q_2 & r_2 & p_2 & \cdots & & \\ & & \ddots & & & & \\ & & & q_{a-1} & r_{a-1} & p_{a-1} & \\ 0 & \cdots & \cdots & 0 & 0 & 1 & \end{Vmatrix}$$

Again **p_i** (or q_i), i = 1, 2, 3, ... a-1, denotes the probability of **player A's fortune increasing** (or decreasing) by '**1**' at the subsequent trail when his present **fortune is 'i'** and r_i is the probability of a draw. In accordance with the Markov Chain given in the above transition matrtix, when the **Player A's fortune** (the state of the process) reaches the **state '0' or 'a'**, it remains in the same state forever. Thus, the **player A is ruined** when the state of the process **reaches '0'** and the **player B is ruined** when the state of the proces **reaches 'a'.**

<u>Classification of Random Walk Processes:</u>

We classify the different proesses by the **nature of the '0' state**. Consider the random walk process described by the TP matrix

$$\mathbf{P} = \begin{Vmatrix} r_0 & p_0 & 0 & 0 & \cdots & & & \\ q_1 & r_1 & p_1 & 0 & \cdots & & & \\ 0 & q_2 & r_2 & p_2 & \cdots & & & \\ & \ddots & & & & & & \\ & & 0 & q_i & r_i & p_i & 0 & \\ & & & \ddots & & & & \ddots \end{Vmatrix},$$

<u>Random Walk Processes with Reflecting Barrier:</u>

If **p_0 = 1** and therefore, **r_0 = 0**, we have a situation where the '**0' state** acts like a **reflecting barrier**. Whenever the process reaches the **state '0'**, in the next transition, automatically it **returns to the state '1'**. This corresponds to the physical process where an **elastic wall exists** at the **state '0'** and the process bounces off with no after-effects.

<u>Random Walk Processes with Absorbing Barrier:</u>

If $\mathbf{p_0 = 0}$ **and** $\mathbf{r_0 = 1}$, then the **state '0'** acts as an **absorbing barrier**. Once the process reaches the **state '0'**, it remains there **forever**.

<u>Random Walk with Reflecting, Absorbing, Or Partially Reflecting Barriers:</u>

If $\mathbf{p_0 > 0}$ **and** $\mathbf{r_0 > 0}$**,** then the **state '0'**, particullarly, is a **reflecting barrier**. When the random walk is restricted to a finite number of states 'S' say 0, 1, 2, ..., a, then **both the states '0' and 'a'** independently and in any combination may be **reflecting, absorbing, or partially reflecting** barriers.

<u>Random Walk Processes with two Absorbing Barriers:</u> Suppose a Gamblers Ruin is with **two adversaries with finite resources**, then its random walk is confined to the state space S where **'0' and 'a' are absorbing** states.

Gambler's Ruin Problem

Gambler's Ruin Problem is an example for the **Random Walk with two absorbing states 0 & k**. The transition probability matrix of the Random Walk can be writen as

$$P = \begin{array}{c} \\ 0 \\ k \\ 1 \\ 2 \\ \cdot \\ \cdot \\ k-1 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & k & 1 & 2 & \cdots & k-1 \end{array} \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & & 0 \\ 0 & 1 & 0 & 0 & & 0 \\ q & 0 & 0 & p & & 0 \\ 0 & 0 & q & 0 & & \\ & \cdots & & & & \cdots \\ & \cdots & & & & \cdots \\ 0 & p & 0 & 0 & & 0 \end{array} \right) \end{array}$$

so that

$$Q = \begin{array}{c} \\ 1 \\ 2 \\ \cdots \\ k-1 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & & \cdots & & k-1 \end{array} \\ \left(\begin{array}{cccccc} 0 & p & 0 & \cdots & 0 & 0 \\ q & 0 & p & \cdots & 0 & 0 \\ & \cdots & & \cdots & \cdots & \\ 0 & 0 & 0 & & q & 0 \end{array} \right) \end{array}$$

is a $(k-1) \times (k-1)$ matrix and

$$R = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ k-1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & k \end{array} \\ \left(\begin{array}{cc} q & 0 \\ 0 & 0 \\ & \cdots \\ 0 & p \end{array} \right) \end{array}$$

is a $(k-1) \times 2$ matrix.

Using the formula $a_{ik} = p_{ik} + \sum_{j\epsilon T} p_{ij}\, a_{jk}$ we get the absorption probability from a transient state 'i' to the absorbing state '0' as

$$a_{i0} = p_{i0} + \sum_{j=1}^{k-1} p_{ij}\, a_{j0}$$

and putting $i = 1, 2, \ldots, l, \ldots, k-1$, we get

$$a_{1,0} = q + p\, a_{2,0}$$

$$a_{2,0} = q\, a_{1,0} + p\, q_{3,0}$$

$$\cdots \qquad \cdots$$

$$a_{k-1,0} = q\, a_{k-2,0}$$

This can be written as

$$\left.\begin{aligned} a_{i,0} &= q\, a_{i-1,0} + p\, a_{i+1,0}, \quad 1 \le i \le k-1 \\ a_{0,0} &= 1,\ a_{k,0} = 0 \end{aligned}\right\}$$

with

For $1 \le i \le k-1$, the above difference equation admits the solution:

$$a_{i,0} = \frac{\left(\frac{p}{q}\right)^{k-i} - 1}{\left(\frac{p}{q}\right)^{k} - 1}, \quad p \ne q$$

$$= 1 - \frac{i}{k}, \qquad p = q$$

We get the absorption probabilities from a transient state i, $1 \le i \le k-1$ to the absorbing state k as

$$a_{i,k} = 1 - a_{i,0}, \qquad 1 \le i \le k-1.$$

It follows that, as $k \to \infty$

$$a_{i,0} \to 1, \quad \text{when } p \le \frac{1}{2}$$

$$\to (q/p)^i, \quad \text{when } p > \frac{1}{2}.$$

This implies that absorption at 0 (gambler's ruin) is certain if $p \le 1/2$, when k is large (the adversary is infinitely rich).

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