Unit	Topic
I	**1.1** Introduction to Stochastic Processes **1.2** Definition – Classification of Stochastic processes according to state space and time domain **1.3** Markov process **1.4** Markov chain **1.5** Countable State Markov Chain **1.6** Transition Probability Matrix **1.7** Chapman-Kolmogrov Equations **1.8** Calculation of 'n' step transition probability matrix.

Introduction:

In many social science and engineering sciences, it has been found that the phenomena that obeys Probabilistic laws are found to be more realistic rather than the phenomena that obey deterministic laws. In later stages it has also been found that such random phenomena are more suitable when they are considered over varying time points. Consequently, the concept of stochastic process has been developed for such situations in which phenomena are considered for varying time points.

A stochastic process is defined as a random phenomena that arises through a process which is developing in time in a manner controlled by probability laws.

From a Mathematical point of view a stochastic process is defined as a collection of r.v's that are indexed by some parameters, usually, times, some time space.

(Ex): Coin tossing problem:

Suppose a coin is tossed 'n' number of times and the number of successes (heads) is observed.

If 'X_n' represents the no. of successes in the first n trials, then clearly sequence of $\{X_n\}$ is a collection of r.v's which is a stochastic process.

Definition of Stochastic process:

A stochastic process is defined as a collection of r.v's that are indexed by some parameters usually time, some time space. This time points constituted a set called Indexed set (or) parameter set, denoted by T.

The r.v x_t indexed by time parameter 't' will have a specific value at time 't' say $x_t = \omega$, this possible value of a random variable is called state of the stochastic process at time 't'. All these states constitute a set called 'state space' denoted by 'S'.

Thus, the r.p is a function of two arguments namely, time point 'T' and state space 'S'.

Hence, Stochastic process can be specified in the form $\{x(t, \omega); t \in T, \omega \in S\}$

But, in general, the stochastic process is specified as $\{x_t; t \in T\}$.

classification of stochastic processes. (1)

A stochastic process $\{x_t ; t \in T\}$ over a state space 'S' can be classified into four types depending on the continuous (or) discrete nature of their state space 'S' and index set 'T'.

1) Discrete random sequence: If both 'S' and 'T' are discrete, the random process (or) stochastic process is called discrete random sequence.

Example: Let 'x_n' denote the outcome of the n th trial of a fair die. Here, $S = \{1, 2, 3, 4, 5, 6\}$ $T = \{1, 2, 3 \ldots\}$ and $\{x_t ; t = 1, 2, 3 \ldots\}$ is discrete random sequence.

2) Discrete random process: If 'S' is discrete and T is continuous, then the random process is called discrete random process.

Example: Let 'x_t' denote the number of telephone calls received in the interval $(0, t)$.
Here, $S = \{1, 2, 3 \ldots\}$, $T = \{t ; t \geq 0\}$ and collection of $\{x(t)\}$ is a discrete random process.

3) Continuous random sequence: If 'S' is continuous and T is discrete, the random process is called the continuous random sequence.

Example: let x_n = temperature at the end of the n^{th} hour of a day. Then 'S' is a continuous set (all possible values of temperature)
$T = \{1, 2, 3, \ldots\}$
$\therefore \{x_t ; t = 1, 2, \ldots 24\}$ is continuous random sequence.

(4)

4) Continuous random process: If both S and T are continuous, the random process is called continuous random process.

Example: let x_t = Maximum temperature at a place in an interval $(0, t)$. Here, 'S' is continuous set (representing temperature values), $\{x_t\}$ is a continuous random process.

Types of Stochastic Processes:

The random variables in a stochastic process may satisfy some sort of dependent relationship among themselves based on the various types of stochastic process.

They are
1) process with independent increments
2) Markov process
3) Normal process
4) Renewal process
5) Branching process
6) Stationary process..

Process with independent increments:

Consider a stochastic process $\{x_t : t \in T\}$ over a state space 's' for a finite set of time points, $t_1, t_2, \dots t_n$, $t_1 \le t_2 \le t_3 \dots \le t_n$. Consider the random variables $x_{t_1}, x_{t_2}, x_{t_3} \dots x_{t_n}$. If the increments $x_{t_2} - x_{t_1}, x_{t_3} - x_{t_2}, \dots x_{t_n} - x_{t_{n-1}}$ are all independent, then the process is said to be process with independent increments.

2) Markov process:

A Markov process is a stochastic process $\{x_t : t \in T\}$ satisfying Markovian dependent conditions (that is the value of the random variable at any time depends only on the most recently known value.

Suppose that, for any a, b, c, such that $a < b < c$ the random variable 'x_c' depends only on 'x_b', but not on 'x_a' for

Example: in tossing a coin, 'x_n' represents the number of heads occured in 'n' trails, then clearly 'x_n' depends only on x_{n-1}.

$\therefore$ the process $\{x_n\}$ is a Markov Process.

A Markov process classified on the basis of the index set 'T' and state space 'S'. the Markov process is said to be Markov Chain if its state space is countable (or) discrete.

The general classification of Markov process is given in the following table:

		State space (S)	
		Discrete	Continuous
Index set (T)	Discrete	Discrete time MC	Discrete time MP
	Continuous	Continuous time MC	Continuous time MP

Discrete Markov chain (countable) :-

A discrete time markov chain is a discrete time markov process, defined over a state space 'S' : $\{0, 1, 2 \ldots n\}$, if a discrete time, 't' defined over a discrete state space satisfy the Markov dependent condition, namely, the value of the random variable at any time depends only on the recently known value. In otherwords, it is defined as process satisfying the condition that for any time points, t_i where $i: 0, 1, 2, \ldots n$

$$P[x_{t_n} = a_n / x_{t_0} = a_0, x_{t_1} = a_1 \ldots x_{t_{n-1}} = a_{n-1}]$$
$$= P[x_{t_n} = a_n / x_{t_{n-1}} = a_{n-1}]$$

(Ex) In insurance risk problem, if x_n represents R company at the n^{th} year so that the x_{n+1} depends only on x_n is a discrete time MC - Process $\{x_n\}$

Absolute and Transition probabilities
(unconditional) (conditional)

consider, a discrete MC $\{x_n; n = 0, 1, 2 \ldots\}$ over a state space $S = \{0, 1, 2, \ldots\}$ consisting of integers. Suppose the process starts from the initial state $i \in S$ at time '0'. (ie). then the prob $P[x_0 = i] = P_i(0)$ is called Initial Probability. There probability of the process at same specified points are called Absolute (or) unconditional Probabilities which are given by

$P[X_n = j] = P_j^{(n)}$; $n = 0, 1, 2 \ldots$ (1)

$j \in S$

ie) $P_j(n)$ gives the prob that the process is in the state j at time 'n', at n^{th} state

consider the probabilies of the position of the process subject to the condition that they were started from some other state then their conditional probabilies are

consider the following:

$P[X_1 = j / X_0 = i] = P_{ij}^{(0,1)}$ or $P_{ij}^{(1)}$

$P[X_2 = j / X_0 = i] = P_{ij}^{(0,2)}$ or $P_{ij}^{(2)}$

$\vdots$

$P[X_n = j / X_0 = i] = P_{ij}^{(0,n)}$ or $P_{ij}^{(n)}$

The above equations gives the probability that the process having started from ($X_0 = i$) makes a shift to j ($X_n = j$) $n = 1, 2, 3 \ldots$ in first, second ... n^{th} step respectively. Since the give the probabilities of transition from one step to another, they are called transition probabilities.

If the transition probabilities do not depend on time at which the process is started, but depends only on time difference then such a property is said to be stationary property and the transition probabilities are called stationary probabilities.

Homogenous MC:

The MC with stationary transition probability is said to be time homogenous (or) Homogenous MC.

Transition Matrix:

For a discrete time MC $\{X_n ; n=0,1,2 \ldots\}$ over a state space $\{S=0,1,2\ldots\}$ the one step transition probabilities are given by $P_{ij} = P[X_1=j / X_0=i]$ (or) $= P[X_{m+1}=j / X_m=i]$

These probabilities satisfy, $\left.\begin{array}{l} P_{ij} \geq 0 \\ \sum_j P_{ij} = 1 \end{array}\right\} \rightarrow ①$

These probabilities constitute a matrix called transition probability matrix (or) simply transition matrix or stochastic matrix and is given by the equation ① which implies that each row total of the matrix is unity, if each column total is also unity (ie) $\sum_i P_{ij} = 1$, $\forall j$ then the matrix is called W stoch matrix (or) transition probability matrix.

Higher Order transition Probabilities:

The transition prob $P_{ij} = P[X_1=j / X_0=i]$ which states that having started from the state i, the process reaches the state 'j' in one step is known as one step transition prob. of the process, suppose it reaches state j in $2, 3 \ldots$ steps then we will have 2nd or 3rd etc step transition prob. The n-th step transition probability is given by $P_{ij}^{(n)} = P[X_n=j / X_0=i]$ and the corresponding matrix consisting n step transition prob is called n step transition matrix.

(5) Chapman - Kolmogrov Equations:

Statement:

If P is TPM Transition Probabilities Matrix (TPM) of a homogenous MC, then the 'n' step TPM P^n is equal to $P^{(n)}$

(ie) $\left[p_{ij}^{(n)}\right] = \left[p_{ij}\right]^n$ $P = (p_{ij})$

Proof:

Let $P = [P_{ij}]$ here $P_{ij}^{(1)}$

here P_{ij} is the prob of reaching state 'j' from state 'i' in one step.

Now, $P_{ij}^{(2)}$ is the Prob of reaching the state 'j' from state 'i' in two steps, through an intermediate state.

Now, we know that

$P(A/B\cap C) \times P(B/C)$

$= \frac{P(A\cap B\cap C)}{P(B\cap C)} \times \frac{P(B\cap C)}{P(C)}$

$= \frac{P(A\cap B\cap C)}{P(C)} = P[A\cap B/C]$

therefore $P_{ij}^{(2)} = P[X_2 = j / X_0 = i]$

$= P[X_2 = j, X_1 = k / X_0 = i]$

Using Markovian Property $P_{ij}^{(2)} = P[X_2 = j / X_1 = k, X_0 = i] \cdot P[X_1 = k / X_0 = i]$

$= P[X_2 = j / X_1 = k] \cdot P[X_1 = k / X_0 = i]$

$= P_{ik}^{(1)} \cdot P_{kj}^{(1)}$

(10) Now, the transition from state 'i' to state j can take place through any of the intermediate steps, and these various cases are mutually exclusive. Thus 'k' can assume values 1, 2, 3 ... etc, therefore

$P_{ij}^{(2)} = \sum_k P_{ik} P_{kj}$. by the definition of product of matrices, this means that i, j th element of 2-step TPM equal to the i, j th element of $P \times P$

$$P_{ij}^{(2)} = P^2$$

Similarly $P_{ij}^{(3)} = P[X_3 = j / X_0 = i]$

$$= \sum_k P[X_3 = j / X_2 = k] \cdot P[X_2 = k / X_0 = i]$$

$$P_{ij}^{(3)} = \sum_k P_{kj}^{(1)} P_{ik}^{(2)}$$

$$P_{ij}^{(3)} = P \cdot P^2 = P^3$$

Similarly we get $P^{(n)} = P^n$

Problem: ①

Three boys A, B, C are throwing a ball two each other. A always throws the ball to B, and B always throws the ball to C, But C is just as likely to throw the ball to B As A find the transition matrix and classify the states.

Sol:

$$P = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix} \end{array}$$

(4)

$$P^2 = P \times P$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0+0+0 & 0+0+0 & 0+1+0 \\ 0+0+1/2 & 0+0+1/2 & 0+0+0 \\ 0+0+0 & 1/2+0+0 & 0+1/2+0 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

$$P^3 = P^2 \times P$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0+0+1/2 & 0+0+1/2 & 0+0+0 \\ 0+0+0 & 1/2+0+0 & 0+1/2+0 \\ 0+0+1/4 & 0+0+1/4 & 0+1/2+0 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

Problem (2):

The one-step TPM of MC $\{X_n ; n = 0, 1, 2 \cdots\}$ having the state space $S = \{1, 2, 3\}$ is
$$\begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$
and the initial distribution is $\pi_0 = (0.7, 0.2, 0.1)$

i) Find $P[X_2 = 3 / X_0 = 1]$

ii) $P[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2]$

iii) $P[X_2 = 3]$

Sol:

$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix} \end{array}$$

$$P^2 = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

$$= \begin{pmatrix} 0.01 + 0.3 + 0.12 & 0.05 + 0.1 + 0.16 & 0.04 + 0.1 + 0.12 \\ 0.06 + 0.12 + 0.06 & 0.3 + 0.04 + 0.08 & 0.24 + 0.04 + 0.06 \\ 0.03 + 0.24 + 0.09 & 0.15 + 0.08 + 0.12 & 0.12 + 0.08 + 0.09 \end{pmatrix}$$

$$P^2 = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.25 & 0.29 \end{pmatrix} \end{array}$$

i) $P[X_2 = 3 / X_0 = 1]$

= element in $(1, 3)^{th}$ position in P^2

= 0.26.

ii) $P[X_3 = 2;\ X_2 = 3;\ X_1 = 3,\ X_0 = 2]$

$P[X_1 = 3 / X_0 = 2] = (2,3)^{rd}$ element in $P = 0.2$

$P[X_1 = 3,\ X_0 = 2] = P[X_1 = 3 / X_0 = 2] \cdot P[X_0 = 2]$

$[P(A \cap B) = P(A/B) \cdot P(B)]$

$= 0.2 \times 0.2 = 0.04$

$P[X_2 = 3,\ X_1 = 3,\ X_0 = 2]$

$= P[X_2 = 3 / X_1 = 3,\ X_0 = 2]$

$= P[X_2 = 3 / X_1 = 3]\ \ P[X_1 = 3,\ X_0 = 2]$ $\quad P(A/B\cap C) = P(A \cap B \cap C)$

$= 0.3 \times 0.04$

$= 0.012$

$P[X_3 = 2,\ X_2 = 3,\ X_1 = 3,\ X_0 = 2]$

$= P[X_3 = 2 / X_2 = 3,\ X_1 = 3,\ X_0 = 2] \cdot P[X_2 = 3, X_1 = 3, X_0 = 2]$

$= P[X_3 = 2 / X_2 = 3] \times 0.012$

$= 0.4 \times 0.012$ (Using Markov property)

$= 0.0048$

iii) $P[X_2 = 3]$

Now, $\pi_2 = \pi_0 P^2$

$$= (0.7\ \ 0.2\ \ 0.1)\begin{pmatrix} 0.43 & 0.31 & [illegible] \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & [illegible] \end{pmatrix}$$

$= (0.301 + 0.048 + 0.036\quad 0.217 + 0.084 + 0.035\quad 0.182 + 0.068 + [illegible])$

$= (0.385\quad 0.336\quad 0.279)$

$\therefore P[X_2 = 3]$

$= 0.279.$

(14) The TPM of a MC with 3 state $\{0, 1, 2\}$

$$P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}$$

and the Initial state distn of the chain is

$P[X_0 = i] = 1/3$,

i) Find $P[X_2 = 2]$

ii) $P[X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2]$

iii) $P[X_2 = 1, X_0 = 0]$

Sol:

$$P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix} \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}$$

$$= \begin{pmatrix} 9/16 + 1/16 + 0 & 3/16 + 1/8 + 0 & 0 + 1/16 + 0 \\ 3/16 + 1/8 + 0 & 1/16 + 1/4 + 3/16 & 0 + 1/8 + 1/16 \\ 0 + 3/16 + 0 & 0 + 3/8 + 3/16 & 0 + 3/16 + 1/16 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 10/16 & 5/16 & 1/16 \\ 5/16 & 8/16 & 3/16 \\ 3/16 & 9/16 & 4/16 \end{pmatrix}$$

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i) Find $P[X_2=2] = \pi_2 = \pi_0 (P^2)$

$$= \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{pmatrix} 10/16 & 5/16 & 1/16 \\ 5/16 & 8/16 & 3/16 \\ 3/16 & 9/16 & 4/16 \end{pmatrix}$$

$$= \left(\frac{10}{48} + \frac{5}{48} + \frac{3}{48} \quad \frac{5}{48} + \frac{8}{48} + \frac{9}{16} \quad \frac{1}{48} + \frac{3}{48} + \frac{4}{48} \right)$$

$$= \left(\frac{18}{48} \quad \frac{22}{48} \quad \frac{8}{48} \right) = \left(\frac{9}{24} \quad \frac{11}{24} \quad \frac{1}{6} \right)$$

$= 1/6.$

ii) $P[X_3=1, X_2=2, X_1=1, X_0=2]$

Consider,

$P[X_1=1 | X_0=2]$

$= P[X_1=1 / X_0=2] \cdot P[X_0=2] = 3/4 \times 1/3 = \frac{3}{12}$

$P[X_2=2/X_1=1] P[X_1=1/X_0=2] = 1/4 \times \frac{3}{12} = \frac{3}{48} = \frac{1}{16}$

$P[X_3=1, X_2=2, X_1=1, X_0=2]$

$= P[X_3=1 / X_2=2, X_1=1, X_0=2] \times P[X_2=2, X_1=1, X_0=2]$

$P[X_3=1 / X_2=2] \quad P[X_2=2, X_1=1, X_0=2]$

$= 3/4 \times 1/16 = \frac{3}{64}.$

iii) $P[X_2=1, X_0=0] = P[X_2=1 | X_0=0] \cdot P(X_0=0).$

$= 5/16 \times 1/3 = 5/48.$