Unit IV	Non- Central distributions
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4.1 Non-central t-distribution

t-distribution (Central t-distribution)

Let $x_i, i = 1,2,\ldots,n$ be a random sample size 'n' drawn from normal population with $N(\mu, \sigma^2)$ then the Student 't' is defined by

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

This statistic follows a student 't' distribution with $(n-1)$ df the pdf g 't' is given by

$$f(t) = \frac{1}{\sqrt{\gamma}} \cdot \frac{1}{\beta(1/2, \gamma/2)} \frac{1}{\left(1+\frac{t^2}{\gamma}\right)^{\frac{\gamma+1}{2}}} \quad ; -\infty < t < \infty$$

and $\gamma = n-1$

If $\gamma = 1$, the pdf g 't' reduced to

$$f(t) = \frac{1}{\beta(1/2 \cdot 1/2)} \cdot \frac{1}{1+t^2} \quad ; -\infty < t < \infty$$

$$= \frac{1}{\pi} \cdot \frac{1}{1+t^2} \quad , -\infty < t < \infty$$

Non-central t-distribution:

The non-central 't' distribution is the distribution the ratio of the normal variate with non-zero mean and unit variance to the root of an independent χ^2 variate defined by it density function.

ie, $x \sim N(\mu, 1)$ and $y \sim \chi^2_n$, then

$t = x/\sqrt{y/n}$ is said to have a non-central 't' distribution with 'n' d.f and the non-centrality parameter μ.

As $x \sim N(\mu, 1)$ we have

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}}$$

Also we have

$$f(y) = \frac{1}{2^{n/2}\Gamma n/2} \cdot e^{-y/2} \cdot y^{n/2 - 1}, \quad 0 \le y \le \infty$$

Since x and y are independent, the joint pdf x and y is $f(x, y) = f(x) \cdot f(y)$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}} \cdot \frac{1}{2^{n/2}\Gamma n/2} \cdot e^{-y/2} \cdot y^{n/2 - 1} \quad -①$$

Now

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-1/2(x^2+\mu^2-2x\mu)}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-1/2(x^2+\mu^2)} e^{x\mu}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-1/2(x^2+\mu^2)} \left\{ 1 + x\mu + \frac{(x\mu)^2}{2!} + \dots \right\}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-1/2(x^2+\mu^2)} \sum_{r=0}^{\infty} \frac{(x\mu)^r}{r!}$$

put this in ①, we get

$$f(x,y) = \frac{1}{\sqrt{2\pi}} e^{-1/2(x^2+\mu^2)} \sum_{r=0}^{\infty} \frac{(x\mu)^r}{r!} \frac{1}{2^{n/2}\Gamma n/2} e^{-y/2} y^{n/2-1} \quad -③$$

Let us introduce the variables 't' and z

$$t' = \frac{x}{\sqrt{y/n}} \quad \text{and } z^2 = y$$

$$\Rightarrow x = \frac{\sqrt{y}\, t'}{\sqrt{n}} \quad \text{and } y = z^2$$

$$\Rightarrow x = \frac{z t'}{\sqrt{n}} \quad \text{and } y = z^2$$

The Jacobian transformation is

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial t'} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial t'} & \frac{\partial y}{\partial z} \end{vmatrix} = \begin{vmatrix} z/\sqrt{n} & t/\sqrt{n} \\ 0 & 2z \end{vmatrix}$$

$$= \frac{2z^2}{\sqrt{n}}$$

∴ The joint pdf of t' and z from ② is

$$f(t',z) = \frac{1}{\sqrt{2\pi}} e^{-1/2\left[\frac{t'z^2}{n}+\mu^2\right]} \sum_{r=0}^{\infty} \frac{\left(\frac{z t' u}{\sqrt{n}}\right)}{r!} \frac{e^{-z^2/2}}{2^{n/2}\Gamma n/2} (z^2)^{n/2-1}$$

$$= \frac{1}{\sqrt{2\pi}\, 2^{n/2}\Gamma n/2} e^{-\mu^2/2} e^{-1/2\left[\frac{t'z^2}{n}+z^2\right]} [z^2]^{n/2} \sum_{r=0}^{\infty} \frac{\mu^r (t'z)}{n(r/2+1/2) r!}$$

$$= \frac{1}{\sqrt{2\pi}\, 2^{n/2} \Gamma \frac{n}{2}} e^{-\mu^2/2} \cdot e^{-z^2/2[t'^2/n+1]} \cdot [z^2]^{n/2} \sum_{r=0}^{\infty} \frac{\mu^r (t'z)^r}{n^{(r/2+1/2)} r!}$$

$$= \frac{1}{\sqrt{\pi}} \cdot \frac{1}{2^{\frac{n+1}{2}-1}} \frac{1}{\Gamma \frac{n}{2}} \cdot e^{-\mu^2/2} \cdot e^{-z^2/2[t'^2/n+1]} [z^2]^{n/2} \sum_{r=0}^{\infty} \frac{\mu^r (t'z)^r}{n^{\frac{r+1}{2}} r!}$$

$$= \frac{1}{\sqrt{\pi}} \frac{1}{2^{\frac{n+1}{2}-1}} \frac{1}{\Gamma \frac{n}{2}} e^{-\mu^2/2} e^{-z^2/2(t'^2/n+1)} \frac{\sum_{r=0}^{\infty} \mu^r t'^r z^{r+n}}{n^{(\frac{r+1}{2})} r!}$$

Integrate out z in the interval 0 to ∞.

The pdf of t' is

$$df(t') = \frac{e^{-\mu^2/2}}{\sqrt{\pi}\, 2^{\frac{n+1}{2}} \Gamma \frac{n}{2}} \cdot \sum \frac{\mu^r t'^r}{n^{r+1/2} r!} \int_0^{\infty} e^{-z^2/2 [\frac{t'^2}{n}+1]} z^{r+n} \, dt' \quad -(5)$$

$$= \int_0^{\infty} e^{-z^2/2(t'^2/n+1)} z^{r+1} \, dz$$

put $z^2 = u$; $2z\, dz = du \Rightarrow dz = \frac{du}{2z} = \frac{du}{2\sqrt{u}}$

$$= \frac{1}{2} \int_0^{\infty} e^{-u/2 (\frac{t'^2}{n}+1)} u^{(\frac{r+n}{2} - \frac{1}{2})} \, du$$

Let $\lambda = \frac{1}{2}\left(\frac{t'^2}{n}+1\right)$

$$df(t') = \frac{e^{-u^2/2}}{\sqrt{\pi}\, 2^{n+1/2} \Gamma \frac{n}{2}} \sum_{r=0}^{\infty} \frac{\mu^r t'^r}{n^{r+\frac{1}{2}} r!} \int_0^{\infty} e^{-\lambda} u^{\frac{r+n+1}{2}} \, du$$

$$= \frac{e^{-u^2/2}}{\sqrt{\pi}\, 2^{n+1/2} \Gamma \frac{n}{2}} \sum_{r=0}^{\infty} \frac{\mu^r t'^r}{n^{r+\frac{1}{2}} r!} \times \frac{\Gamma \frac{r+n+1}{2}}{\lambda^{r+n+1/2}}$$

$$= \frac{e^{-u^2/2}}{\sqrt{\pi}\, 2^{n+1/2} \Gamma \frac{n}{2}} \sum_{r=0}^{\infty} \frac{\mu^r (t')^r}{n^{r+1/2} r!} \frac{\Gamma \frac{r+n+1}{2}}{\left[\frac{1}{2}\left(\frac{t'^2}{n}+1\right)\right]^{\frac{n+r+1}{2}}}$$

Which is the pdf of non-central 't' distribution which 'n' d.f and non centrality parameter μ.

4.2 Non-central F-distribution

The ratio of two independent χ^2 variates divided by the corresponding d.f has a non-central F distribution if the numerator has a non-central χ^2 distribution.

Thus if x has a non-central χ^2 distribution with n_1 d.f

ie, $X \sim \chi'^2_{n_1}$ and y is an independent χ^2 variate with n_2 d.f ie, $y \sim \chi^2_{n_2}$ then the non-central F statistic is defined has

$$F' = \frac{X/n_1}{Y/n_2} = \frac{n_2 X}{n_1 Y}$$

Probability distribution function of F'

Since x and y are independent their j.p.d.f is given by $f(x,y) = f(x) \cdot f(y)$

$$f(x) = \frac{1}{2^{\left(\frac{n_1+2r}{2}\right)} \Gamma\frac{n_1+2r}{2}} e^{-x/2} x^{n_1/2+r-1} \sum_{r=0}^{\infty} \frac{e^{-\lambda}\lambda^r}{r!}, \ 0 \le x < \infty$$

$$f(y) = \frac{1}{2^{(n_2/2)} \Gamma n_2/2} e^{-y/2} y^{\frac{n_2}{2}-1}$$

$$f(x,y) = \frac{1}{2^{\left(\frac{n_1+2r}{2}\right)} \Gamma\frac{n_1+2r}{2}} e^{-x/2} x^{n_1/2+r-1} \sum_{r=0}^{\infty} \frac{e^{-\lambda}\lambda^r}{r!} \times \frac{1}{2^{n_2/2}\Gamma n_2/2} e^{-y/2} y^{n_2/2-1}$$

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda}\lambda^r}{r!} \frac{e^{-x/2} x^{n_1/2+r-1}}{2^{\left(\frac{n_1+2r}{2}\right)} \Gamma\frac{n_1+2r}{2}} \cdot \frac{e^{-y/2} y^{n_2/2-1}}{2^{n_2/2}\Gamma n_2/2}$$

Let us define the variate F' and v as

$$F' = \frac{n_2 x}{n_1 y} \quad \& \quad y = v$$

$$\Rightarrow x = \frac{n_1 F' y}{n_2} \quad \& \quad y = v$$

$$\Rightarrow x = \frac{n_1 F' v}{n_2} \quad \& \quad y = v$$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial F'} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial F'} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{n_1 v}{n_2} & \frac{n_1}{n_2} F' \\ 0 & 1 \end{vmatrix} = \frac{n_1}{n_2} v$$

The joint density is

$$f(F'v) = \sum_{r=0}^{\infty} \frac{e^{-\lambda}\lambda^r}{r!} \frac{e^{-\left(\frac{n_1}{2n_2} F'v\right)} \left(\frac{n_1}{n_2} F'v\right)^{n_1/2 + r - 1}}{2^{\left(\frac{n_1+2r}{2}\right)} \Gamma\left(\frac{n_1+2r}{2}\right)} \cdot \frac{e^{-v/2} (v)^{n_2/2 - 1}}{2^{n_2/2} \Gamma\frac{n_2}{2}} \cdot \frac{n_1}{n_2} v$$

$$= \sum_{r=0}^{\infty} \frac{\frac{e^{-\lambda}\lambda^r}{r!} \cdot e^{-\frac{v}{2}\left[\frac{n_1}{n_2} F' + 1\right]} \left(\frac{n_1}{n_2}\right)^{n_1/2 + r} F'^{\frac{n_1}{2}+r-1} \cdot v^{\frac{n_1+n_2}{2}+r-1}}{2^{n_1/2 + n_2/2 + r} \Gamma\frac{n_1+2r}{2} \Gamma\left(\frac{n_2}{2}\right)}$$

Integrating "x" in the range 0 to ∞ we get

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda}\lambda^r}{r!} \frac{\left(\frac{n_1}{n_2}\right)^{n_1/2 + r} (F')^{n_1/2 + r - 1}}{2^{\frac{n_1+n_2+2r}{2}} \Gamma\frac{n_1+2r}{2} \Gamma\frac{n_2}{2}} \int_0^\infty e^{-\frac{v}{2}\left(1+\frac{n_1}{n_2}F'\right)} v^{\frac{n_1+n_2}{2}+r-1} dv$$

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda}\lambda^r}{r!} \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}+r} (F')^{\frac{n_1}{2}+r-1}}{2^{\frac{n_1+n_2+2r}{2}} \Gamma\frac{n_1+2r}{2} \Gamma\frac{n_2}{2}} \frac{\Gamma\frac{n_1+n_2}{2}}{\left\{\frac{1}{2}\left(1+\frac{n_1}{n_2}F'\right)\right\}^{\frac{n_1+n_2+2r}{2}}}$$

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda}\lambda^r}{r!} \frac{\left(\left(\frac{n_1}{n_2}\right) F'\right)^{\frac{n_1+n_2}{2}+r-1} \left(\frac{n_1}{n_2}\right)}{\left(1+\frac{n_1}{n_2}F'\right)^{\frac{n_1+n_2}{2}+r} B\left(\frac{n_1}{2}+r, \frac{n_2}{2}\right)}$$

Which is the pdf of non-central F-distribution with non centrality parameter λ.

4.3 Non-central χ^2-distribution

The χ^2 distribution is defined as the sum of squares of independent standard normal variates and this often refered to as central χ^2 distribution.

The distribution of sum of squares of independent normal variates having unit variance but with non-zero means is known as non-central χ^2 distribution.

Thus if x_i's are independent normal $N(\mu_i, 1)$ random variable's and $\chi'^2 = \sum_{i=1}^{n} x_i^2$ as the non-central x_i^2 distribution with n degrees of freedom. This distribution would see to depend upon the n parameters $\mu_1, \mu_2, \dots \mu_n$ but it will be seen that it depends on these parameters only through the non-centrality parameter

$$\lambda = \frac{1}{2}(\mu_1^2 + \mu_2^2 + \dots + \mu_n^2) \text{ and we write}$$

$$\chi^2 \sim \chi^2(n, \lambda)$$

Non central χ^2 distribution with non-centrality parameter λ

pdf is given by

$$f(\chi_n'^2, \lambda) = \sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!} \left(p[\chi^2_{n+2j}]\right)$$

where $p(\chi_n'^2, \lambda)$ is a mixture of N.C χ^2 distribution with degrees of freedom $n, n_2, \dots n$

The corresponding weights being the terms of the poisson distribution with parameter λ.

Derivation of pdf of Non-central $\chi^2(\chi'^2)$

We obtain the pdf of non-central χ^2 distribution through mgf by using uniqueness theorem of mgf

If $X \sim N(\mu, 1)$, then $\mu_{x^2}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx^2} e^{-\frac{(x-\mu)^2}{2}} dx$ —①

$$\exp\left\{ tx^2 - \frac{1}{2}(x-\mu)^2 \right\} = \exp\left\{ -\left(\frac{1}{2} - t\right)x^2 - \mu x + \frac{\mu^2}{2} \right\}$$

Add and subtract $\frac{\mu^2}{(1-2t)^2}$ we get

$$\exp\left\{ -\left(\frac{1-2t}{2}\right)\left(x^2 - \frac{2\mu x}{1-2t} + \frac{\mu^2}{1-2t}\right) \right\}$$

$$= \exp\left\{ -\left(\frac{1-2t}{2}\right)\left(x^2 - \frac{2\mu x}{1-2t} + \frac{\mu^2}{1-2t}\right) + \frac{\mu^2}{(1-2t)^2} - \frac{\mu^2}{(1-2t)^2} \right\}$$

$$= \exp\left\{-\left(\frac{1-2t}{2}\right)\left[x^2 - \frac{2\mu x}{1-2t} + \frac{\mu^2}{1-2t} + \frac{\mu^2}{(1-2t)^2} - \frac{\mu^2}{(1-2t)^2}\right]\right\}$$

$$= \exp\left\{-\left(\frac{1-2t}{2}\right)\left[\left(x - \frac{\mu}{1-2t}\right)^2 + \frac{\mu^2}{1-2t} - \frac{\mu^2}{(1-2t)^2}\right]\right\}$$

$$= \exp\left\{-\left(\frac{1-2t}{2}\right)\left[\left(x - \frac{\mu}{1-2t}\right)^2 + \left(\frac{-2\mu^2 t}{(1-2t)^2}\right)\right]\right\}$$

$$= \exp\left[-\left(\frac{1-2t}{2}\right)\left(x - \frac{\mu}{1-2t}\right)^2 + \frac{\mu^2 t}{1-2t}\right]$$

$$= \exp\left(\frac{t\mu^2}{1-2t}\right)\exp\left[-\left(\frac{1-2t}{2}\right)\left(x - \frac{\mu}{1-2t}\right)^2\right]$$

$$M_{x^2}(t) = \exp\left(\frac{t\mu^2}{1-2t}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left\{-\left(\frac{1-2t}{2}\right)\left(x - \frac{\mu}{1-2t}\right)^2\right\}dx \quad -(1)$$

If $u = (1-2t)^{1/2}\left(x - \frac{\mu}{1-2t}\right)$

$$\frac{du}{dx} = (1-2t)^{1/2}$$

$$du = (1-2t)^{1/2}\,dx$$

$$= \exp\left(\frac{t\mu^2}{1-2t}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-u^2/2}\frac{du}{(1-2t)^{1/2}}$$

$$M_{x^2}(t) = \exp\left(\frac{t\mu^2}{1-2t}\right)\cdot\frac{1}{\sqrt{2\pi}}\cdot\frac{\sqrt{2\pi}}{(1-2t)^{1/2}}$$

$$= \frac{1}{(1-2t)^{1/2}}\exp\left(\frac{t\mu^2}{1-2t}\right)$$

$$= (1-2t)^{-1/2}\exp\left(\frac{t\mu^2}{1-2t}\right)$$

If x_i $(i=1,2,\dots,n)$ are independent normal $(\mu_i, 1)$ then m.g.f of non-central χ^2 variate $\chi'^2 = \sum_{i=1}^{n} x_i^2$ is given by

$$M_{\chi'}(t) = M_{\Sigma x_i^2}(t)$$

$$= \prod_{i=1}^{n} M_{x_i^2}(t)$$

$$= \prod_{i=1}^{n}(1-2t)^{-1/2}\, e^{(t\mu_i^2/1-2t)}$$

$$= (1-2t)^{-n/2} e^{\left(t/1-2t \sum_{i=1}^{n} \mu_i^2\right)}$$

$$= (1-2t)^{-n/2} e^{(2\lambda t/1-2t)} \quad ; \quad t < 1/2 \text{ where } \lambda = 1/2 \sum_{i=1}^{n} \mu_i^2$$

is the non-centrality parameter

$$= (1-2t)^{-n/2} e^{\{\lambda(-1+1/1-2t)\}}$$

$$= (1-2t)^{-n/2} e^{-\lambda} \cdot e^{\lambda/1-2t}$$

$$= (1-2t)^{-n/2} e^{-\lambda} \sum_{r=0}^{\infty} \left(\frac{\lambda}{1-2t}\right)^r \frac{1}{r!} \quad ; t < 1/2$$

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda}\lambda^r}{r!} (1-2t)^{-n/2 - r}$$

$$= (1-2t)^{-(n/2+r)} \sum_{r=0}^{\infty} \frac{e^{-\lambda}\lambda^r}{r!} \quad ; t < 1/2$$

Thus the m.g.f of non-central chi-square distribution is seen to be the complex combination of the x^2 m.g.f with $n, n_2+2, n+4, \ldots$ degrees of freedom. The coefficients appearing in the convex combination of the poisson probability

Hence by uniqueness theorem of m.g.f the p.d.f of non-central x^2 distribution with n-degrees of freedom and with non-centrality parameter λ is given by

$$f(x^2, \lambda) = \sum_{r=0}^{\infty} \frac{e^{-\lambda}\lambda^r}{r!} p(x^2_{n+2r})$$

$$\text{where } p\{x^2_{n+2r}) = \frac{1}{2^{(n+2r)/2} \left\lceil \frac{n+2r}{2} \right.} e^{-1/2 x^2} (x^2)^{n/2+r-1} \quad ; 0 \le x^2 < \infty$$

Is the pdf of central chi-square distribution with $n+2r$ degrees of freedom

Additive property (or) Reproductive property of non-central χ^2 distribution:

If Y_i $(i=1,2,\ldots,k)$ are independent non-central χ^2 variate with n_i d.f and non-centrality element λ_i, $\sum_{i=1}^{k} y_i$ is also a non-central χ^2 variate with, $n=\sum_{i=1}^{k} n_i$, $\lambda=\sum_{i=1}^{k} \lambda_i$

proof:

Let $y_i \sim \chi^2$ with n_i d.f and non-centrality parameter λ_i, then the m.g.f of y_i is

$$M_{\sum_{i=1}^{k} y_i}(t) = M_{y_1}(t) + M_{y_2}(t) + \cdots + M_{y_n}(t)$$

$$= \prod_{i=1}^{k} M_{y_i}(t)$$

$$= \prod_{i=1}^{k} (1-2t)^{-n_i/2} e^{(2t\lambda_i/1-2t)}$$

$$= (1-2t)^{-1/2 \sum_{i=1}^{k} n_i} \exp\left\{\frac{2t}{1-2t} \sum_{i=1}^{k} \lambda_i\right\}$$

which is the mgf of non-central χ^2 variate with Σn_i df and non-centrality parameter λ_i

Hence by uniqueness theorem of mgf

$\sum_{i=1}^{k} y_i \sim \chi^2$ with Σn_i df and non-centrality parameter

$$M_{\chi^2}(t) = (1-2t)^{-n/2} \exp\left\{2t\lambda/_{1-2t}\right\}$$

where $n=\sum_{i=1}^{k} n_i$ and $\lambda=\sum_{i=1}^{k} \lambda$

Cumulant generating function:

$$K_{\chi^2}(t) = \log M_{\chi^2}(t)$$

$$= \log\left((1-2t)^{-n/2} \exp\left\{\left(2t\lambda/_{1-2t}\right)\right\}\right)$$

$$= \frac{-n}{2}\log(1-2t) + 2\lambda t(1-2t)^{-1}$$

$$= -\frac{n}{2}\left[-2t - \frac{(2t)^2}{2} - \frac{(2t)^3}{3} + \ldots\right] + 2t\lambda\left[1 + 2t + (2t)^2 + \ldots\right)$$

$$= \left[nt + nt^2 + \frac{4nt^3}{3} + \frac{8nt^4}{4} + \ldots\right] + \left[2t\lambda + 4t^2\lambda + 8t^3\lambda + \ldots\right]$$

$$= t(n+2\lambda) + t^2(n+4\lambda) + t^3\left(\frac{4n}{3} + 8\lambda\right) + \ldots + t^r\left(\frac{2^{r-1}n}{r} + 2^r\lambda\right) + \ldots$$

The coefficient of $t^r = \dfrac{n2^{r-1}}{r} + 2^r\lambda$

$$= 2^{r-1}\left(\frac{n}{r} + 2\lambda\right)$$

The coefficient of $\dfrac{t^r}{r!} = 2^{r-1}\left(\dfrac{n\,r!}{r} + 2\lambda r!\right)$

$$= 2^{r-1}(r-1)!\left[n + \lambda r\right]$$

The r^{th} cumulant

k_r = coeff of $\dfrac{t^r}{r!}$ is $k_{\chi^2}(t) = 2^{r-1}(r-1)!\,(n + \lambda r)$

Put $r = 1 \Rightarrow k_1 = n + 2\lambda$

put $r = 2 \Rightarrow k = 2(n + 4\lambda) = 2n + 8\lambda$

Put $r = 3 \Rightarrow k = 4\cdot 2!(n + 6\lambda) = 8n + 48\lambda$

4.4 Relationships

Relation between F and χ^2. In $F\,(n_1, n_2)$ distribution if we let $n_2 \to \infty$, then $\chi^2 = n_1 F$ follows χ^2-distribution with n_1 d.f.

Proof. We have

$$p(F) = \frac{(n_1/n_2)^{n_1/2} F^{(n_1/2)-1}}{\Gamma(n_1/2)\,\Gamma(n_2/2)} \cdot \frac{\Gamma[(n_1+n_2)/2]}{\left[1 + \frac{n_1}{n_2}F\right]^{(n_1+n_2)/2}},\ 0 < F < \infty$$

In the limit as $n_2 \to \infty$, we have

$$\frac{\Gamma[(n_1+n_2)/2]}{n_2^{n_1/2}\,\Gamma(n_2/2)} \to \frac{(n_2/2)^{n_1/2}}{n_2^{n_1/2}} = \frac{1}{2^{n_1/2}}$$

$$\left[\because \frac{\Gamma(n+k)}{\Gamma(n)} \to n^k \text{ as } n \to \infty. (\textit{c.f.} \text{ Remark below.})\right]$$

Also
$$\lim_{n_2\to\infty}\left[1 + \frac{n_1}{n_2}F\right]^{(n_1+n_2)/2} = \lim_{n_2\to\infty}\left[\left(1 + \frac{n_1}{n_2}F\right)^{n_2}\right]^{1/2} \times \lim_{n_2\to\infty}\left(1 + \frac{n_1}{n_2}F\right)^{n_1/2}$$

$$= \exp(n_1F/2) = \exp(\chi^2/2) \qquad (\because n_1F = \chi^2)$$

Hence in the limit, the p.d.f. of $\chi^2 = n_1 F$ becomes

$$dP(\chi^2) = \frac{(n_1/2)^{n_1/2}\, e^{-\chi^2/2}}{\Gamma(n_1/2)} \cdot \left(\frac{\chi^2}{n_1}\right)^{(n_1/2)-1} d\left(\frac{\chi^2}{n_1}\right)$$

$$= \frac{1}{2^{n_1/2}\,\Gamma(n_1/2)} \cdot e^{-\chi^2/2}\,(\chi^2)^{(n_1/2)-1}\,d\chi^2,\ 0 < \chi^2 < \infty$$

which is the p.d.f. of chi-square distribution with n_1 d.f.

Relation between t and F distributions. In F-distribution with (ν_1, ν_2) d.f. [c.f. 14·5 (*a*)], take $\nu_1 = 1$, $\nu_2 = \nu$ and $t^2 = F$, *i.e.*, $dF = 2t\, dt$. Thus the probability differential of F transforms to

$$dG(t) = \frac{(1/\nu)^{1/2}}{B\left(\frac{1}{2}, \frac{\nu}{2}\right)} \cdot \frac{(t^2)^{\frac{1}{2}-1}}{\left[1 + \frac{t^2}{\nu}\right]^{(\nu+1)/2}}\, 2t\, dt, \quad 0 \le t^2 < \infty$$

$$= \frac{1}{\sqrt{\nu}\, B\left(\frac{1}{2}, \frac{\nu}{2}\right)} \cdot \frac{1}{\left[1 + \frac{t^2}{\nu}\right]^{(\nu+1)/2}}\, dt, \; -\infty < t < \infty$$

the factor 2 disappearing since the total probability in the range $(-\infty, \infty)$ is unity. This is the probability function of Student's t-distribution with ν d.f. Hence we have the following relation between t and F distributions.

'*If a statistic t follows Student's t distribution with n d.f., then t^2 follows Snedecor's F-distribution with $(1, n)$ d.f. Symbolically,*

$$\left.\begin{array}{ll} \text{if} & t \sim t_{(n)} \\ \text{then} & t^2 \sim F_{(1,n)} \end{array}\right\} \qquad \ldots(1)$$

Aliter Proof of (1). If $\xi \sim N(0, 1)$ and $X \sim \chi^2_{(n)}$ are independent *r.v.*'s then :

$$U = \xi^2 \sim \chi^2_{(1)} \qquad \text{[Square of a S.N.V.]}$$

and
$$t = \frac{\xi}{\sqrt{X/n}} \sim t_{(n)}$$

$$\Rightarrow \qquad t^2 = \frac{\xi^2}{(X/n)} = \frac{(\xi^2/1)}{(X/n)},$$

being the ratio of two independent chi-square variates divided by their respective degrees of freedom is $F(1, n)$ variate.

Hence $\qquad t^2 \sim F(1, n)$

With the help of relation (14·19), all the uses of t-distribution can be regarded as the applications of F-distribution also, *e.g.*, for test for a single mean, instead of computing

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}},$$

we may compute

$$F = t^2 = \frac{n(\bar{x} - \mu)^2}{S^2}$$

and then apply F-test with $(1, n)$ d.f. and so on.

4.5 Sampling Distribution of Simple correlation co-efficient for null case

Sampling distribution of correlation coefficient (r):

Let x and y follows Bivariate normal distribution with parameter $\mu_x = \mu_y = 0,\ \sigma_x^2, \sigma_y^2$ and $\rho = 0$

Let us select a random sample of size 'n' from the above population. Let $\bar{x}$ and $\bar{y}$ be the mean, S_x^2 and S_y^2 be the variance of x and y respectively in the sample.

Let r be the correlation coefficient between x and y and is given by,

$$r = \frac{1}{n} \frac{\Sigma(x-\bar{x})(y-\bar{y})}{S_x S_y} \quad -① $$

Let us introduce a new variable η_i using the orthogonal transformation $\eta = ay$

where $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ and $a = (a_{ij})_{n \times n}$

Such that $\sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} \eta_i^2 \quad -②$

Let m assume that $a_{11} = a_{12} = \cdots = a_{1n} = \frac{1}{\sqrt{n}}$

so that $\eta_1 = \frac{1}{\sqrt{n}} y_1 + \frac{1}{\sqrt{n}} y_2 + \cdots + \frac{1}{\sqrt{n}} y_n$

$$= \frac{1}{\sqrt{n}}(y_1 + y_2 + \cdots + y_n) = \frac{n\bar{y}}{\sqrt{n}} = \sqrt{n}\,\bar{y}$$

or $\eta_1^2 = n\bar{y}^2 \quad -③$

Consider $nS_y^2 = \sum_{i=1}^{n}(y_i - \bar{y})^2$

$$\therefore \Sigma y_i^2 - n\bar{y}^2 = \sum_{i=1}^{n} \eta_i^2 - \eta_1^2 = \sum_{i=1}^{n} \eta_i^2 \qquad \because \Sigma a_i^2 = 1 = \Sigma\left(\frac{1}{\sqrt{n}}\right)^2$$

We know that

$$\frac{nS_y^2}{\sigma_y^2} \sim \chi^2_{(n-1)} \quad \text{ie, } \sum_{i=1}^{n} \frac{\eta_i^2}{\sigma_y^2} \sim \chi^2_{(n-1)}$$

From ① $\sqrt{n}\, r\, S_y = \frac{1}{\sqrt{n}} \frac{\Sigma(x-\bar{x})(y-\bar{y})}{S_x}$

$$= \frac{\Sigma(x-\bar{x})y_i}{\sqrt{n}\,S_x} \qquad \because \Sigma(x-\bar{x})\bar{y} = 0$$

$$\Rightarrow \sqrt{n}\, r s_y = n\eta_2 \text{ (say)}$$

$$\Rightarrow \eta_2^2 = n r^2 s_y^2$$

We know that, $\dfrac{\eta_2^2}{\sigma_y^2} = \dfrac{r^2 n s_y^2}{\sigma_y^2} = \chi^2_{(1)}$ as η_2 is also normal

Consider, $n s_y^2 = (r^2 + 1 - r^2) n s_y^2 = r^2 n s_y^2 + (1-r^2) n s_y^2$ - ⑤

$$\frac{n s_y^2}{\sigma_y^2} = \frac{\sigma_n^2 s_y^2}{\sigma_y^2} + \frac{(1-r^2) n s_y^2}{\sigma_y^2}$$

$$\chi^2_{(n-1)} = \chi_1^2 + \chi_{n-2} \quad -⑥$$

By the converse of the additive property of χ^2 consider

$$r^2 = \frac{r^2 n s_y^2}{n s_y^2} = \frac{r^2 n s_y^2}{r^2 n s_y^2 + (1-r^2) n s_y^2} \quad \text{from ⑤}$$

$$= \frac{\chi_1^2}{\chi_1^2 + \chi_{n-2}^2} \quad \text{from ⑥}$$

We know that

$$\frac{\chi_1^2}{\chi_1^2 + \chi_{n-2}^2} \sim \beta_1\left(\frac{1}{2}, \frac{n-2}{2}\right)$$

Sampling distribution of $r^2 = f(r^2)\, dr^2$

$$= \frac{(r^2)^{1/2 - 1} (1-r^2)^{\frac{n-2}{2} - 1}}{\beta_1\left(\frac{1}{2}, \frac{n-2}{2}\right)} dr^2 \quad ; \; 0 \le r^2 \le 1$$

$$= \frac{(1-r^2)^{\frac{n-4}{2}}\, dr}{\beta_1\left(\frac{1}{2}, \frac{n-2}{2}\right)} \quad ; \; -1 \le r \le 1 \text{ of which the pdf of } (r)$$

$$\therefore \text{note } t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}} \sim t(n-1)$$

4.6 Sampling Distribution of Simple Regression co-efficient.

Let x and y have bivariate normal distribution with S.D σ_x σ_y respectively.

Let us select a random sample of size 'n' from the above population. we know that

$b = r \cdot \frac{s_y}{s_x}$ is the regression coefficient of y on x in the sample.

$$b^2 = \frac{r^2 s_y^2}{s_x^2} = \frac{n r^2 s_y^2}{n s_x^2} = \frac{n r^2 s_y^2/\sigma_y^2}{n r^2 s_x^2/\sigma_x^2}$$

$$= \frac{\chi_1^2}{\chi_{n-1}^2} \sim \beta_2\left(\frac{1}{2}, \frac{n-1}{2}\right)$$

The distribution of $b^2 \frac{s_y^2}{s_x^2}$ as

$$f\left(b^2 \frac{\sigma_y^2}{\sigma_x^2}\right) d b^2 \frac{\sigma_y^2}{\sigma_x^2} = \frac{1}{\beta\left(\frac{1}{2}, \frac{n-1}{2}\right)} \frac{b^2 \frac{\sigma_y^2}{\sigma_x^2}}{\left(1 + \frac{b^2 \sigma_y^2}{\sigma_x^2}\right)} \left(\frac{1}{2}, \frac{n-1}{2}\right). \quad db^2\left(\frac{\sigma_y^2}{\sigma_x^2}\right)$$

pdf of b as

$$f(b)\,db = \frac{1}{\beta\left(\frac{1}{2}, \frac{n-1}{2}\right)} \frac{\left(b^2 \frac{\sigma_y^2}{\sigma_x^2}\right)^{1/2 - 1}}{\left(1 + b^2 \frac{\sigma_y^2}{\sigma_x^2}\right)^{\frac{-1}{2}, \frac{n-1}{2}}} \cdot \frac{b\,\sigma_y^2}{\sigma_x^2}\, 2b\,db ;\ 0 \le b \le \infty, \ , 0 \le b^2 \frac{\sigma_y^2}{\sigma_x^2} < \infty$$

$$= \frac{1}{\beta\left(\frac{1}{2}, \frac{n-1}{2}\right)} \frac{(\sigma_x^2/\sigma_y^2)^{1/2}}{\left(1 + b^2 \frac{\sigma_y^2}{\sigma_x^2}\right)^{n/2}}\, db \ ; 0 \le b \le \infty$$

$$= \frac{1}{\beta\left(\frac{1}{2}, \frac{n-1}{2}\right)} \frac{(\sigma_x \cdot \sigma_y)\, \sigma_x^n}{(\sigma_x^2 + b\,\sigma_y^2)^{n/2}}\, db = \frac{1}{\beta} \frac{\sigma_x^{n+1}/\sigma_y}{\sigma_x^2 + b^2 \sigma_y^2}\, db.$$

$$f(b_{xy})\, db_{xy} = \frac{1}{\beta\left(\frac{1}{2}, \frac{n-1}{2}\right)} \cdot \frac{\sigma_x \sigma_y^{n-1}}{(\sigma_y^2 + b^2 \sigma_n^2)^{n/2}}$$

$$\text{Mean of } (b) = \int_{-\infty}^{\infty} b f(b)\, db = \int \frac{b}{\beta\left(\frac{1}{2}, \frac{n-1}{2}\right)} \frac{\sigma_x \sigma_y^{n-1}}{(\sigma_y^2 + b^2 \sigma_x^2)^{n/2}}\, db$$

$$= \frac{\sigma_x \sigma_y^{n-1}}{\beta\left(\frac{1}{2}, \frac{n-1}{2}\right)} \int \frac{b}{(\sigma_y^2 + \sigma_x^2)^{n/2}} \, db$$

$= 0 = \beta$ [$\because$ The integrant is odd]

Mean of $f(r)$

$$E(r) = \int_{-1}^{1} r f(r)\, dr = \int \frac{r(1-r^2)^{\frac{n-4}{2}}}{\beta\left(\frac{1}{2}, \frac{n-1}{2}\right)} dr = 0 = \rho$$

$\therefore$ integrant is an odd $= 0$

Variance :

$$V(r) = E(r-\rho)^2 = E(r^2) - \rho = 0$$

$$= \int_{-1}^{1} r^2 f(r) dr = \int_{-1}^{1} \frac{r^2 (1-r^2)^{\frac{n-4}{2}}}{\beta\left(\frac{1}{2}, \frac{n-2}{2}\right)} dr$$

$$= \frac{2}{\beta\left(\frac{1}{2}, \frac{n-2}{2}\right)} \int_0^1 r^2 (1-r^2)^{\frac{n-4}{2}} dr$$

$$= \frac{2}{\beta\left(\frac{1}{2}, \frac{n-2}{2}\right)} \int_0^1 \theta (1-\theta)^{\frac{n-4}{2}} \frac{d\theta}{2\sqrt{\theta}}$$

$$= \frac{1}{\beta\left(\frac{1}{2}, \frac{n-2}{2}\right)} \int_0^1 \theta^{1/2} (1-\theta)^{\frac{n-4}{2}} d\theta$$

$\therefore$ Put $r^2 = \theta$

$2r\,dr = d\theta \Rightarrow dr = \frac{d\theta}{2r} = \frac{d\theta}{2\sqrt{\theta}}$

$$= \frac{1}{\beta\left(\frac{1}{2}, \frac{n-2}{2}\right)} \int_0^1 \theta^{3/2 - 1} (1-\theta)^{\frac{n-4}{2} - 1} d\theta$$

$$= \frac{\beta\left(\frac{3}{2}, \frac{n-2}{2}\right)}{\beta\left(\frac{1}{2}, \frac{n-2}{2}\right)}$$

$$= \frac{\Gamma\frac{3}{2}\, \Gamma\frac{n-2}{2} \Big/ \Gamma\left(\frac{3}{2} + \frac{n-2}{2}\right)}{\Gamma\frac{1}{2}\, \Gamma\frac{n-2}{2} \Big/ \Gamma\left(\frac{1}{2} + \frac{n-2}{2}\right)}$$

$$= \frac{\Gamma 3/2 \,/\, \Gamma \frac{n+1}{2}}{\Gamma 1/2 \; \Gamma \frac{n-1}{2}} = \frac{\frac{1}{2}\Gamma\frac{1}{2}\,\Gamma\frac{n-1}{2}}{\Gamma\frac{1}{2}\;\Gamma\frac{n+1}{2}}$$

$$= \frac{\frac{1}{2}\,\Gamma\frac{n-1}{2}}{\frac{n-1}{2}\,\Gamma\frac{n-1}{2}} = \frac{1}{2}\,\frac{2}{n-1} = \frac{1}{n-1}$$

Mode (b)

$$f(b') = \frac{\sigma_x \sigma_y^{n-1}}{\beta(\frac{1}{2}, \frac{n-1}{2})} = \frac{\sigma_x \sigma_y^{n-1}}{\beta(\frac{1}{2}, \frac{n-1}{2})} \left(-\frac{n}{2}\right)\left(\sigma_y^2 + b^2\sigma_x^2\right)^{-\frac{n}{2}-1}$$

$$2b\sigma_x^2 = 0 \text{ (say)}$$

$b = 0$. which is the mode

$\therefore$ mean = mode = 0. The curve of b is symmetric