UNIT - I

Distribution function - Def

Let x be a random variable. The function F defined for all real x by

$$F(x) = P[X \leq x] = P[\omega : X(\omega) \leq x], \quad -\infty < x < \infty$$

is called the distribution function (d.f) of the random variable X.

Properties.

1. If F is the d.f of the r.v X and if $a < b$ then $P(a < X \leq b) = F(b) - F(a)$

Proof: The events '$a < X \leq b$' and '$X \leq a$' are disjoint and their union is the event '$X \leq b$'. Hence by addition theorem of probability:

$$P(a < X \leq b) + P(X \leq a) = P(X \leq b)$$

$$\Rightarrow \quad P(a < X \leq b) = P(X \leq b) - P(X \leq a)$$

$$= F(b) - F(a)$$

Cor. 1:
$$P(a \leq X \leq b) = P[(X = a) \cup (a < X \leq b)]$$
$$= P(X = a) + P(a < X \leq b)$$
$$= P(X = a) + F(b) - F(a).$$

Similarly, we get

$$P(a < X < b) = P(a < X \leq b) - P(X = b)$$
$$= F(b) - F(a) - P(X = b)$$

$$P(a \leq X < b) = P(a < X < b) + P(X = a)$$
$$= F(b) - F(a) - P(X = b) + P(X = a).$$

2. If F is d.f of one-dimensional r.v. X, then
i) $0 \leq F(x) \leq 1$, ii) $F(x) \leq F(y)$ if $x < y$.

In otherwords, all the distribution functions are monotonically non-decreasing and lie between 0 and 1.

3. If F is d.f of one-dimensional r.v X, then

$$F(-\infty) = \lim_{x \to \infty} F(x) = 0 \quad \text{and} \quad F(\infty) = \lim_{x \to \infty} F(x) = 1$$

Proof: Let us express the whole sample space S as a countable union of disjoint as follows;

$$S = \left\{ \bigcup_{n=1}^{\infty} (-n < X \leq -n+1) \right\} \cup \left\{ \bigcup_{n=0}^{\infty} (n < X \leq n+1) \right\}$$

$$\Rightarrow \quad P(S) = \sum_{n=1}^{\infty} P(-n < X \leq -n+1) + \sum_{n=0}^{\infty} P(n < X \leq n+1)$$

$$1 = \lim_{a \to \infty} \sum_{n=1}^{a} \{F(-n+1) - F(-n)\} + \lim_{b \to \infty} \sum_{n=0}^{b} \{F(n+1) - F(n)\}$$

$$1 = \lim_{a\to\infty}\left[F(0) - F(-a)\right] + \lim_{b\to\infty}\left[F(b+1) - F(0)\right]$$

$$= \left[F(0) - F(-\infty)\right] + \left[F(\infty) - F(0)\right]$$

$$1 = F(\infty) - F(-\infty) \qquad \text{——— (1)}$$

Since $-\infty < \infty$, $F(-\infty) \leq F(\infty)$. Also $F(-\infty) \geq 0$ and $F(\infty) \leq 1$

$$\therefore\ 0 \leq F(-\infty) \leq F(\infty) \leq 1 \qquad \text{——— (2)}$$

From (1) and (2), we get

$$F(-\infty) = 0 \quad \text{and} \quad F(\infty) = 1.$$

Remarks: i) Discontinuities of $F(x)$ are at most countable.

ii) $F(a) - F(a-0) = \lim_{h\to 0} P(a-h \leq x \leq a),\ h<0$

and $F(a+0) - F(a) = \lim_{h\to 0} P(a \leq x \leq a+h) = 0,\ h>0.$

Transformation of one - Dimensional Random variable.

Given the probability density function of a r.v X, to determine the probability density function of a new r.v $Y = g(X)$', the following simple theorem is used.

Theorem: Let X be a continuous r.v with p.d.f $f_X(x)$.

Let $Y = g(x)$ be strictly monotonic (increasing or decreasing) function of x. Assume that $g(x)$ is differentiable (and hence continuous) for all x. Then the p.d.f of $h(\cdot)$ of the r.v Y is given by

$$h_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

where x is expressed in terms of y and the range of Y is determined from the given range of the variable X, on using the transformation $y = g(x)$.

Proof: case i) $y = g(x)$ is strictly increasing function of x (i.e, $\frac{dy}{dx} > 0$). The d.f of Y is given by

$$H_Y(y) = P[Y \leq y] = P[g(X) \leq y] = P[X \leq g^{-1}(y)]$$

the inverse exists and is unique, since $g(\cdot)$ is strictly increasing.

$$\therefore\ H_Y(y) = F_X(g^{-1}(y)], \text{ where } F \text{ is the d.f of } X.$$

$$= F_X(x)$$

Differentiating w.r.t. y, we get

$$h_Y(y) = \frac{d}{dy}\left[F_X(x)\right] = \frac{d}{dx}\left[F_X(x)\right]\frac{dx}{dy} = f_X(x)\cdot\frac{dx}{dy} \quad (1)$$

Case ii) $y = g(x)$ is strictly monotonic decreasing function of x.

$$H_Y(y) = P[Y \le y] = P[g(x) \le y] = P[x \geq g^{-1}(y)]$$

$$= 1 - P[x \le g^{-1}(y)] = 1 - F_X[g^{-1}(y)]$$

$$= 1 - F_X(x),$$

where $x = g^{-1}(y)$, the inverse exists and is unique.

Differentiating w.r.t to y, we get

$$h_Y(y) = \frac{d}{dx}\left[1 - F_X(x)\right]\frac{dx}{dy} = -f_X(x)\cdot\frac{dx}{dy}$$

$$= f_X(x) - \frac{dx}{dy} \quad (2)$$

Note that the algebraic sign (−ve) obtained in (2) is correct, since y is a decreasing function of x $\Rightarrow$ x is a decreasing function of y or $\frac{dx}{dy} < 0$.

The results (1) and (2) can be combined to give

$$h_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|.$$

Transformation of two dimensional Random variable.

Let us consider the problem of change of variables in the two dimensional case.

Let U and V be two random variables. Let these two random variables be transformed to the r.v's X and Y by the transformation

$$u = u(x, y)$$

$$v = v(x, y),$$

where u and V are continuously differentiable functions for which Jacobian transformation

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

is either > 0 or < 0 throughout the (x, y) plane so that the inverse transformation is uniquely given by $x = x(u, v)$, $y = y(u, v)$.

Theorem:

The joint p.d.f $g_{uv}(u,v)$ of the transformed variables U and V is

$$g_{UV}(u,v) = f_{xy}(x,y)\ |J|$$

where $|J|$ is the modulus value of the Jacobian of transformation and $f(x,y)$ is expected in terms of u and v.

Proof: $P(x < X \leq x+dx,\ y < y \leq y+dy) =$

$$P(u < U \leq u+du,\ v < v \leq v+dv)$$

$$\Rightarrow f_{xy}(x,y)\,dx\,dy = g_{UV}(u,v)\,du\,dv$$

$$\Rightarrow g_{UV}(u,v)\,du\,dv = f_{xy}(x,y)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$g_{UV}(u,v) = f_{xy}(x,y)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = f_{xy}(x,y)\ |J|.$$

Distribution of Sum of Random variables.

Theorem: If X and Y are independent continuous random variables, then the p.d.f of $U = X+Y$ is given by

$$h(u) = \int_{-\infty}^{\infty} f_x(v)\, f_y(u-v)\,dv \qquad (1)$$

Proof: Let X and Y be independent continuous random variables with joint p.d.f $f_{xy}(x,y)$.

Let us make the transformation

$$u = x+y,\quad v = x$$

$$\Rightarrow x = v \quad \text{and} \quad y = u - x$$

$$= u - v$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$$

Thus the joint p.d.f of r.v's U and V is given by

$$g_{UV}(u,v) = f_{xy}(x,y)\ |J|$$

$= f_X(x) \cdot f_Y(y) \cdot |J|$ since X and Y are independent

$= f_X(v) \cdot f_Y(u-v)$

note: The marginal density of U is given by

$$h(u) = \int_{-\infty}^{\infty} g_{UV}(u,v)\, dv = \int_{-\infty}^{\infty} f_X(v)\, f_Y(u-v) \cdot dv.$$

Distribution of the Difference of Two Random Variables.

Theorem: If X and Y are independent continuous random variables, then the p.d.f of $U = X - Y$ is given by

$$h(u) = \int_{-\infty}^{\infty} f(x, x+u)\, dx = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(x+u)\, dx.$$

Proof: Let X and Y are two independent random variables. Let us transform (X, Y) to (U, V) by the transformation:

$x - y = u$ and $y = v \Rightarrow x = u + y$, $y = v$.
$\qquad = u + v$

The Jacobian of transformation is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

Hence the joint p.d.f of U and V becomes,

$$g_{UV}(u,v) = f_{XY}(x,y)\, |J|$$

$$= f(u+v, v) \cdot 1$$

Integrating w.r.t v, the p.d.f of U is given by

$$g_1(u) = \int_{-\infty}^{\infty} f(u+v, v)\, dv$$

If X and Y are independent random variables, then the p.d.f of $U = X - Y$ is given by

$$g(u) = \int_{-\infty}^{\infty} f_X(u+v) \cdot f_Y(v) \cdot dv.$$

Distribution of the Product of two random variables.

Theorem: Let X and Y be two independent continuous random variables. Then the p.d.f of their product $U = XY$ is given by

$$h(u) = \int_{-\infty}^{\infty} f_{XY}\left(x, \frac{u}{x}\right) \frac{dx}{|x|}$$

Proof: Let us transform (x, y) to (u, v) by the transformation

$U = XY$ and

$V = X$

So that $x = v$ and $y = \frac{u}{x} = \frac{u}{v}$ —(1)

Jacobian transformation is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix}$$

$$= -\frac{1}{v}$$

Hence the joint p.d.f of U and V is given by

$$g(u,v) = f(x,y)\,|J| = f\left(u, \frac{u}{v}\right).\left|\frac{1}{v}\right|$$

Integrating w.r.t. v, the p.d.f of $U = XY$ is given by

$$g(u) = \int_{-\infty}^{\infty} f\left(v, \frac{u}{v}\right) \frac{dv}{|v|} = \int_{-\infty}^{\infty} f\left(x, \frac{u}{x}\right) \frac{dx}{|x|}$$

$$= \int_{-\infty}^{\infty} f_X(x).\, f_Y\left(\frac{u}{x}\right). \frac{dx}{|x|}$$ if X and Y are independent.

The range of U is determined from the given range of X and Y on using the information given in (1)

Distribution of the Quotient of Two Random Variables.

If X and Y are independent continuous random variables, then the p.d.f of the quotient $Z = X/Y$, is given by

$$g(z) = \int_{-\infty}^{\infty} g(vz, v)\,|v|\,dv$$

$$= \int_{-\infty}^{\infty} f_X(vz)\, f_Y(v)\,|v|\,dv \quad (1)$$

Proof: Let us transform (x,y) to (z,v) by the transformation $z = x/y$ and $v = y$ so that

$$z = \frac{x}{y}, \quad v = y \implies \begin{array}{l} x = vz \\ y = v \end{array} \qquad (\alpha)$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & 0 \\ z & 1 \end{vmatrix} = v$$

Hence the joint p.d.f of z and v given by

$$g(z,v) = f(x,y)\,|J|$$

$$= f(vz, v).\,|v|$$

Integrating w.r.t. v, the p.d.f of $z = x/y$ is given by

$$g(z) = \int_{-\infty}^{\infty} f(vz, v)\,|v|\,dv$$

$$= \int_{-\infty}^{\infty} f_X(vz).\,f_Y(v).\,|v|\,dv \quad \text{if } X \text{ and } Y \text{ are independent.}$$

The new range of the variable z is determined from the range of the variables X and Y, on using the transformation (α).

Example. 1 Let X and Y are two independent standard normal variates. Find the distribution of their sum $Z = X+Y$.

Sol: It is given that $X \sim N(0,1)$ and $Y \sim N(0,1)$

$$\therefore f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{and} \quad f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

$$-\infty < x, y < \infty$$

Consider the joint density of x & y

$$f(x,y) = f(x).\,f(y) \quad \text{since } x \text{ and } y \text{ are independent.}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} . \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

The distribution of $z = x+y$ is

$$f(z) = P(Z \le z) = \int_{-\infty}^{z} f(z)\,dz$$

$$= \int_{-\infty}^{\infty} f(x+y)\,dz$$

$$= \int_{-\infty}^{z} \int_{-\infty}^{\infty} f(x, u-x)\,dx \quad \text{(as per theorem)}$$

$$= \int_{-\infty}^{z}\int_{-\infty}^{\infty} f(x).\, f(z-x)\, dx.\,dz$$ since x and y are independent.

$$= \int_{-\infty}^{z}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \,.\, \frac{1}{\sqrt{2\pi}} \,.\, e^{-\frac{(z-x)^2}{2}}\, dx.\,dz$$

$$= \frac{1}{2\pi}\int_{-\infty}^{z}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \,.\, e^{-\frac{1}{2}(z^2-2zx+x^2)}\, dx\, dz$$

$$= \frac{1}{2\pi}\int_{-\infty}^{z}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(2x^2+z^2-2zx)}\, dx\, dz$$

$$= \frac{1}{(\sqrt{2\pi})^2}\int_{-\infty}^{z}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{z^2}{2}\right)} \,.\, e^{-\frac{1}{2}\left(\sqrt{2}x - \frac{z}{\sqrt{2}}\right)^2}\, dx.\,dz$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}\,.\,\frac{z^2}{2}}\, dz \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\left(\sqrt{2}x-\frac{z}{\sqrt{2}}\right)^2}}{\sqrt{2\pi}}\, dx$$

Let $u = \sqrt{2}x - \frac{z}{\sqrt{2}}$ $\quad du = \sqrt{2}\,x\,.\,dx \Rightarrow dx = \frac{du}{\sqrt{2}}$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}\frac{z^2}{2}}\,.\, dz \int_{-\infty}^{\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \,.\, \frac{du}{\sqrt{2}}\,.$$

$$= \frac{1}{\sqrt{2}}\int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z^2}{2}\right)}\, dz \,.\, \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-u^2/2}\,.\, du$$

$$f(z). = \frac{1}{\sqrt{2}\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}\left(\frac{z^2}{2}\right)}\, dz$$

2. If X and Y are two independent standard normal variates; find the distribution function of their difference $Z = x - y$.

Sol: Given $X \sim N(0,1)$ and $Y \sim N(0,1)$ and are independent.

The joint density function is given by

$$\int f(x,y)\,dx\,dy = \int f(x).f(y).\,dx\,dy$$

$$= \int \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} . \frac{1}{\sqrt{2\pi}} . e^{-y^2/2} \, dx\,dy.$$

To find the distribution function of $z = x - y$, take

$$t = x - y \Rightarrow x = t + y \Rightarrow dx = dt$$

$$f(z) = \int_{-\infty}^{z}\int_{-\infty}^{\infty} \frac{e^{-\frac{y^2}{2}} . e^{-\frac{1}{2}(t+y)}}{\sqrt{2\pi}.\sqrt{2\pi}} \, dy\,dt$$

$$= \int_{-\infty}^{z}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[t^2 + 2yt + y^2 + y^2\right]} . \, dy\,dt$$

$$= \int_{-\infty}^{z} \frac{e^{-\frac{1}{2}t^2/2}}{\sqrt{2\pi}} . \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[\sqrt{2}y + \frac{t}{\sqrt{2}}\right)^2} \, dy\,dt$$

Let $u = \sqrt{2}y + \frac{t}{\sqrt{2}} \Rightarrow du = \sqrt{2}dy \Rightarrow dy = \frac{du}{\sqrt{2}}$

$$= \int_{-\infty}^{z} \frac{e^{-\frac{1}{2}t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} . e^{-u^2/2} . \frac{du}{\sqrt{2}} . dt$$

$$= \frac{1}{\sqrt{2}} \int_{-\infty}^{z} \frac{e^{-\frac{1}{2}\left(\frac{t^2}{2}\right)}}{\sqrt{2\pi}} . dt . \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

$$= \frac{1}{\sqrt{2}} \int_{-\infty}^{z} \frac{e^{-\frac{1}{2}\left(\frac{t^2}{2}\right)}}{\sqrt{2\pi}} dt.$$

$$\therefore f(z) = \frac{1}{\sqrt{2}} \frac{e^{-\frac{1}{2}\left(\frac{z^2}{2}\right)}}{\sqrt{2\pi}}$$

3. If X and Y follows gamma distribution, find distribution function of $Z = X+Y$. Also, find its density function.

Sol: Let $X \sim$ gamma distribution with parameter α.
and $Y \sim$ gamma distribution with parameter β.

$$f(x) = \frac{e^{-x} x^{\alpha-1}}{\Gamma\alpha}, \quad f(y) = \frac{e^{-y} y^{\beta-1}}{\Gamma\beta} \qquad 0 < x, y < \infty$$
$$\alpha, \beta > 0.$$

Since x and y are independent, their joint distribution function is given by

$$f(x,y) = f(x) \cdot f(y)$$
$$= \frac{e^{-x} x^{\alpha-1}}{\Gamma\alpha} \cdot \frac{e^{-y} y^{\beta-1}}{\Gamma\beta}.$$

Let $z = x+y$.
The distribution function of z is given by

$$F(z) = \int_{x+y<z} f(x) f(y)\, dx\, dy$$
$$= \int_{-\infty}^{z} \int_{0}^{\infty} f(x) \cdot f(y)\, dx\, dy$$

Let $y = t - x$

$$F(z) = \int_{-\infty}^{z} \int_{0}^{\infty} f(x) \cdot f(t-x)\, dx\, dt$$
$$= \int_{-\infty}^{z} \int_{0}^{\infty} \frac{e^{-x} x^{\alpha-1}}{\Gamma\alpha} \cdot \frac{e^{-(t-x)} \cdot (t-x)^{\beta-1}}{\Gamma\beta}\, dx\, dt$$
$$= \int_{-\infty}^{z} \frac{e^{-t} \cdot t^{\beta-1}}{\Gamma\alpha \Gamma\beta} \cdot dt \int_{0}^{\infty} e^{-x} x^{\alpha-1} \cdot e^{x} \left(1 - \frac{x}{t}\right)^{\beta-1} dt$$
$$= \int_{-\infty}^{z} \frac{e^{-t} t^{\beta-1}}{\Gamma\alpha \Gamma\beta} \int_{0}^{\infty} x^{\alpha-1} \left(1 - \frac{x}{t}\right)^{\beta-1} dt\, dx$$

Let $x = ut \Rightarrow dx = t\,du$

$$= \int_{-\infty}^{z} \frac{e^{-t} t^{\beta-1}}{\Gamma\alpha \Gamma\beta} \int_{0}^{\infty} (ut)^{\alpha-1} (1-u)^{\beta-1} \cdot t \cdot du\, dt$$
$$= \int_{-\infty}^{z} \frac{e^{-t} t^{\beta-1} \cdot t^{\alpha-1+1}}{\Gamma\alpha \Gamma\beta} \int_{0} u^{\alpha-1} (1-u)^{\beta-1} \cdot du\, dt$$

$$= \int_{-\infty}^{z} \frac{e^{-t} t^{\alpha+\beta-1}}{\Gamma\alpha \Gamma\beta} dt \left[\int_0^1 u^{\alpha-1} (1-u)^{\beta-1} du + \int_1^{\infty} u^{\alpha-1} (1-\alpha)^{\beta-1} du \right]$$

$$= \int_{-\infty}^{z} \frac{e^{-t} t^{\alpha+\beta-1}}{\Gamma a \Gamma\beta} \cdot dt \left[\beta(\alpha,\beta) + 0 \right]$$

$$= \int_{-\infty}^{z} \frac{e^{-t} t^{\alpha+\beta-1}}{\cancel{\Gamma a} \cancel{\Gamma\beta}} \cdot dt \; \frac{\cancel{\Gamma\alpha} \cancel{\Gamma\beta}}{\Gamma\alpha+\beta}$$

$$F(z) = \int_{-\infty}^{z} \frac{e^{-t} t^{\alpha+\beta-1}}{\Gamma\alpha+\beta} dt$$

$$\therefore f(z) = \frac{e^{-z} z^{\alpha+\beta-1}}{\Gamma\alpha+\beta} \quad \text{for} \quad z : 0 \text{ to } \infty$$

Assignment No. 1

4. If X has the Beta distribution of first kind with parameter α and β and Y has gamma distribution with parameter $\alpha+\beta$ then, find the distribution and the density function of $z = XY$, where X and Y are independent rvs.

Sol: Given X and Y are independent random variables.
$X \sim$ Beta distribution of I kind with parameters α and β.

$$f(x) = \frac{x^{\alpha-1} (1-x)^{\beta-1}}{\beta(\alpha,\beta)} \qquad 0 \le x \le 1$$

$Y \sim$ gamma distribution with parameter $\alpha+\beta$,

$$f(y) = \frac{e^{-y} y^{(\alpha+\beta)-1}}{\Gamma\alpha+\beta}, \qquad 0 < y < \infty$$

The joint p.d.f can be written as

$$f(x,y) = f(x) \cdot f(y)$$

$$= \frac{x^{\alpha-1} (1-x)^{\beta-1}}{\beta(\alpha,\beta)} \cdot \frac{e^{-y} y^{\alpha+\beta}}{\Gamma\alpha+\beta}.$$

Let $z = xy \Rightarrow y = z/x, \quad x = x$

The joint distribution of (x,z) is given by the theorem

$$g(u) = \int f_x(x) \cdot f_y(z/x) \cdot \frac{dx}{|x|}$$

$$F(z) = \int_{-\infty}^{z} \int_0^1 f(x) \cdot f(z/x) \cdot \frac{1}{x} \cdot dx \cdot dz$$

$$= \int_{-\infty}^{z}\int_{0}^{1} \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \cdot \frac{e^{-z/x}\,(z/x)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)\cdot x}\, dx\, dz$$

$$= \int_{-\infty}^{z} \frac{z^{\alpha+\beta-1}}{B(\alpha,\beta)\,\Gamma(\alpha+\beta)} \int_{0}^{1} \frac{x^{\alpha-1}\, e^{-z/x}\,(1-x)^{\beta-1}\, e^{-z/x}}{x^{\alpha+\beta}}\, dx\, dz$$

$$= \int_{-\infty}^{z} \frac{z^{\alpha+\beta-1}}{B(\alpha,\beta)\,\Gamma(\alpha+\beta)} \int_{0}^{1} \frac{x^{\beta-1}\left(\frac{1}{x}-1\right)^{\beta-1} e^{-z/x}}{x^{\beta+1}}\, dz\, dx$$

$$= \int_{-\infty}^{z} \frac{z^{\alpha+\beta-1}}{B(\alpha,\beta)\,\Gamma(\alpha+\beta)} \int_{0}^{1} \frac{1}{x^{2}}\left(\frac{1}{x}-1\right)^{\beta-1} e^{-z/x}\, dx\, dz$$

Assuming $\theta = \frac{1}{x} - 1 \Rightarrow d\theta = \frac{-dx}{\theta^{2}}$

$$= \int_{-\infty}^{z} \frac{z^{\alpha+\beta-1}}{B(\alpha,\beta)\,\Gamma(\alpha+\beta)} \int^{\infty} \theta^{\beta-1} \cdot \frac{1}{\cancel{x^{2}}}\, e^{-z(1+\theta)}\, \cancel{x^{2}}\, d\theta\, dz$$

$$= \int_{-\infty}^{z} \frac{z^{\alpha+\beta-1}\cdot e^{-z}}{B(\alpha,\beta)\,\Gamma(\alpha+\beta)} \int_{0}^{\infty} \theta^{\beta-1}\, e^{-z\theta}\, d\theta \cdot dz$$

Let $\theta = \frac{u}{z} \Rightarrow d\theta = \frac{du}{z}$

$$= \int_{-\infty}^{z} \frac{z^{\alpha+\beta-1}\, e^{-z}}{B(\alpha,\beta)\,\Gamma(\alpha+\beta)} \int_{0}^{\infty} \left(\frac{u}{z}\right)^{\beta-1} e^{-\theta\cdot\frac{u}{\theta}} \cdot \frac{du}{z}\, dz$$

$$= \int_{-\infty}^{z} \frac{z^{\alpha+\beta-1}\, e^{-z}}{B(\alpha,\beta)\,\Gamma(\alpha+\beta)\, z^{\beta-1}\cdot z} \int_{0}^{\infty} u^{\beta-1}\, e^{-u}\, du \cdot dz$$

$$= \int_{-\infty}^{z} \frac{z^{\alpha-1}\cdot e^{-z}}{B(\alpha,\beta)\,\Gamma(\alpha+\beta)} \int_{0}^{\infty} u^{\beta-1} e^{-u}\, du \cdot dz$$

$$= \int_{-\infty}^{z} \frac{z^{\alpha-1}\, e^{-z}\, \Gamma(\alpha+\beta)\cdot \cancel{\Gamma\beta}}{\Gamma\alpha\, \cancel{\Gamma\beta}\, \cancel{\Gamma(\alpha+\beta)}}\, dz$$

$$F(z) = \int_{-\infty}^{z} \frac{e^{-z}\, z^{\alpha-1}}{\Gamma\alpha}\, dz$$

$$f(z) = e^{-z} z^{\alpha-1} / \Gamma\alpha$$

5. If X and Y are two independent gamma variates. Find the density function and distribution function of $z = x/y$.

Sol: Let X and Y are two independent gamma variates.

$$f(x) = \frac{e^{-x} x^{\alpha-1}}{\Gamma\alpha}, \quad f(y) = \frac{e^{-y} y^{\beta-1}}{\Gamma\beta}, \quad 0 < x < \infty$$

The joint density is given by

$$f(x,y) = f(x) \cdot f(y) \quad \text{since } x \text{ and } y \text{ are independent.}$$

$$= \frac{e^{-x} x^{\alpha-1}}{\Gamma\alpha} \cdot \frac{e^{-y} y^{\beta-1}}{\Gamma\beta}.$$

$$z = \frac{x}{y} \Rightarrow x = zy \text{ and } y = y, \quad |J| = y$$

Based on the theorem,

$$f(z) = \int_0^\infty f(zy) \cdot f(y) \cdot y \, dy$$

$$= \int_0^\infty \frac{e^{-zy} (zy)^{\alpha-1}}{\Gamma\alpha} \cdot \frac{e^{-y} y^{\beta-1}}{\Gamma\beta} \cdot y \, dy$$

$$= \int_0^\infty \frac{e^{-y(z+1)} \cdot z^{\alpha-1} y^{\alpha+\beta-1}}{\Gamma\alpha \, \Gamma\beta} dy$$

$$= \frac{z^{\alpha-1}}{\Gamma\alpha \, \Gamma\beta} \int_0^\infty e^{-y(z+1)} y^{\alpha+\beta-1} dy$$

Let $u = y(z+1) \Rightarrow y = \frac{u}{z+1} \Rightarrow \frac{du}{z+1} = dy$

$\Rightarrow du = (z+1) dy$

$$\therefore f(z) = \frac{z^{\alpha-1}}{\Gamma\alpha \, \Gamma\beta} \int_0^\infty e^{-u} \left(\frac{u}{z+1}\right)^{\alpha+\beta-1} \cdot \frac{du}{z+1}$$

$$= \frac{z^{\alpha-1}}{(z+1)^{\alpha+\beta} \Gamma\alpha \, \Gamma\beta} \int_0^\infty e^{-u} u^{(\alpha+\beta)-1} du$$

$$= \frac{z^{\alpha-1}}{(z+1)^{\alpha+\beta} \Gamma\alpha \, \Gamma\beta} \Gamma\alpha+\beta \qquad \therefore f(z) = \frac{z^{\alpha-1}}{B(\alpha,\beta)(z+1)^{\alpha+\beta}}$$

The distribution function is given by

$$F(z) = \int_{-\infty}^{z} \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta} \cdot \beta(\alpha, \beta)} \, dt.$$

6. If x and y are two independent standard normal variates then find the density function of $z = x/y$

Sol: Given $X \sim N(0,1) \Rightarrow f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $-\infty < x < \infty$

$Y \sim N(0,1) \Rightarrow f(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$, $-\infty < y < \infty$

$z = \frac{x}{y}$, $y = y$, $|J| = y$

Based the theorem,

$$f(z) = \int_{-\infty}^{\infty} f(z, y)\, f(y) \cdot y \, dy$$

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{z^2y^2}{2}}}{\sqrt{2\pi}} \cdot \frac{e^{-y^2/2}}{\sqrt{2\pi}} \cdot y \, dy$$

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2} y^2 (z^2+1)}}{2\pi} \cdot y \, dy$$

Let $u = \frac{y^2(1+z^2)}{2}$, $\quad 2u = y^2(1+z^2)$

$$\frac{2u}{1+z^2} = y^2$$

$$\frac{2\,du}{1+z^2} = 2y \cdot dy$$

$$\Rightarrow y\,dy = \frac{\cancel{2}\,du}{\cancel{2}(1+z^2)}$$

$$\therefore = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-u}\,du}{1+z^2}$$

$$= \frac{2}{2\pi} \int_{0}^{\infty} \frac{e^{-u}\,du}{1+z^2} = \frac{1}{\pi} \cdot \frac{1}{1+z^2}.$$

$$\therefore f(z) = \frac{1}{\pi} \cdot \frac{1}{1+z^2}, \quad 0 \le z \le \infty.$$

This is the p.d.f of Cauchy distribution.

7. X and Y are independent random variables with common pdf, $f(x) = e^{-x}, \quad x \geq 0$

$= 0 \quad x < 0$

Find pdf of $X - Y$.

Sol: Given X and Y are identically independently distributed.

The joint p.d.f is given by

$$f(x,y) = f(x) \cdot f(y) \, , = e^{-(x+y)} \, , \quad x, y \geq 0$$

Let $u = x - y \Rightarrow x = u + v$ $\quad v > -u$ if $-\infty < u < 0$

$v = y \Rightarrow v = x - u$ $\quad v > 0$ if $u > 0$

$$|J| = 1$$

$$g(u) = \int_{-\infty}^{\infty} f(u+v, v) \, dv \quad \text{(by theorem)}$$

$$= \int_{-u}^{\infty} e^{-(u+2v)} \, dv$$

$$= e^{-u} \int_{-u}^{\infty} e^{-2v} \, dv = e^{-u} \cdot \left[\frac{e^{-2v}}{-2} \right]_{-u}^{\infty}$$

$$= \frac{e^{-u}}{2} \left[0 + e^{2u} \right] = \frac{1}{2} e^{u} \quad \text{for } u > 0$$

$$g(u) = \int_{0}^{\infty} g(u,v) \, dv = e^{-u} \left. \frac{e^{-2v}}{-2} \right|_{0}^{\infty}$$

$$= \frac{1}{2} e^{-u}$$

P.d.f of $U = X - Y$ is

$$g(u) = \frac{1}{2} e^{u} \quad -\infty < u < 0$$

$$= \frac{1}{2} e^{-u} \quad u > 0.$$

Assignment

2. Let $X_1, X_2 \ldots X_n$ be random sample from a population with continuous density. Show that

$$Y_1 = \min(X_1 X_2 \ldots X_n) \text{ is exponential with parameter } n\theta$$

3. Let (X, Y) be two-dimensional non-negative continuous random variable having the joint density

$$f(x,y) = 4xy\, e^{-(x^2+y^2)} \quad ; \quad x \geq 0, y \geq 0$$
$$= 0 \quad \text{elsewhere.}$$

Prove that the density function of $U = \sqrt{x^2+y^2}$ is

$$h(u) = 2u^3 e^{-u^2}, \quad 0 \leq u < \infty$$
$$= 0, \quad \text{elsewhere.}$$

4. Let the p.d.f of the random variable (X, Y) be;

$$f(x,y) = \alpha^{-2} e^{-(x+y)/\alpha}, \quad \begin{matrix} x, y > 0 \\ \alpha > 0 \end{matrix}$$
$$= 0 \quad \text{elsewhere.}$$