$$E(X^2) = 1^2 \cdot \left\{ \frac{\theta}{2N} + \frac{\alpha}{2}\left(1 - \frac{\theta}{N}\right) \right\} + 2^2 \cdot \left\{ \frac{\theta}{4N} + \frac{1-\alpha}{2}\left(1 - \frac{\theta}{N}\right) \right\}$$

$$= \frac{3\theta}{2N} + \left(1 - \frac{\theta}{N}\right)\left[\frac{\alpha}{2} + 2(1-\alpha)\right] = \frac{3\theta}{2N} + \left(1 - \frac{\theta}{N}\right)\left(2 - \frac{3\alpha}{2}\right)$$

$$\Rightarrow \qquad \mu_2' = 2 - \frac{\theta}{2N} - \frac{3}{2}\alpha\left(1 - \frac{\theta}{N}\right) \qquad \ldots(**)$$

The sample frequency distribution is :

x	0	1	2
f	27	38	10

$$\mu_1' = \frac{1}{N}\Sigma fx = \frac{1}{75}(38 + 20) = \frac{58}{75}, \qquad \mu_2' = \frac{1}{N}\Sigma fx^2 = \frac{1}{75}(38 + 40) = \frac{78}{75}$$

Equating the sample moments to theoretical moments, we get

$$1 - \frac{\alpha}{2}\left(1 - \frac{\theta}{N}\right) = \frac{58}{75} \quad \Rightarrow \quad \frac{\alpha}{2}\left(1 - \frac{\theta}{N}\right) = 1 - \frac{58}{75} = \frac{17}{75} \qquad \ldots(***)$$

Substituting in (**), we get $\quad 2 - \dfrac{\theta}{2N} - 3 \times \dfrac{17}{75} = \dfrac{78}{75} \quad \Rightarrow \quad \hat{\theta} = \dfrac{42}{75}N$

Substituting in (***), we get $\quad \dfrac{\alpha}{2}\left(1 - \dfrac{42}{75}\right) = \dfrac{17}{75} \quad \Rightarrow \quad \hat{\alpha} = \dfrac{34}{33}$

17·6·4. Method of Least Squares. The principle of least squares is used to fit a curve of the form : $\quad y = f(x, a_0, a_1, \ldots, a_n) \qquad \ldots(17·62)$

where a_i's are unknown parameters, to a set of n sample observations (x_i, y_i); $i = 1, 2, \ldots, n$ from a bivariate population. It consists in minimising the sum of squares of residuals, *viz.*,

$$E = \sum_{i=1}^{n} \{y_i - f(x_i, a_0, a_1, \ldots, a_n)\}^2 \qquad \ldots(17·63)$$

subject to variations in $a_0, a_1, \ldots, a_n$.

The normal equations for estimating $a_0, a_1, \ldots, a_n$ are given by :

$$\frac{\partial E}{\partial a_i} = 0; \; i = 1, 2, \ldots, n. \qquad \ldots(17·64)$$

Remarks. 1. In chapter 10, we have discussed in detail the method of least squares for fitting linear regression, polynomial regression and the exponential family of curves reducible to linear regression. In chapter 11, we have discussed the method of fitting multiple linear regression (§ 11·12·1).

2. If we are estimating $f(x, a_0, a_1, \ldots, a_n)$ as a linear function of the parameters $a_0, a_1, \ldots, a_n$, the x's being known given values, the least square estimators obtained as linear functions of the y's will be MVU estimators.

17·7. CONFIDENCE INTERVAL AND CONFIDENCE LIMITS

Let x_i, $(i = 1, 2, \ldots, n)$ be a random sample of n observations from a population involving a single unknown parameter θ, (say). Let $f(x, \theta)$ be the probability function of the parent distribution from which the sample is drawn and let us suppose that this distribution is continuous. Let $t = t(x_1, x_2, \ldots, x_n)$, a function of the sample values be an estimate of the population parameter θ, with the sampling distribution given by $g(t, \theta)$.

Having obtained the value of the statistic t from a given sample, the problem is, "Can we make some reasonable probability statements about the unknown parameter θ in the population, from which the sample has been drawn?" This question is very well answered by the technique of *Confidence interval* due to Neyman and is obtained below :

We choose once for all some small value of α (5% or 1%) and then determine two constants say, c_1 and c_2 such that : $\quad P(c_1 < \theta < c_2 \mid t) = 1 - \alpha \qquad$ ...(17·65)

The quantities c_1 and c_2, so determined, are known as the *confidence limits* or *fiducial limits* and the interval $[c_1, c_2]$ within which the unknown value of the population parameter is expected to lie, is called the *confidence interval* and $(1 - \alpha)$ is called the *confidence coefficient.*

Thus if we take $\alpha = 0·05$ (or $0·01$), we shall get 95% (or 99%) confidence limits.

How to find c_1 and c_2. ? Let T_1 and T_2 be two statistics such that

$$P(T_1 > \theta) = \alpha_1 \qquad \text{...(17·66)}$$

and

$$P(T_2 < \theta) = \alpha_2 \qquad \text{...(17·66}a\text{)}$$

where α_1 and α_2 are constants independent of θ. (17·66) and (17·66*a*) can be combined to give

$$P(T_1 < \theta < T_2) = 1 - \alpha, \qquad \text{...(17·66}b\text{)}$$

where $\alpha = \alpha_1 + \alpha_2$. Statistics T_1 and T_2 defined in (17·66) and (17·66*a*) may be taken as c_1 and c_2 defined in (17·65).

For example, if we take a large sample from a normal population with mean μ and standard deviation σ, then

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

and $\quad P(-1·96 \le Z \le 1·96) = 0·95 \qquad$ (From Normal Probability Tables)

$$\Rightarrow P\left(-1·96 \le \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \le 1·96\right) = 0·95 \quad \Rightarrow \quad P\left(\bar{x} - 1·96\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + 1·96\frac{\sigma}{\sqrt{n}}\right) = 0·95$$

Thus $\bar{x} \pm 1·96 \dfrac{\sigma}{\sqrt{n}}$ are 95% confidence limits for the unknown parameter μ, the population mean and the interval $\left(\bar{x} - 1·96\dfrac{\sigma}{\sqrt{n}},\ \bar{x} + 1·96\dfrac{\sigma}{\sqrt{n}}\right)$ is called the 95% confidence interval.

Also $\quad P(-2·58 \le Z \le 2·58) = 0·99 \quad$ or $\quad P\left(-2·58 \le \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}} \le 2·58\right) = 0·99$

$$\Rightarrow \qquad P\left(\bar{x} - 2·58\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + 2·58\frac{\sigma}{\sqrt{n}}\right) = 0·99$$

Hence 99% confidence limits for μ are : $\bar{x} \pm 2·58\dfrac{\sigma}{\sqrt{n}}$ and

99% confidence interval for μ is $\left(\bar{x} - 2·58\dfrac{\sigma}{\sqrt{n}},\ \bar{x} + 2·58\dfrac{\sigma}{\sqrt{n}}\right)$.

***Remarks* 1.** Usually σ^2 is not known and its unbiased estimate S^2 obtained from the samples, is used. However if n is small, $Z = \dfrac{\bar{x} - \mu}{S/\sqrt{n}}$ is not $N(0, 1)$ and in this case the confidence limits and confidence intervals for μ are obtained by using Student's 't' distribution.

2. It can be seen that in many cases there exist more than one set of confidence intervals with the same confidence coefficient. Then the problem arises as to which particular set is to be regarded as better than the others in some useful sense and in such cases we look for the shortest of all the intervals.

Example 17·46. *Obtain 100 $(1-\alpha)$% confidence intervals for the parameters (a) θ and (b) σ^2, of the normal distribution :*

$$f(x, \theta; \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\theta}{\sigma}\right)^2\right\}, -\infty < x < \infty$$

Solution. Let X_i, $(i = 1, 2, \ldots, n)$ be a random sample of size n from the density $f(x\,;\theta, \sigma)$ and let : $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, $\quad s^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$, $\quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

(*a*) The statistic $t = \dfrac{\bar{X}-\theta}{S/\sqrt{n}}$ follows student's t-distribution with $(n-1)$ degrees of freedom. Hence $100(1-\alpha)$% confidence limits for θ are given by :

$$P(|\,t\,| \le t_\alpha) = 1-\alpha \quad \Rightarrow \quad P\left(|\,\bar{X}-\theta\,| \le \frac{S}{\sqrt{n}}\, t_\alpha\right) = 1-\alpha$$

$$\therefore \quad P\left(\bar{X} - t_\alpha \cdot \frac{S}{\sqrt{n}} \le \theta \le \bar{X} + t_\alpha \cdot \frac{S}{\sqrt{n}}\right) = 1-\alpha \qquad \ldots(17\cdot67)$$

where t_α is the tabulated value of t for $(n-1)$ *d.f.* at significance level 'α'. Hence the required confidence interval for θ is : $\left(\bar{X} - t_\alpha \frac{S}{\sqrt{n}}, \bar{X} + t_\alpha \frac{S}{\sqrt{n}}\right)$

(*b*) *Case (i)* θ *is known and equal to* μ *(say).*

Then
$$\frac{\Sigma(X_i-\mu)^2}{\sigma^2} = \frac{ns^2}{\sigma^2} \sim \chi^2_{(n)}$$

If we define χ_α^2 as the value of χ^2 such that $P(\chi^2 > \chi_\alpha^2) = \int_{\chi_\alpha^2}^{\infty} p(\chi^2)d\chi^2 = \alpha \qquad \ldots(*)$

where $p(\chi^2)$ is the *p.d.f.* of χ^2-distribution with n *d.f.*, then the required confidence interval is given by :

$$P\{\chi^2_{1-(\alpha/2)} \le \chi^2 \le \chi^2_{\alpha/2}\} = 1-\alpha \quad \Rightarrow \quad P\left\{\chi^2_{1-(\alpha/2)} \le \frac{ns^2}{\sigma^2} \le \chi^2_{\alpha/2}\right\} = 1-\alpha \qquad \ldots(**)$$

Now $\dfrac{ns^2}{\sigma^2} \le \chi^2_{\alpha/2} \Rightarrow \dfrac{ns^2}{\chi^2_{\alpha/2}} \le \sigma^2$ and $\chi^2_{1-(\alpha/2)} \le \dfrac{ns^2}{\sigma^2} \Rightarrow \sigma^2 \le \dfrac{ns^2}{\chi^2_{1-(\alpha/2)}}$

Hence (**) gives :
$$P\left\{\frac{ns^2}{\chi^2_{\alpha/2}} \le \sigma^2 \le \frac{ns^2}{\chi^2_{1-(\alpha/2)}}\right\} = 1-\alpha, \qquad \ldots(***)$$

where $\chi^2_{\alpha/2}$ and $\chi^2_{1-(\alpha/2)}$ are obtained from (*) by using n *d.f.*

Thus *e.g.*, 95% confidence interval for σ^2 is : $P\left(\dfrac{ns^2}{\chi^2_{0\cdot025}} \le \sigma^2 \le \dfrac{ns^2}{\chi^2_{0\cdot975}}\right) = 0\cdot95$

Case (ii). θ is unknown. In this case the statistic : $\dfrac{\Sigma(X_i-\bar{X})^2}{\sigma^2} = \dfrac{ns^2}{\sigma^2} \sim \chi^2_{(n-1)}$.

Here also confidence interval for σ^2 is given by (***), where now χ^2_α is the significant value of χ^2 [as defined in (*)] for $(n-1)$ *d.f.* at the significance level 'α'.

Example 17·47. *Show that the largest observations L of a sample of n observations from a rectangular distribution with density function :*

$$f(x, \theta) = \begin{cases} \frac{1}{\theta}, & 0 \le x \le \theta \\ 0, & otherwise \end{cases} \quad ...(*)$$

has the distribution : $dG(L) = n\left(\frac{L}{\theta}\right)^{n-1} \cdot \frac{dL}{\theta}, 0 \le L \le \theta$

Show that the distribution of $V = L/\theta$ *is given by p.d.f. :* $h(v) = nv^{n-1}, 0 \le v \le 1$

Hence deduce that the confidence limits for θ *corresponding to confidence coefficient* α *are*

$$L \quad and \quad \frac{L}{(1-\alpha)^{1/n}}.$$

Solution. Let $X_1, X_2 ..., X_n$ be a random sample of size n from the population (*) and let $L = \max(X_1, X_2, ..., X_n)$. The distribution of L is given by :

$dG(L) = n[F(L)]^{n-1}. f(L)\, dL$, where $F(.)$ is the distribution function of X given by :

$$F(L) = \int_0^L f(x, \theta)dx = \frac{L}{\theta} \qquad \therefore dG(L) = n\left(\frac{L}{\theta}\right)^{n-1} \cdot \frac{dL}{\theta}, 0 \le L \le \theta$$

If we take $V = L/\theta$, the Jacobian of transformation is θ. Hence *p.d.f.* $h(.)$ of V is :

$$h(v) = nv^{n-1} \cdot \frac{1}{\theta} \mid J \mid = nv^{n-1}, 0 \le v \le 1,$$

which is independent of θ.

To obtain the confidence limits for θ, with confidence coefficient α, let us define v_α such that

$$P(v_\alpha < V < 1) = \alpha \quad \Rightarrow \quad \int_{v_\alpha}^1 h(v)\, dv = \alpha \quad ...(**)$$

$$\Rightarrow \quad n\int_{v_\alpha}^1 v^{n-1}\, dv = \alpha \quad \Rightarrow \quad 1 - v_\alpha^{\ n} = \alpha \quad \Rightarrow \quad v_\alpha = (1-\alpha)^{1/n} \quad ...(***)$$

From (**) and (***), $P\{(1-\alpha)^{1/n} < V < 1\} = \alpha \quad \Rightarrow \quad P\left\{(1-\alpha)^{1/n} < \frac{L}{\theta} < 1\right\} = \alpha$

$$\therefore \qquad P\left\{L < \theta < \frac{L}{(1-\alpha)^{1/n}}\right\} = \alpha$$

Hence the required confidence limits for θ are L and $L/(1-\alpha)^{1/n}$.

Example 17·48. *Given a random sample from a population with p.d.f. :*

$$f(x, \theta) = \frac{1}{\theta}, \; 0 \le x \le \theta$$

show that 100 (1 – α)% confidence interval for θ is given by R and R/ψ, where ψ is given by

$$\psi^{n-1}[n - (n-1)\psi] = \alpha, \textit{ and R is the sample range.}$$

Solution. The joint *p.d.f.* of $x_1, x_2, ..., x_n$ is given by : $L = \frac{1}{\theta^n}, 0 \le x_i \le \theta$

If $x_{(1)}, x_{(2)}, ..., x_{(n)}$ is the ordered sample then the joint *p.d.f.* of $x_{(n)}$ and $x_{(1)}$ is :

$$g[x_{(1)}, x_{(n)}] = \frac{n(n-1)}{\theta^n}\left[x_{(n)} - x_{(1)}\right]^{n-2}, 0 \le x_{(1)} \le x_{(n)} \le \theta$$

To obtain the distribution of the sample range R, let us make the transformation of variables :

$$R = x_{(n)} - x_{(1)} \text{ and } v = x_{(1)} \quad \Rightarrow \quad v = x_{(n)} - R \le \theta - R.$$

The Jacobian of transformation is $|J| = 1$ and the joint $p.d.f.$ of R and V becomes :

$$h(R, v) = \frac{n(n-1)}{\theta^n} R^{n-2}, 0 < v < \theta - R$$

The marginal density of R is given by :

$$h_1(R) = \int_0^{\theta - R} \frac{n(n-1)}{\theta^n} . R^{n-2} \, dv \quad = \frac{n(n-1) R^{n-2} (\theta - R)}{\theta^n}, 0 \le R \le \theta$$

The $p.d.f.$ $h_2(.)$ of $U = R/\theta$ is :

$$h_2(u) = h_1(R) \left| \frac{dR}{du} \right| = \frac{n(n-1) R^{n-2} (\theta - R)}{\theta^n} . \theta = n(n-1)u^{n-2}(1-u), 0 \le u \le 1$$

100 $(1 - \alpha)$% confidence interval for θ is given by : $P(\psi \le U \le 1) = 1 - \alpha$ ...(*)

where ψ is obtained from the equation $\int_0^{\psi} h_2(u) du = \alpha$

$$\Rightarrow \quad n(n-1) \int_0^{\psi} u^{n-2}(1-u) du = \alpha \quad \Rightarrow \quad \left| nu^{n-1} - (n-1) u^n \right|_0^{\psi} = \alpha$$

$$\therefore \quad \psi^{n-1} \{ n - (n-1)\psi \} = \alpha \quad \text{...(**)}$$

From (*), we get

$$P\left(\psi \le \frac{R}{\theta} \le 1\right) = 1 - \alpha \quad \Rightarrow \quad P\left(R \le \theta \le \frac{R}{\psi}\right) = 1 - \alpha$$

Hence the required limits for θ are given by R and R/ψ where ψ is given by (**).

Example 17·49. *Given one observation from a population with p.d.f. :*

$$f(x, \theta) = \frac{2}{\theta^2}(\theta - x), \; 0 \le x \le \theta,$$

obtain 100 $(1 - \alpha)$% confidence interval for θ.

Solution. The density of $u = x/\theta$ is given by :

$$g(u) = f(x, \theta) . \left| \frac{dx}{du} \right| = \frac{2}{\theta^2}(\theta - x). \theta = 2(1-u), \; 0 \le u \le 1$$

To obtain 100 $(1 - \alpha)$% confidence interval for θ, we choose two quantities u_1 and u_2 such that

$$\left. \begin{aligned} P(u_1 \le u \le u_2) &= 1 - \alpha \\ \text{and} \quad P(u < u_1) &= P(u > u_2) = \tfrac{1}{2} \alpha \end{aligned} \right\} \quad \text{...(*)}$$

$$P(u < u_1) = \frac{\alpha}{2} \quad \Rightarrow \quad \int_0^{u_1} 2(1-u) \, du = \frac{\alpha}{2} \quad \Rightarrow \quad u_1^2 - 2u_1 + \frac{\alpha}{2} = 0 \quad \text{...(**)}$$

$$\text{and} \quad P(u > u_2) = \frac{1}{2}\alpha \quad \Rightarrow \quad \int_{u_2}^{1} 2(1-u) \, du = \frac{\alpha}{2} \cdot \quad \Rightarrow \quad u_2^2 - 2u_2 + \left(1 - \frac{\alpha}{2}\right) = 0 \quad \text{...(***)}$$

From (*), we get $\quad P\left(u_1 \le \frac{x}{\theta} \le u_2\right) = 1 - \alpha \quad \Rightarrow \quad P\left(\frac{x}{u_2} \le \theta \le \frac{x}{u_1}\right) = 1 - \alpha$

Hence the required interval for θ is $\left(\frac{x}{u_2}, \frac{x}{u_1}\right)$, where u_1 and u_2 are given by (**) and (***) respectively.

17·7·1. Confidence Intervals for Large Samples. It has been proved that under certain regularity conditions, the first derivative of the logarithm of the likelihood function w.r.to parameter θ *viz.*, $\frac{\partial}{\partial \theta} \log L$, is asymptotically normal with mean zero and variance given by :

$$\text{Var}\left(\frac{\partial}{\partial \theta} \log L\right) = E\left(\frac{\partial}{\partial \theta} \log L\right)^2 = E\left(-\frac{\partial^2}{\partial \theta^2} \log L\right)$$

Hence for large n,

$$Z = \frac{\frac{\partial}{\partial \theta} \log L}{\sqrt{\text{Var}\left(\frac{\partial}{\partial \theta} \log L\right)}} \sim N(0, 1) \qquad \ldots(17\cdot68)$$

The result enables us to obtain confidence interval for the parameter θ in large samples. Thus for large samples, the confidence interval for θ with confidence coefficient $(1 - \alpha)$ is obtained by converting the inequalities in

$$P(|Z| \le \lambda_\alpha) = 1 - \alpha \qquad \ldots(17\cdot69)$$

where λ_α is given by $\frac{1}{\sqrt{2\pi}} \int_{-\lambda_\alpha}^{\lambda_\alpha} \exp\left(-u^2/2\right) du = 1 - \alpha$ ...[17·69(*a*)]

Example 17·50. *Obtain 100 (1 – α)% confidence limits (for large samples) for the parameter λ of the Poisson distribution :*

$$f(x, \lambda) = \frac{e^{-\lambda} . \lambda^x}{x!} ; x = 0, 1, 2, \ldots$$

Solution. We have

$$\frac{\partial}{\partial \lambda} \log L = \frac{\partial}{\partial \lambda} \left\{-n\lambda + \left(\sum_{i=1}^{n} x_i\right) \log \lambda - \sum_{i=1}^{n} \log (x_i)\right\} = -n + \frac{\Sigma x_i}{\lambda} = n\left(\frac{\bar{x}}{\lambda} - 1\right)$$

$$\text{Var}\left(\frac{\partial}{\partial \lambda} \log L\right) = E\left(-\frac{\partial^2}{\partial \lambda^2} \log L\right) = E\left(\frac{n\bar{x}}{\lambda^2}\right) = \frac{n}{\lambda^2} E(\bar{x}) = \frac{n}{\lambda} \qquad [\because E(\bar{x}) = \lambda]$$

$$\therefore \quad Z = \frac{n\left(\frac{\bar{x}}{\lambda} - 1\right)}{\sqrt{n/\lambda}} = \sqrt{(n/\lambda)}\, (\bar{x} - \lambda) \sim N(0, 1) \qquad \text{[Using (17·68)]}$$

Hence 100 $(1 - \alpha)$% confidence interval for λ is given by (for large samples)

$$P\left\{\left|\sqrt{(n/\lambda)}\, (\bar{x} - \lambda)\right| \le \lambda_\alpha\right\} = 1 - \alpha$$

Hence the required limits for λ are the roots of the equation :

$$\left|\sqrt{n/\lambda}\, (\bar{x} - \lambda)\right| = \lambda_\alpha \quad \Rightarrow \quad n(\bar{x} - \lambda)^2 - \lambda . \lambda^2_\alpha = 0$$

$$\Rightarrow \lambda^2 - \lambda\left(2\bar{x} + \frac{\lambda_\alpha^2}{n}\right) + \bar{x}^2 = 0 \Rightarrow \lambda = \frac{\left(2\bar{x} + \frac{\lambda_\alpha^2}{n}\right) \pm \left\{\left(2\bar{x} + \frac{\lambda_\alpha^2}{n}\right)^2 - 4\bar{x}^2\right\}^{1/2}}{2} \qquad \ldots(*)$$

For example, 95% confidence interval for λ is given by taking $\lambda_\alpha = 1\cdot96$ in (*), thus giving :

$$\lambda = \frac{1}{2}\left(2\bar{x} + \frac{3\cdot84}{n}\right) \pm \left(\frac{3\cdot84\bar{x}}{n} + \frac{3\cdot69}{n^2}\right)^{1/2} = \bar{x} \pm 1\cdot96 \sqrt{\bar{x}/n}, \text{ to the order } n^{-1/2}.$$

Example 17·51. *Show that for the distribution :* $dF(x) = \theta\, e^{-x\theta} ; 0 < x < \infty$, *central confidence limits for large samples with 95% confidence coefficient are given by*

$$\theta = \left(1 \pm \frac{1\cdot 96}{\sqrt{n}}\right) / \bar{x}.$$

Solution. Here $L = \theta^n \exp\left(-\theta \sum_{i=1}^{n} x_i\right)$

$$\frac{\partial}{\partial\theta}\log L = \frac{\partial}{\partial\theta}\left(n\log\theta - \theta\Sigma x_i\right) = \frac{n}{\theta} - \sum_{i=1}^{n} x_i = n\left(\frac{1}{\theta} - \bar{x}\right)$$

and $\frac{\partial^2}{\partial\theta^2}\log L = -\frac{n}{\theta^2} \Rightarrow \text{Var}\left(\frac{\partial}{\partial\theta}\log L\right) = E\left(-\frac{\partial^2}{\partial\theta^2}\log L\right) = \frac{n}{\theta^2}$

Hence, for large samples, using (17·68), we have

$$Z = \frac{n\left(\frac{1}{\theta} - \bar{x}\right)}{\sqrt{n/\theta^2}} \sim N(0, 1) \quad \Rightarrow \quad \sqrt{n}\,(1 - \theta\bar{x}) \sim N(0, 1)$$

Hence 95% confidence limits for θ are given by :

$$P[-1\cdot 96 \le \sqrt{n}\,(1 - \theta\bar{x}) \le 1\cdot 96] = 0\cdot 95 \quad \ldots(*)$$

$$\sqrt{n}\,(1 - \theta\bar{x}) \le 1\cdot 96 \quad \Rightarrow \quad \left(1 - \frac{1\cdot 96}{\sqrt{n}}\right)\frac{1}{\bar{x}} \le \theta \quad \ldots(**)$$

and $-1\cdot 96 \le \sqrt{n}\,(1 - \theta\bar{x}) \quad \Rightarrow \quad \theta \le \left(1 + \frac{1\cdot 96}{\sqrt{n}}\right)\frac{1}{\bar{x}} \quad \ldots(***)$

Hence, from (*), (**) and (***), the central 95% confidence limits for θ are given by :

$$\theta = \left(1 \pm \frac{1\cdot 96}{\sqrt{n}}\right) \cdot \frac{1}{\bar{x}}.$$

CHAPTER CONCEPTS QUIZ

1. Comment on the following statements :

(*i*) In the case of Poisson distribution with parameter λ, $\bar{x}$ is sufficient for λ.

(*ii*) If $(X_1, X_2, \ldots X_n)$ be a sample of independent observation from the uniform distribution on $(\theta, \theta + 1)$, then the maximum likelihood estimator of θ is unique.

(*iii*) A maximum likelihood estimator is always unbiased.

(*iv*) Unbiased estimator is necessarily consistent

(*v*) A consistent estimator is also unbiased.

(*vi*) An unbiased estimator whose variance tends to zero as the sample size increases is consistent.

(*vii*) If t is a sufficient statistic for θ then $f(t)$ is a sufficient statistic for $f(\theta)$.

(*viii*) If t_1 and t_2 are two independent estimators of θ, then $t_1 + t_2$ is less efficient than both t_1 and t_2.

(*ix*) If T is consistent estimator of a parameter θ, then $aT + b$ is a consistent estimator of $a\theta + b$, where a and b are constants.

(*x*) If x is the number of successes in n independent trials with a constant probability p of success in each trial, then x/n is a consistent estimator of p.