UNIT - II

Uniformly Minimum - Variance Unbiased Estimator (UMVUE) ①

Since estimators with uniformly minimum mean - squared error rarely exist, a reasonable procedure is to restrict the class of estimating functions and look for estimators with uniformly minimum mean - squared error within the restricted class.

One way of restricting the class of estimating functions would be to consider only the unbiased estimators and then among the class of unbiased estimators search for an estimator with minimum mean squared error.

The mean - squared error of an estimator T of $\tau(\theta)$ can be written as

$$E_\theta [T - \tau(\theta)]^2 = V(T) + [\tau(\theta) - E_\theta(T)]^2$$

and if T is an unbiased estimator of $\tau(\theta)$, then $E_\theta(T) = \tau(\theta)$ and so $E_\theta [T - \tau(\theta)]^2 = V_\theta(T)$. Hence, seeking an estimator with uniformly minimum mean - squared error among unbiased estimators is tantamount to seeking an estimator with uniformly minimum variance among unbiased estimators.

Definition.

Let $x_1, x_2 \ldots x_n$ be a random sample drawn from $f(x, \theta)$. An estimator $T^* = t^*(x_1, x_2 \ldots x_n)$ of $\tau(\theta)$ is defined to be a uniformly minimum - variance unbiased estimator of $\tau(\theta)$ if and only if

i) $E_\theta(T^*) = \tau(\theta)$

i.e. T^* is unbiased and

ii) $\text{var}_\theta(T^*) \le \text{var}_\theta(T)$ for any other estimator $T = t(x_1, x_2 \ldots x_n)$ of $\tau(\theta)$ which satisfies $E_\theta(T) = \tau(\theta)$.

There are two methods of finding UMVUE. one method is through finding lower bounds for the variance of unbiased estimator and the other is using the concept of completeness in conjunction with sufficiency.

Lower Bound for variance.

It has been established that when the variance of any unbiased estimator is taken as a tool to make some criteria, the variance of such estimator will not fall below certain limits known as the lower bound of the estimator. The various bounds are

i) Cramer - Rao lower bound
ii) Bhattacharya's lower bound.
iii) Chapmann's Robin's bound.

Regularity conditions

Let us assume that θ is the single parameter ranging over the parameter space Θ. Let us denote the likelihood function as

$$L(x_1, x_2 \ldots x_n ; \theta) = L(x, \theta)$$

For convenience, let us denote

$$\int \ldots \int (\quad) \prod dx_i = \int (\,) dx$$

Beside these, we make the following assumptions:

1. Θ is either on the real line or an interval on the real line.
2. For almost all $x_1, x_2 \ldots x_n$

$$\frac{\partial L(x_1, x_2 \ldots x_n ; \theta)}{\partial \theta} \text{ exist } \forall \theta \in \Theta$$

3. $$\frac{\partial}{\partial \theta} \int_A L\,dx = \int_A \frac{\partial L}{\partial \theta} dx$$

where A is the domain of positive probability density.

4. $$\frac{\partial}{\partial \theta} \int_A t\, L\,dx = \int_A t \cdot \frac{\partial L}{\partial \theta} dx \text{ exists.}$$

5. $$E_\theta \left[\frac{\partial \log L}{\partial \theta}\right]^2 \text{ exist.}$$

Among the above, (3) & (4) are important one.

Cramer - Rao inequality.

If T is an unbiased estimator of $\gamma(\theta)$, under certain regularity conditions

$$V_{\theta}(T) \geq \frac{[\gamma'(\theta)]^2}{E_{\theta}\left[\frac{\partial \log L}{\partial \theta}\right]^2}$$

3

Proof: By defn $\int_A L\,dx = 1$

Differentiating partially w.r.t θ, we get

$$\frac{\partial}{\partial \theta} \int_A L\,dx = 0$$

Applying regularity condition ③, we get

$$\int_A \frac{\partial L}{\partial \theta}\,dx = 0$$

This may be written as

$$\int \frac{1}{L} \frac{\partial L}{\partial \theta} \cdot L\,dx = 0$$

ie $$\int \left(\frac{\partial \log L}{\partial \theta}\right) \cdot L\,dx = 0 \quad \text{—— ①}$$

$$\Rightarrow E\left(\frac{\partial \log L}{\partial \theta}\right) = 0$$

It is given that T is an unbiased estimator of $\gamma(\theta)$.

$$\therefore \gamma(\theta) = \int t\,L\,dx$$

Differentiating partially on bothsides w.r.t. θ, we have

$$\gamma'(\theta) = \frac{\partial}{\partial \theta} \int t\,L\,dx$$

using condition (iv), the above integral may be written as

$$\gamma'(\theta) = \int t \frac{\partial L}{\partial \theta}\,dx$$

$$= \int t \cdot \frac{1}{L} \frac{\partial L}{\partial \theta} \cdot L\,dx$$

$$\gamma'(\theta) = \int t \frac{\partial \log L}{\partial \theta} \cdot L\,dx \quad \text{—— ②}$$

Multiplying equation ① by $\gamma(\theta)$ and subtracting the result from ②, we have

$$\gamma'(\theta) = \int [t - \gamma(\theta)] \frac{\partial \log L}{\partial \theta} \cdot L\,dx \quad \text{—— ③}$$

$$\Rightarrow \gamma'(\theta) = \operatorname{cov}\left(t, \frac{\partial \log L}{\partial \theta}\right) \quad \text{since } E\left(\frac{\partial \log L}{\partial \theta}\right) = 0$$

know that Cauchy - Schwartz inequality is given by

$$[cov(x,y)]^2 \leq var(x) \cdot var(y)$$

Treating $x = t$ and $y = \frac{\partial \log L}{\partial \theta}$ and applying Cauchy - Swartz, we have

$$[\gamma'(\theta)]^2 \leq var(T) \cdot E\left[\frac{\partial \log L}{\partial \theta}\right]^2 \quad \because E\left(\frac{\partial \log L}{\partial \theta}\right) = 0$$

4

$\therefore$ C. R. Rao inequality is

$$V(T) \geq \frac{[\gamma'(\theta)]^2}{E\left[\frac{\partial \log L}{\partial \theta}\right]^2}$$

Hence the proof.

case i) If $\gamma(\theta) = \theta$, then $\gamma'(\theta) = 1$,

$$\therefore V(T) = \frac{1}{E\left(\frac{\partial \log L}{\partial \theta}\right)^2}$$

case ii) The cramer - rao inequality may also be given as

$$Var(T) \geq \frac{[\gamma'(\theta)]^2}{-E\left[\frac{\partial^2 \log L}{\partial \theta^2}\right]}$$

Fisher's information.

Let us recall the equation

$$0 = \int_A \frac{\partial \log L}{\partial \theta} \cdot L\, dx$$

Differentiating partially w.r.t. θ, we get

$$0 = \int_A \frac{\partial \log L}{\partial \theta} \cdot \frac{\partial L}{\partial \theta} \cdot dx + \int_A L \frac{\partial^2 \log L}{\partial \theta^2} \cdot dx$$

$$= \int_A \left(\frac{\partial \log L}{\partial \theta}\right)^2 L\, dx + \int_A \frac{\partial^2 \log L}{\partial \theta^2} L\, dx$$

i.e. $E\left(\frac{\partial \log L}{\partial \theta}\right)^2 = -E \frac{\partial^2 \log L}{\partial \theta^2}$

Substituting the above in C. R. Rao inequality, we get

$$Var(T) \geq \frac{[\gamma'(\theta)]^2}{E\left(\frac{\partial \log L}{\partial \theta}\right)^2}$$

Scanned with CamScanner

$$\therefore \quad var(T) \geq \frac{[\gamma'(\theta)]^2}{-E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right)}$$

(5)

Case iii) when $x_1, x_2 \dots x_n$ are i.i.d random variables.

Let $x_1, x_2 \dots x_n$ be i.i.d random variables. Then the likelihood function becomes

$$L = \prod_{i=1}^{n} f(x_i, \theta)$$

$$= f(x_1, \theta)\, f(x_2\, \theta) \dots f(x_n\, \theta)$$

$$\log L = \Sigma \log f(x_i\, \theta)$$

$$\frac{\partial \log L}{\partial \theta} = \Sigma \frac{\partial \log f(x_i\, \theta)}{\partial \theta}$$

Squaring both sides and taking expectation, we have

$$E\left[\frac{\partial \log L}{\partial \theta}\right]^2 = E\left[\sum_{1}^{n} \frac{\partial \log f(x_i\, \theta)}{\partial \theta}\right]^2$$

$$= E \sum_{1}^{n}\left(\frac{\partial \log f(x_i\, \theta)}{\partial \theta}\right)^2 + \sum_{i \neq j}\left(\frac{\partial \log f(x_i\, \theta)}{\partial \theta}\right)\left(\frac{\partial \log f(x_j\, \theta)}{\partial \theta}\right)$$

$$= n\, E\, \frac{\partial \log f(x, \theta)}{\partial \theta} + 0$$

$$\therefore E\left(\frac{\partial \log L}{\partial \theta}\right)^2 = n\, E\left(\frac{\partial \log f(x, \theta)}{\partial \theta}\right)^2$$

$\therefore$ Cramer Rao inequality can be written as

$$var(T) \geq \frac{[\gamma'(\theta)]^2}{n\, E\left(\frac{\partial \log f(x, \theta)}{\partial \theta}\right)^2}$$

$$\geq \frac{[\gamma'(\theta)]^2}{-n\, E\, \frac{\partial^2 \log f(x, \theta)}{\partial \theta^2}}$$

Minimum Variance Bound Estimator (MVB)

An estimator T of $\gamma(\theta)$ for which Cramer-Rao lower bound is attained is called a minimum variance bound (MVB) estimator.

. Minimum variance unbiased estimator is unique in the sense that if T_0 and T_1 are two MVUE, then $T_0 = T_1$ a.e for all $\theta \in \Theta$

Proof: It is given that T_0 and T_1 are two minimum VUE. This means that (6)

$$E_\theta(T_0) = E_\theta(T_1) = \gamma(\theta) \quad \text{--- (1)}$$

and $$V_\theta(T_0) = V_\theta(T_1) \quad \text{--- (2)}$$

Now consider a new estimator
$$T = \frac{T_0 + T_1}{2}, \quad \text{which is unbiased.}$$

Consider
$$V(T) = V\left[\frac{T_0+T_1}{2}\right]$$
$$= \frac{1}{4}\left[V(T_0) + V(T_1) + 2\rho_\theta \sqrt{V(T_0)} \cdot \sqrt{V(T_1)}\right]$$
$$= \frac{1}{4}\left[V(T_0) + V(T_0) + 2\rho_\theta V(T_0)\right] \text{ by (2)}$$
$$= \frac{V(T_0)}{2}\left[1 + \rho_\theta\right].$$

ie $$\frac{V(T)}{V(T_0)} = \frac{1+\rho_\theta}{2}$$

Since T_0 is a MVUE, we have
$$\frac{1+\rho_\theta}{2} \geq 1 \quad \text{since } \frac{V(T)}{V(T_0)} \geq 1.$$

$\Rightarrow \rho_\theta \geq 2 - 1 = 1$, ie $\rho_\theta \geq 1$

$\Rightarrow \rho_\theta = 1$ since $-1 \leq \rho_\theta \leq +1$.

The correlation coefficient between T_0 and T_1 is unity. If the two variables T_0 and T_1 are linearly dependent.

ie $$T_1 = a(\theta) + b(\theta) \cdot T_0 \quad \text{--- (3)}$$

where $a(\theta)$ and $b(\theta)$ are independent of x but may be depend on θ.

Then $$E_\theta(T_1) = a(\theta) + b(\theta) \cdot E_\theta(T_0)$$
$$\gamma(\theta) = a(\theta) + b(\theta) \cdot \gamma(\theta) \quad \text{--- (4)}$$

and $$V_\theta(T_1) = b^2(\theta) \cdot V_\theta(T_0)$$

since T_0 and T_1 are MVUE

$\Rightarrow b^2(\theta) = 1$

$\Rightarrow b(\theta) = 1$ since ρ is +ve

Substituting the value of $b(\theta)$ in (4)

$$\gamma(\theta) = a(\theta) + \gamma(\theta) \Rightarrow a(\theta) = 0.$$

∴ the equation (3) becomes

$T_1 = T_0$ i.e.

Hence MVUE is unique.

Theorem: The correlation coefficient between an MVUE and an unbiased estimator is the square root of its relative efficiency.

or

Let T_0 be the most efficient estimator and T_1 be an unbiased estimator. Then the correlation coefficient between T_0 and T_1 is

$\rho_\theta = \sqrt{e_\theta}$ where ρ_θ is the correlation coefficient

and $e_\theta = \dfrac{V(T_0)}{V(T_1)}$.

Proof: consider a new estimator T

Let $(1 - 2\sqrt{e_\theta}\,\rho_\theta + e_\theta)T = (1 - \sqrt{e_\theta}\,\rho_\theta)T_0 + \sqrt{e_\theta}\,(\sqrt{e_\theta} - \rho_\theta)T_1$ —(1)

The variance of the above equation is

$$(1 - 2\sqrt{e_\theta}\cdot\rho_\theta + e_\theta)^2 V_\theta(T) = (1 - \sqrt{e_\theta}\cdot\rho_\theta)^2 V_\theta(T_0) + e_\theta(\sqrt{e_\theta} - \rho_\theta)^2 V(T_1)$$

$$+ 2(1 - \sqrt{e_\theta}\,\rho_\theta)T_0\cdot\sqrt{e_\theta}\,(\sqrt{e_\theta} - \rho_\theta)\rho_\theta\sqrt{V(T_0)}\,\sqrt{V(T_1)}$$ —(2)

We know that the relative efficiency

$$e_\theta = \frac{V(T_0)}{V(T_1)} \Rightarrow V(T_1) = \frac{V(T_0)}{e_\theta}$$

Substituting the above in (2), we get

$$(1 - 2\sqrt{e_\theta}\cdot\rho_\theta + e_\theta)^2 V(T) = (1 - \sqrt{e_\theta}\cdot\rho_\theta)^2 V(T_0) + \cancel{e_\theta}(\sqrt{e_\theta} - \rho_\theta)^2\frac{V(T_0)}{\cancel{e_\theta}}$$

$$+ 2(1 - \sqrt{e_\theta}\cdot\rho_\theta)\rho_\theta\cancel{\sqrt{e_\theta}}(\sqrt{e_\theta} - \rho_\theta)\cdot\sqrt{V(T_0)}\cdot\sqrt{\frac{V(T_0)}{\cancel{e_\theta}}}$$

$$= (\cancel{1 + e_\theta\rho_\theta^2 - 2\sqrt{e_\theta}\,\rho_\theta + e_\theta + \rho_\theta^2 - 2\sqrt{e_\theta}\,\rho_\theta}$$

$$\cancel{+ 2\sqrt{e_\theta} - 2\rho_\theta + 2e_\theta\rho_\theta} + 2\sqrt{e_\theta}\,\rho_\theta)V(T_0)$$

$$= \cancel{1 - e_\theta\rho_\theta^2 + \rho_\theta}$$

$$= 1 - 2\sqrt{e_\theta}\,\rho_\theta + e_\theta\rho_\theta^2 + e_\theta - \rho_\theta^2 - 2\sqrt{e_\theta}\,\rho_\theta$$

$$+ 2\rho_\theta\sqrt{e_\theta} - 2\rho_\theta^2 - 2e_\theta\rho_\theta^2 + 2\sqrt{e_\theta}\,\rho_\theta^3$$

$$= \Big[1 + e_\theta - 2\sqrt{e_\theta}\,\rho_\theta - e_\theta\rho_\theta^2 - \rho_\theta^2$$

$$+ 2\sqrt{e_\theta}\,\rho_\theta^3\Big]V(T)$$

(8)

$$= (1+e_\theta - 2\sqrt{e_\theta}\,\rho_\theta)(1-\rho_\theta^2)(1+e_\theta - 2\sqrt{e_\theta}\,\rho_\theta)$$

$$= (1+e_\theta - 2\sqrt{e_\theta}\,\rho_\theta)(1-\rho_\theta^2)\, v(T_0)$$

$$\therefore v(T) = \frac{(1-\rho_\theta)^2\, v(T_0)}{(1+e_\theta - 2\sqrt{e_\theta}\,\rho_\theta)}$$

$$= \frac{(1-e_\theta^2)}{(1-e_\theta^2)(\sqrt{e_\theta}-\rho_\theta)^2}\, v(T_0).$$

The term on RHS $(1-\rho_\theta^2)$ and $(\sqrt{e_\theta}-\rho_\theta)^2$ are positive. This implies that $v(T_0)$ is greater than or equal to $v(T)$. But this is impossible since T_0 is MVUE. This is possible if $\rho_\theta = \sqrt{e_\theta}$ i.e. the correlation between most efficient estimator and unbiased estimator is the square root of its relative efficiency.

$\therefore$ A necessary and sufficient condition for the existence MVB estimator is that the distribution must belong to exponential family. (or) A necessary and sufficient condition for C.R inequality to become equality the distribution of T must belong to exponential family.

Proof: Consider C.R. Rao inequality

$$V(T) \geq \frac{[\gamma'(\theta)]^2}{E\left[\frac{\partial \log L}{\partial \theta}\right]^2}$$

To make the above inequality into an equality, we must have (from Cauchy Schwartz inequality),

$$[\operatorname{Cov}(x,y)]^2 = \operatorname{Var}(x) \cdot \operatorname{Var}(y)$$

Here $x = \frac{\partial \log L}{\partial \theta}$ and $y = T$.

i.e the correlation between x and y must be either +1 or -1.

That is the two variables x and y must be linearly dependent.

i.e $\frac{\partial \log L}{\partial \theta} = \lambda\,[T - \gamma(\theta)] + h$

where λ and h are independent of x but may depend on θ.

Taking expectation on both sides,

$$E\left[\frac{\partial \log L}{\partial \theta}\right] = \lambda\,[E(T) - \gamma(\theta)] + h$$

$$0 = \lambda\,[\gamma(\theta) - \gamma(\theta)] + h$$

$$\Rightarrow h = 0$$

$$\therefore \frac{\partial \log L}{\partial \theta} = \lambda(\theta)\,[T - \gamma(\theta)]$$

$$= \lambda(\theta) \cdot T - \lambda(\theta) \cdot \gamma(\theta)$$

Integrating w.r.t θ we get

$$\partial \log L = A(\theta) \cdot B(x) + D(\theta) + C(x)$$

$$\Rightarrow L = e^{A(\theta) \cdot B(x) + D(\theta) + C(x)}$$

$\Rightarrow$ the distribution of T must belong to exponential family.

Theorem: Necessary and sufficient condition for the existence of MVB estimator.

Proof: In deriving C.R. inequality, we used

$$[\gamma'(\theta)]^2 \leq E[t-\gamma(\theta)]^2 . E\left[\frac{\partial \log L}{\partial \theta}\right]^2$$

The sign of equality will hold in C.R inequality if and only if the sign of equality holds in Cauchy Schwartz inequality. But the equality of Cauchy Schwartz inequality will hold if and only if the variables $[t-\gamma(\theta)]$ and $\frac{\partial \log L}{\partial \theta}$ are proportional to each other.

$$\text{ie} \quad \frac{t-\gamma(\theta)}{\frac{\partial \log L}{\partial \theta}} = \lambda = \lambda(\theta)$$

where λ is a constant independent of x but may depend on θ.

$$\therefore \frac{\partial \log L}{\partial \theta} = \frac{t-\gamma(\theta)}{\lambda(\theta)}$$

$$= [t-\gamma(\theta)] . A(\theta) \quad \text{—— (1)}$$

$$\text{Where} \quad A(\theta) = \frac{1}{\lambda(\theta)}$$

Hence the necessary and sufficient condition for an unbiased estimator T to attain the lower bound 8 its variance is given by (1).

Further the C.R minimum variance bound is given by

$$\text{var}(T) = [\gamma'(\theta)]^2 / E\left(\frac{\partial \log L}{\partial \theta}\right)^2 \quad \text{—— (2)}$$

But from (1) $E\left[\frac{\partial \log L}{\partial \theta}\right]^2 = E[A(\theta).(T-\gamma(\theta))]^2$

$$= [A(\theta)]^2 . E[T-\gamma(\theta)]^2$$

$$= [A(\theta)]^2 . \text{var}(T). \quad \text{—— (3)}$$

Substituting ③ in ②, we get

$$\text{var}(T) = \frac{[\gamma'(\theta)]^2}{[A(\theta)]^2 \cdot \text{var}(T)}$$ ⑪

$$\Rightarrow [A(\theta)]^2 \cdot \text{var}(T) = \frac{[\gamma'(\theta)]^2}{V(T)}$$

$$\Rightarrow (\text{var}\, T)^2 = \frac{[\gamma'(\theta)]^2}{[A(\theta)]^2}$$

$$\Rightarrow \text{var}(T) = \frac{\gamma'(\theta)}{A(\theta)}$$

Example.

1. Obtain MVB estimator for λ of Poisson distribution.

Sol: Consider p.m.f of Poisson distribution

$$p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad \lambda > 0, \; x = 0, 1, 2, \ldots$$

Let $x_1, x_2, \ldots x_n$ be a random sample taken from the population that follows Poisson distribution.

Consider

$$L = \prod p(x_i)$$
$$= \prod_{i=1}^{n} e^{-\lambda}\lambda^{x_i} / x_i!$$
$$= e^{-n\lambda}\lambda^{\Sigma x_i} / \prod x_i!$$

$$\log L = -n\lambda + \Sigma x_i \log\lambda + c$$
$$= -n\lambda + n\bar{x}\log\lambda + c$$

$$\frac{\partial \log L}{\partial \lambda} = -n + \frac{n\bar{x}}{\lambda} + 0 \quad \text{—— ①}$$
$$= \frac{n}{\lambda}[\bar{x} - \lambda]$$

But the necessary and sufficient condition for the existence of MVB estimator is

$$\frac{\partial \log L}{\partial \theta} = A(\theta)[T - \gamma(\theta)] \quad \text{—— ②}$$

and $\text{var}(T) = \frac{\gamma'(\theta)}{A(\theta)}$.

Comparing the two equations, we get

$$Var(T) = \frac{1}{n/\lambda} = \frac{\lambda}{n}$$ (12)

5. Obtain UMVUE / MVB estimator for μ of $N(\mu, \sigma^2)$ when σ^2 is known.

Sol: Let $x_1, x_2, \ldots x_n$ be a random sample of size n taken from the normal population $N(\mu, \sigma^2)$.

$$f(x_i, \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2}$$

$$L = \prod f(x_i, \mu, \sigma^2)$$

$$= \prod_{i=1}^{n} \left(\frac{1}{\sigma\sqrt{2\pi}}\right) e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2}$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{\Sigma(x_i-\mu)^2}{2\sigma^2}}$$

$$\log L = c - \frac{1}{2\sigma^2}\Sigma(x_i-\mu)^2$$

$$\frac{\partial \log L}{\partial \mu} = +\frac{2}{2\sigma^2}\Sigma(x_i-\mu)$$

$$= +\frac{1}{\sigma^2}\Sigma(x_i-\mu)$$

$$= \frac{n(\bar{x}-\mu)}{\sigma^2} \quad \text{--- (1)}$$

The necessary and sufficient condition for the existence of MVB estimator is

$$\frac{\partial \log L}{\partial \mu} = A(\theta)\left[T - \gamma(\theta)\right] \quad \text{--- (2)}$$

and $V(T) = \frac{\gamma'(\theta)}{A(\theta)}$

Comparing (1) & (2) we have

$A(\theta) = n/\sigma^2$, $\gamma(\theta) = \mu$, the parameter.

$$\therefore V(\bar{x}) = \frac{1}{n/\sigma^2} = \sigma^2/n = Var(T).$$

$\therefore \bar{x}$ is a MVB estimator.

MVB estimator for p of Binomial distribution.

Let $x_1, x_2 \ldots x_n$ be a random sample of size n taken from a population having binomial distribution. 13

$$p(x) = \binom{n}{x} p^x q^{n-x}, \quad \text{for } x = 0, 1, 2, \ldots, \quad 0 < p, q < 1$$

Then the likelihood function is given by

$$L = \prod p(x_i)$$
$$= \prod_{i=1}^{n} \binom{n}{x_i} p^{x_i} q^{n-x_i}$$
$$= \prod_{i=1}^{n} \binom{n}{x_i} p^{x_i} (1-p)^{n-x_i}$$
$$= \prod \binom{n}{x_i} p^{\Sigma x_i} \cdot (1-p)^{\Sigma(n-x_i)}$$
$$= \prod \binom{n}{x_i} p^{n\bar{x}} \cdot q^{n^2 - n\bar{x}}$$

Taking log on both sides

$$\log L = \Sigma \log \binom{n}{x_i} + n\bar{x} \log p + n^2 - n\bar{x} \log q$$
$$= \Sigma \log \binom{n}{x_i} + n\bar{x} \log p + (n^2 - n\bar{x}) \log(1-p)$$

Differentiate w.r.t. p

$$\frac{\partial \log L}{\partial p} = 0 + \frac{n\bar{x}}{p} + \frac{(n^2 - n\bar{x})}{1-p}(-1)$$
$$= \frac{n\bar{x}}{p} - \frac{n^2 - n\bar{x}}{q}$$
$$= \frac{n\bar{x}}{p} + \frac{n\bar{x}}{q} - \frac{n^2}{q}$$
$$= \frac{q n\bar{x} + n\bar{x}p}{pq} - \frac{n^2}{q}$$
$$= \frac{n\bar{x}(p+q)}{pq} - \frac{n^2}{q}$$
$$= \frac{n\bar{x}}{pq} - \frac{n^2}{q}$$
$$= \frac{n}{pq}\left[\bar{x} - np\right] \quad \text{--- (1)}$$

Scanned with CamScanner

By nec and Suff. condition for the existence of MVB estimator, we have

$$\frac{\partial \log L}{\partial \theta} = A(\theta).[T - \gamma(\theta)] \quad \text{—} ②$$

and $V(T) = \dfrac{\gamma'(\theta)}{A(\theta)}$.

(14)

comparing ① & ②, we have

$$V(T) = \frac{n}{n/pq} = pq$$

But in Binomial distribution, if $X \sim B(n,p)$, then

$V(X) = npq$.

$$V(\bar{x}) = V\left(\frac{\Sigma x}{n}\right) = \frac{\Sigma V(x_i)}{n^2} = \frac{n^2 pq}{n^2} = pq.$$

Since in B.D, $\bar{x} = pq$

$\therefore$ MVB = estimator for p is $\bar{x}$.

2. Find MVB estimator for p of binomial distribution.

Sol: The p.m.f of B.D is

$$p(x) = \binom{n}{x} p^x q^{n-x}$$

Since the likelihood function and pmf are the same

$$L = \binom{n}{x} p^x q^{n-x}$$

now $\log L = \log \binom{n}{x} + x \log p + (n-x) \log q$

$$\frac{\partial \log L}{\partial p} = \frac{x}{p} + \frac{n-q}{q} x - 1$$

$$= \frac{qx - np + px}{pq} = \frac{x - np}{pq}$$

$$= \frac{n}{pq}\left[\frac{x}{n} - p\right]$$

The necessary and sufficient condition for MVB estimator is

$$\frac{\partial \log L}{\partial \theta} = A(\theta).[T - \gamma(\theta)], \quad V(T) = \frac{\gamma'(\theta)}{A(\theta)}.$$

Here $\gamma'(\theta) = 1$

now $V(\bar{x}) = V\left(\frac{x}{n}\right) = \dfrac{1}{n/pq} = \dfrac{pq}{n}$

But in B-D, $V(\bar{x}) = \dfrac{pq}{n}$. $\therefore$ $\bar{x}$ is MVB estimator for p.

Suppose $x_1, x_2, \ldots x_n$ is a random sample drawn from Normal distribution with known mean μ and unknown variance σ^2. Show that $s_0^2 = \frac{1}{n}\Sigma(x_i-\mu)^2$ is a MVBE for σ^2.

Sol: It is given that $x_1, x_2 \ldots x_n$ is a random sample of size n taken from Normal distribution with mean μ (known) and unknown variance σ^2.

The p.d.f is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \quad \text{—} ① \qquad \begin{matrix} -\infty < x < \infty \\ -\infty < \mu < \infty \\ \sigma^2 > 0. \end{matrix}$$

(15)

As σ^2 is to be estimated, let $\theta = \sigma^2$ & $\sqrt{\theta} = \sigma$

$\therefore$ ① is

$$f(x) = \frac{1}{\sqrt{\theta}\sqrt{2\pi}} e^{-\frac{1}{2\theta}(x-\mu)^2}$$

$$\log f = -\frac{1}{2}\log 2\pi - \frac{1}{2}\log\theta - \frac{1}{2\theta}(x-\mu)^2$$

$$\frac{\partial \log f}{\partial\theta} = 0 - \frac{1}{\theta} + \frac{1}{2\theta^2}(x-\mu)^2$$

$$\frac{\partial^2 \log f}{\partial\theta^2} = +\frac{1}{2\theta^2} - \frac{(x-\mu)^2}{\theta^3}$$

$$E\left[\frac{\partial^2\log f}{\partial\theta^2}\right] = \frac{1}{2\theta^2} - \frac{\theta}{\theta^3} = \frac{1}{2\theta^2} - \frac{1}{\theta^2}$$

$$= \frac{1-2}{2\theta^2} = -\frac{1}{2\theta^2}$$

$$\therefore -\frac{1}{nE\left[\frac{\partial^2 \log f}{\partial\theta^2}\right]} = \frac{-1}{-\frac{1}{2\theta^2}} = \frac{2\theta^2}{n} = \frac{2\sigma^4}{n}. \quad \text{—} ②$$

But we have

$$s_0^2 = \frac{\Sigma(x_i-\mu)^2}{n}$$

$$ns_0^2 = \Sigma(x_i-\mu)^2$$

But we know $\frac{ns_0^2}{\sigma^2} \sim \chi^2$ with n d.f

$\Rightarrow ns_0^2 \sim \chi^2\sigma^2$ with n d.f

$$\therefore V\left(\frac{ns_0^2}{\sigma^2}\right) = 2n$$

$$V(s_0^2) = \frac{2\sigma^4 n}{n^2} \quad \frac{2\sigma^4}{n} \quad \text{—} ③$$

C. R. Rao lower bound is

$$V(T) \geq \frac{[\tau'(\theta)]^2}{-nE\left[\frac{\partial^2 \log f}{\partial \theta^2}\right]}$$

from ① $\geq \dfrac{1}{n \cdot \frac{-1}{2\theta^2}} = \dfrac{2\theta^2}{n} = \dfrac{2\sigma^4}{n}$ — ④

(16)

From ③ & ④ s_0^2 is a MVB estimator for σ^2.

5. Let $x_1\, x_2 \ldots x_n$ be a random sample from $f(x;\theta) = \theta e^{-\theta x}$, $0 < x < \infty$. Examine whether ∃ an MVB estimator for $\frac{1}{\theta}$. If so, obtain its variance.

Sol: Given, $x_1\, x_2 \ldots x_n$ be a random sample from density

$$f(x,\theta) = \theta e^{-\theta x}$$

$$L = \prod f(x,\theta)$$

$$= \theta^n \cdot e^{-\theta \Sigma x}$$

$$\log L = n \log \theta - \theta \Sigma x$$

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} - \Sigma x$$

$$= \frac{n}{\theta} - n\bar{x}$$

$$= -n\left(\bar{x} - \frac{1}{\theta}\right) = A(\theta)[T - \tau(\theta)]$$ — ①

* using the $E(x^2) = \int x^2 \theta e^{-\theta x}\, dx = 2/\theta^2$

$E(x) = \int x \theta e^{-\theta x}\, dx = 1/\theta^2$

$$V(\bar{x}) = \frac{1}{n} V(\Sigma x)$$ *

But $V(x_i) = E(x_i^2) - [E(x)]^2$

$$= \frac{2}{\theta^2} - \frac{1}{\theta^2} = \frac{1}{\theta^2}$$

$$\therefore V(\bar{x}) = \frac{1}{n} \cdot \frac{1}{\theta^2} = \frac{1}{n\theta^2}$$ — ⊖

From ①, $V(T) = \dfrac{\tau'(\theta)}{A(\theta)} = \dfrac{-1/\theta^2}{-n} = \dfrac{1}{n\theta^2}$.

$\therefore$ For $\frac{1}{\theta}$, $\bar{x}$ is the MVB estimator.

Scanned with CamScanner

... there exist a parametric function $\varphi(\theta)$ and an ...ased estimator T of $\varphi(\theta)$ such that $V(T)$ equals the C-R ...er bound and if so, obtain $\varphi(\theta)$ and $V(T)$.

Sol: Let $x_1, x_2 \ldots x_n$ be a random sample of size n.

The pdf is

$$f(x,\theta) = \theta x^{\theta-1} \qquad 0 < x < \alpha$$

(17)

$$L = \theta^n . \prod x_i^{\theta-1}$$

$$\log L = n \log \theta + (\theta-1) \Sigma \log x_i$$

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} + \cancel{\theta-1} \Sigma \log x_i$$

$$= -n \left[\frac{-\Sigma \log x_i}{n} - \frac{1}{\theta} \right] \quad \text{— (1)}$$

$$= A(\theta) \left[T - \gamma(\theta) \right]$$

$\therefore - \frac{\Sigma \log x_i}{n}$ is the unbiased estimator for $\frac{1}{\theta}$.

The parametric function is

$$\varphi(\theta) = \frac{1}{\theta}$$

By necessary and sufficient condition, we have

$$\frac{\partial \log L}{\partial \theta} = A(\theta) \left[T - \gamma(\theta) \right]$$

$$V(T) = \frac{\gamma'(\theta)}{A(\theta)} \quad \text{— (2)}$$

Comparing (1) & (2)

$$A(\theta) = -n$$

$$T = -\frac{\Sigma \log x_i}{n}, \qquad \gamma(\theta) = \varphi(\theta) = \frac{1}{\theta}$$

$$\therefore V(T) = \frac{\varphi'(\theta)}{A(\theta)} = \frac{-1/\theta^2}{-n} = \frac{1}{n\theta^2}$$

7. Show that there exist a parametric function $\varphi(\theta)$ in case of the geometric distribution

$$f(x;\theta) = (1-\theta)\,\theta^{x} \; ; \quad x = 0, 1, 2 \ldots$$
$$0 < \theta < 1$$

$\exists$ a MVB unbiased estimator T of $\varphi(\theta)$.

$V(\bar{x})$, Hence

Sol: Let $x_1, x_2 \ldots x_n$ be a random sample of size n from a geometric distribution

(18)

$$f(x,\theta) = (1-\theta)\,\theta^{x}$$

The likelihood function is

$$L = \prod f(x_i,\theta) = \prod (1-\theta)\,\theta^{x}$$
$$= (1-\theta)^{n}\,\theta^{\Sigma x}$$

$$\log L = n\log(1-\theta) + \Sigma x \log\theta$$

$$\frac{\partial \log L}{\partial\theta} = \frac{-n}{(1-\theta)} + \frac{n\bar{x}}{\theta}$$
$$= \frac{n}{\theta}\left[\bar{x} - \frac{\theta}{1-\theta}\right] \quad \text{— (1)}$$

$\therefore$ The parametric function is

$$\varphi(\theta) = \frac{\theta}{1-\theta}$$

By the necessary and sufficient condition

$$\frac{\partial \log L}{\partial\theta} = A(\theta)\left[T - \gamma(\theta)\right] \Rightarrow \quad V(T) = \frac{\gamma'(\theta)}{A(\theta)}$$

$$V(\bar{x}) = V\left(\frac{\Sigma x}{n}\right)$$
$$= \frac{1}{n^2}\,\Sigma V(x)$$
$$= \frac{1}{n^2}\,\Sigma\,\theta(1-\theta)^{-2}$$
$$= \frac{n\theta(1-\theta)^{-2}}{n^2} = \frac{\theta}{n(1-\theta)^{2}}$$

Now $V(T) = V(\bar{x}) = \dfrac{\varphi'(\theta)}{A(\theta)}$

From (1),
$$\varphi(\theta) = \frac{\theta}{1-\theta}$$
$$A(\theta) = n/\theta$$

$$V(T) = \frac{(1-\theta)\cdot 1 + \theta(1)}{(1-\theta)^2} \Big/ A(\theta) = \frac{1-\theta+\theta}{(1-\theta)^2} \Big/ n/\theta = \frac{\theta}{n(1-\theta)^2}$$

Scanned with CamScanner

$$v(\bar{x}) = \frac{\psi'(\theta)}{A(\theta)}.$$

Hence $\exists$ a parametric function $\psi(\theta)$

$\Rightarrow \exists$ a MVB estimator T of $\psi(\theta)$.

8. Examine if sample mean is a MVB estimator for θ in Cauchy distribution.

Sol: Let $x_1, x_2 \ldots x_n$ be a random sample drawn from a population with Cauchy distribution.

(19)

$$L = \prod f(x_i, \mu)$$

$$= \prod \frac{1}{\pi} \cdot \frac{1}{1+(x_i-\mu)^2}$$

$$= \left(\frac{1}{\pi}\right)^n \cdot \prod \frac{1}{1+(x-\mu)^2}$$

$$\log L = k + \Sigma \log \frac{1}{1+(x_i-\mu)^2}$$

$$= k + \Sigma \log \left[1+(x_i-\mu)^2\right]^{-1}$$

$$= k - \Sigma \log \left[1+(x_i-\mu)^2\right]$$

$$\frac{\partial \log L}{\partial \mu} = - \Sigma \frac{1}{1+(x_i-\mu)^2} \cdot 2(x_i-\mu).(-1)$$

$$= 2\Sigma \frac{(x_i-\mu)}{1+(x_i-\mu)^2}$$

This expression cannot be written in the form of

$$\frac{\partial \log L}{\partial \theta} = A(\theta)\left[T - \gamma(\theta)\right]$$

Hence, for Cauchy distribution MVB estimator does not exist.

9. Let $x_1, x_2 \ldots x_n$ be a random sample drawn from $N(0, \sigma^2)$. Obtain MVB estimator for σ^2.

Sol: Let $x_1, x_2 \ldots x_n$ be a random sample drawn from $N(0, \sigma^2)$

$$f(x; 0, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}x^2}$$

$$L = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2\sigma^2}\Sigma x^2}$$

$L = (\sqrt{2\pi})^{-n} \cdot (\sigma^2)^{-n/2} \cdot e^{-\frac{1}{2\sigma^2}\Sigma x^2}$

$\therefore \log L = c - \frac{n}{2}\log\sigma^2 - \frac{\Sigma x^2}{2\sigma^2}$

$$\frac{\partial \log L}{\partial \sigma^2} = \frac{-n}{2\sigma^2} + \frac{\Sigma x^2}{2\sigma^4}$$

$$= \frac{n}{2\sigma^4}\left[\frac{\Sigma x^2}{n} - \sigma^2\right]$$

(28)

By nec & suff. condition

$$\frac{\partial \log L}{\partial \theta} = A(\theta)\,[T - \gamma(\theta)] \qquad V(T) = \frac{\gamma'(\theta)}{A(\theta)}$$

Here $T = \frac{\Sigma x^2}{n}$

$$V(T) = \frac{\gamma'(\theta)}{A(\theta)} = \frac{1}{n/2\sigma^4} = \frac{2\sigma^4}{n}$$

Hence $\frac{\Sigma x^2}{n}$ is a MVBE.

...'s Bound.

The generalisation of C.R lower bound ... give a lower bound for the variance of the unbiased estimator due to Bhattacharya. Here we assume that the parametric function is differentiable k times and if $k=1$, the Bhattacharya lower bound becomes C.R. Rao lower bound.

Regularity conditions:

1. Θ is either on the real line or an interval on the real line

2. For almost all $x_1, x_2 \dots x_n$ (21)

$$\frac{\partial^i L}{\partial \theta^i} \text{ exist } \forall \ \theta \in \Theta$$

3. $$\frac{\partial^i}{\partial \theta^i} \int L\,dx = \int \frac{\partial^i L}{\partial \theta^i}\,dx$$

4. $$\frac{\partial^i}{\partial \theta^i} \int t\,L\,dx = \int t.\frac{\partial^i L}{\partial \theta^i}\,dx$$

5. $V_{ij} = \text{cov}(\beta_i, \beta_j)$ where $\beta_i = \frac{1}{L}.\frac{\partial L}{\partial \theta}$, $|V_{ij}| \geq 0, \neq 0$.

Statement. If T is an unbiased estimator of θ, under certain regularity conditions,

$$V(T) \geq \frac{\begin{vmatrix} V_{22} & V_{23} & \dots & V_{2k} \\ V_{32} & V_{33} & \dots & V_{3k} \\ \vdots & & & \\ V_{k2} & V_{k2} & \dots & V_{kk} \end{vmatrix}}{\begin{vmatrix} V_{11} & V_{12} & V_{13} & \dots & V_{1k} \\ V_{21} & V_{22} & V_{23} & \dots & V_{2k} \\ \vdots & & & & \\ V_{k1} & V_{k2} & \dots & & V_{kk} \end{vmatrix}}$$

Proof: By def $\int L\,dx = 1$

By condition (3), we have

$$\int \frac{\partial^i L}{\partial \theta^i}.dx = 0$$

$$\Rightarrow \int \frac{1}{L}\frac{\partial^i L}{\partial \theta^i}.L\,dx = 0 \quad \text{--- (1)}$$

It is given that T is an unbiased estimator of θ.

$$\Rightarrow \theta = \int T\, L\, dx$$

By applying regularity condition ④, we have

$$\int T \frac{\partial^i L}{\partial \theta^i}\, dx = \begin{cases} 1 & \text{if } i=1 \\ 0 & \text{if } i \geq 2 \end{cases}$$

(22)

This can be re written as

$$\int T \frac{1}{L} \frac{\partial^i L}{\partial \theta^i}\, L\, dx = \begin{cases} 1 & \text{if } i=1 \\ 0 & \text{if } i \geq 2 \end{cases}$$

Multiplying ① on both sides by θ and subtracting the result from ②, we have

$$\int (T-\theta) \cdot \frac{1}{L} \frac{\partial^i L}{\partial \theta^i}\, L\, dx = \begin{cases} 1 & \text{if } i=1 \\ 0 & \text{if } i \geq 2 \end{cases}$$

This is the covariance between T and $\frac{1}{L} \frac{\partial^i L}{\partial \theta^i}$.

ie $cov(T, S_i) = V_{0i} = \begin{cases} 1 & \text{if } i=1 \\ 0 & \text{if } i \geq 1 \end{cases}$

Now consider $(k+1)$ non-random real variables, $y_0, y_1 \ldots y_k$ and we have

$$E\left[y_0(T-\theta) + y_1 S_1 + y_2 S_2 + \ldots + y_k S_k\right]^2 \geq 0$$

This can be written in the quadratic form

$$\sum_{i=0}^{k} \sum_{j=0}^{k} V_{ij}\, y_i y_j \geq 0$$

$$\Rightarrow |V_{ij}| \geq 0$$

ie
$$\begin{vmatrix} V_{00} & V_{01} & V_{02} \ldots & V_{0k} \\ V_{10} & V_{11} & V_{12} \ldots & V_{1k} \\ \vdots & & & \\ V_{k0} & V_{k1} & \ldots\ldots & V_{kk} \end{vmatrix} \geq 0$$

$$V_{00} \begin{vmatrix} V_{11} & V_{12} & \ldots & V_{1k} \\ V_{21} & V_{22} & \ldots & V_{2k} \\ & & & \\ V_{k1} & V_{k2} & \ldots & V_{kk} \end{vmatrix} - V_{01} \begin{vmatrix} V_{10} & V_{12} & \ldots & V_{1k} \\ V_{20} & V_{22} & & V_{2k} \\ \vdots & & & \vdots \\ V_{k0} & V_{k2} & & V_{kk} \end{vmatrix} \geq 0$$

$$\begin{vmatrix} v_{11} & v_{12} & \cdots & v_{1k} \\ v_{21} & v_{22} & \cdots & v_{2k} \\ \vdots & & & \\ v_{k1} & v_{k2} & \cdots & v_{kk} \end{vmatrix} - v_{01} \times v_{10} \begin{vmatrix} v_{22} & \cdots & v_{2k} \\ v_{32} & \cdots & v_{3k} \\ \vdots & & \\ v_{k2} & \cdots & v_{kk} \end{vmatrix} \geq 0$$

This gives

$$V(T) \geq \frac{\begin{vmatrix} v_{22} & v_{23} & \cdots & v_{2k} \\ v_{32} & v_{33} & \cdots & v_{3k} \\ v_{k2} & v_{k3} & \cdots & v_{kk} \end{vmatrix}}{\begin{vmatrix} v_{11} & v_{12} & \cdots & v_{1k} \\ v_{21} & v_{22} & \cdots & v_{2k} \\ \vdots & & & \\ v_{k1} & v_{k2} & \cdots & v_{kk} \end{vmatrix}}$$

Hence the proof.

complete family of densities.

Let $x_1, x_2 \ldots x_n$ denote a random sample from density $f(\cdot, \theta)$ with parameter space Θ, and let $T = t(x_1, x_2 \ldots$ be a statistic.

The family of densities of T is defined to be complete if and only if $E_\theta[\psi(T)] = 0$ for all $\theta \in \Theta$.

It implies that $P_\theta[\psi(T) = 0] \equiv 1 \quad \forall \theta \in \Theta$, where $\psi(T)$ is a statistic.

Also the statistic T is said to be complete if and only if its family of densities is complete.

Sufficiency and completeness.

We continue our search for finding UMVUE. Our first result will show how sufficiency will help in this search. An unbiased estimator which is a function of sufficient statistics has smaller variance than an unbiased estimator which is not sufficient statistics.

In fact, let $f(\cdot, \theta)$ be the density function from which a sample $x_1, x_2 \ldots x_n$ is drawn. We want to estimate $\gamma(\theta)$. Let $T = t(x_1, x_2 \ldots x_n)$ is an unbiased estimator of $\gamma(\theta)$ and that $S = s(x_1, \ldots x_n)$ is a sufficient statistic.

It can be shown that another unbiased estimator T' can be derived from T such that

i) T' is a function of the sufficient statistic S and
ii) T' is an unbiased estimator of $\gamma(\theta)$ with variance less than or equal to the variance of T.

$\therefore$ in our search for UMVUE, we need to consider only unbiased estimators that are functions of sufficient statistics. This idea is formulated in Rao-Blackwell theorem.

Scanned with CamScanner

Show that $X_{(n)}$, the largest order statistic of the rectangular distribution $f(x,\theta) = \frac{1}{\theta}$, $0 \le x \le \theta$ is complete sufficient statistic.

Sol: Let $x_1, x_2 \ldots x_n$ be a random sample from

$$f(x,\theta) = \frac{1}{\theta}, \quad 0 \le x \le \theta.$$

We know that, here, $X_{(n)}$ the largest order statistic is a sufficient statistic for θ.

Let $T = x_{(n)}$

We know T is distributed with p.d.f

$$g(t) = \frac{nt^{n-1}}{\theta^n}, \quad 0 < t < \theta$$

Let $\varphi(T)$ be a measurable function.

Then $E_\theta[\varphi(T)] = 0 \quad \forall\ \theta \in \Theta.$

$$\Rightarrow \frac{n}{\theta^n} \int_0^\theta \varphi(T) \cdot t^{n-1}\, dt = 0 \quad \forall\ \theta \in \Theta$$

$$\Rightarrow \int_0^\theta \varphi(T) \cdot t^{n-1}\, dt = 0 \quad \forall\ \theta \in \Theta$$

$$\Rightarrow \varphi(T) \cdot \theta^{n-1} = 0 \quad \forall\ \theta \in \Theta.$$

$$\Rightarrow \varphi(T) = 0 \quad \forall\ \theta \in \Theta$$

But for $t > 0$,

$\therefore\ \varphi(t) = 0$ for $0 < t < \theta$

$\forall\ \theta \in \Theta$

ie $\varphi(t) = 0$ a.e

$\therefore\ T = x_{(n)}$ is a complete statistic.

2. Show that Σx_i is a complete sufficient statistic for p in binomial distribution.

Sol: Let $x_1, x_2 \ldots x_n$ be a random sample from binomial distribution with p.m.f

$$p(x) = \binom{n}{x} p^x q^{n-x}, \quad x = 0, 1, 2 \ldots$$

Let $T = \Sigma x_i$

We know that the sum Σx_i has the binomial distribution with pmf

$$f_p(t) = \binom{n}{t} p^t (1-p)^{n-t}$$

For any measurable function $\varphi(t)$

$$E_p[\varphi(T)] = \Sigma\, \varphi(t) \cdot f_p(t)$$
$$= \Sigma\, \varphi(t) \cdot \binom{n}{t} p^t (1-p)^{n-t}$$

$= \Sigma\, a(t) \cdot p^{t} (1-p)^{n-t}$

Where $a(t) = \binom{n}{t} \varphi(t)$

Now $E_p[\varphi(t)] = 0 \quad \forall \; p \in (0,1)$

$\Rightarrow \Sigma\, a(t) \cdot p^{t} (1-p)^{n-t} = 0$

$\Rightarrow a(0) \cdot (1-p)^{n} + a(1) \cdot p(1-p)^{n-1} + \dots + a(n)\, p^{n} (1-p)^{n-n} = 0$

or $(1-p)^{n}\left[a(0) + a(1) \cdot \frac{p}{1-p} + a(2) \cdot \frac{p^{2}}{(1-p)^{2}} + \dots + a(n) \cdot \frac{p^{n}}{(1-p)^{n}} \right] = 0$

$\Rightarrow a(0) + a(1)\lambda + a(2)\lambda^{2} + \dots + a(n)\lambda^{n} = 0$

where $\lambda = \frac{p}{1-p}$, But $\lambda \neq 0$

$\Rightarrow$ all the coefficients of λ are zeros

ie $a(t) = 0 \quad \forall \; t = 0, 1, 2 \dots n$

ie $\varphi(t) \cdot \binom{n}{t} = 0 \quad \forall \; t = 0, 1, 2 \dots$

$\Rightarrow \varphi(t) = 0$ a.s

$\Rightarrow T = \Sigma x_i$ is complete.

3. Let $x_1, x_2 \dots x_n$ be a random sample from Poison population with $p(x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$, $x = 0, 1, 2 \dots$

Prove $T = \Sigma x_i$ is a minimal sufficient statistic.

Sol: Given $x_1, x_2 \dots x_n$ be a random sample from

$p(x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$, $x = 0, 1, 2 \dots$

Let $T = \Sigma x_i$

we know that T has distributed like Poison distribution with parameter $n\lambda$.

ie $P_\lambda(t) = \frac{e^{-n\lambda} (n\lambda)^{t}}{t!}$; $t = 0, 1, 2 \dots$

Now $E_\lambda[\varphi(t)] = \sum_{t=0}^{\infty} \varphi(t) \frac{e^{-n\lambda} (n\lambda)^{t}}{t!}$

$= e^{-n\lambda} \sum_{t=0}^{\infty} \frac{\varphi(t) (n\lambda)^{t}}{t!}$

$$= e^{-n\lambda} \sum_{t=0}^{\infty} a(t)\lambda^t, \quad \text{where } a(t) = \frac{\varphi(t).n^t}{t!}$$

Now $E[\varphi(t)] = 0 \quad \forall \lambda \Rightarrow 0<\lambda<\infty$

$$\Rightarrow \sum a(t)\lambda^t = 0 \quad \forall\ 0<\lambda<\infty$$

ie $a(0)\lambda^0 + a(1)\lambda^1 + \dots = 0$

Since $\lambda>0$, all the coefficients of λ are zeros.

ie $a(t)=0 \quad t=0,1,2\dots$

$$\frac{\varphi(t).n^t}{t} = 0 \quad \forall\ t=0,1,2\dots$$

$\Rightarrow \varphi(t)=0$ a.s $\forall\ t=0,1,2\dots$

Hence $t=\sum x_i$ is a complete sufficient statistic.

4. Let $x_1, x_2 \dots x_n$ follow exponential distribution with parameter θ with p.d.f

$$f_\theta(x) = \theta e^{-\theta x} \quad \theta>0$$

Find minimal sufficient statistic for θ

Sol: Let $x_1 x_2 \dots x_n$ be a random sample from exponential distribution $f_\theta(x) = \theta e^{-\theta x}$, $\theta>0$, $0<x<\infty$.

we know that $\sum x_i$ follows gamma distribution with p.d.f

$$f_\theta(t) = \frac{e^{-\theta t}\theta^t t^{n-1}}{\Gamma n}, \quad 0<t<\infty, \ \theta>0.$$

now $E\ \varphi(t) = \int_0^\infty \varphi(t).\ f_\theta(t)\,dt$

$$= \int_0^\infty \varphi(t).\frac{e^{-\theta t}\theta^t t^{n-1}}{\Gamma n}\,dt$$

$$= \int_0^\infty a(t).\ e^{-\theta t}.\ dt = 0$$

where $a(t) = \dfrac{\varphi(t).\theta^t t^{n-1}}{\Gamma n}$

$\Rightarrow a(t).e^{-\theta t} = 0$

$\Rightarrow \dfrac{\varphi(t).\theta^t t^{n-1}}{\Gamma n} = 0$

$\Rightarrow \varphi(t) = 0$ a.s

$\therefore \sum x_i$ is a minimal suff. stat. for θ.

Assignment No-2

1. Examine if Poisson family of distributions is complete

2. " Binomial " "

3. " Cauchy "

4. " Exponential "

5. " $N(0,1)$ is complete

6. " $N(\mu,1)$ "

7. " Gamma distribution is complete.

Note: Any distribution which belong to exponential family is complete.

Rao - Blackwell theorem.

Let $U = u(x_1, x_2 \ldots x_n)$ be an unbiased estimator of parameter θ and let $T = T(x_1, x_2 \ldots x_n)$ be sufficient statistic for θ. Consider the function $\varphi(t)$ of the sufficient statistic defined as

$$\varphi(t) = E(u/t) \quad \text{——} (1)$$

which is independent of θ, since T is sufficient for θ. Then

i) $E[\varphi(t)]$ is an unbiased estimator of θ —— (2)

ii) $V_\theta[\varphi(t)] \le V_\theta(u)$ —— (3)

Proof: consider $E(u/t)$ which is independent of θ since t is sufficient for θ (by the condition of sufficiency).

Then $\varphi(T) = E(u/t)$ is independent of θ.

Now consider

$$E(u) = E[E(u/t)] = E[\varphi(t)] = \theta$$

$\Rightarrow u$ and $\varphi(t)$ are both unbiased estimators of θ.

Consider the variance of u.

$$V(u) = E[u-\theta]^2$$

$$= E[u - \varphi(t) + \varphi(t) - \theta]^2$$

$$= E[u - \varphi(t)]^2 + E[\varphi(t) - \theta]^2 + 0$$

But we know that

$$V[\varphi(T)] = E[\varphi(T) - \theta]^2$$

Scanned with CamScanner

7 distri...

$\therefore$ $var(u) = +ve + V[\phi(T)]$

$\Rightarrow V(u) \geq V[\phi(T)]$

The equality sign holds if $u = \phi(T)$. (25)

consider the cross product term

$E[(u - \phi(T))(\phi(T) - \theta)] = E[[\phi(T) - \theta][u - \phi(T)]]$

$= E[\phi(T) \cdot (u - \phi(T))] - \theta\, E(u - \phi(T))$

$= E\, \phi(T) \cdot E[u - \phi(T)/T] - \theta \cdot E[u - \phi(T)/T]$

$= E\, \phi(T) \cdot [\phi(T) - \phi(T)] - \theta \cdot E[\phi(T) - \phi(T)]$

$= 0.$

Then $\phi(T) = E(u/t)$ has smaller variance than u.

LEHMANN - SCHEFE THEOREM:

Statement: Let X be a random variable with pdf $f_\theta(x)$ and let $T = t(x)$ be complete sufficient statistic. Then the function of T, $\phi(T)$ is unique and has UMVU for its expected value. that is any function of complete sufficient statistic is UMVUE. In otherwords, if a complete sufficient statistic exist, then every function of it is a UMVUE of its expected value.

OR

Statement: If there is a complete sufficient statistic T for θ then for every estimable real parameter $g(\theta)$ has a unique unbiased estimate with minimum variance unbiased, minimum risk, for strictly convex loss function. That is a estimate and minimum unbiased estimator. This estimator is the only function of T.

Proof: let $g(\theta)$ be estimable. i.e. assume that there is atleast one unbiased estimator function of $g(\theta)$.

By Rao - Blackwell theorem, if the statistic $h(t) = E[f(x)/T]$ is unbiased for $g(\theta)$, it has the property

$$V[h(t)/g(\theta)] \le V[f(x)/g(\theta)]$$

with strict inequality if the loss function is convex, holding for atleast one θ unless $f(x)$ can be written as a function of $h(t)$, almost everywhere for each θ.

So, looking for minimum risk estimate based on sufficient statistic. If there are 2 such estimators $h_1(t)$ and $h_2(t)$, then confine our attention to the unbiased estimator

$$E[h_1(t)] = E[h_2(t)] = g(\theta)$$

$$\Rightarrow E[h_1(t) - h_2(t)] = 0 \quad \forall \ \theta.$$

Thus an unbiased estimator based on T are really the same and such an estimator has smaller variance or risk than any other unbiased estimators. Also,

$$V f(x) = E[f(x) - g(\theta)]^2$$

$$= E[\overline{f(x) - h_1(T)} + \overline{h_1(T) - g(\theta)}]^2$$

$$= E[f(x) - h_1(T)]^2 + E[h_1(t) - g(\theta)]^2 + 0$$

$$\Rightarrow V[f(x)] \geq V h_1(t)$$

or

$$V h_1(t) \le V f(x)$$

Hence the proof.

30

Scanned with CamScanner