Matrices

Definition of Matrix

A matrix 'A' is a rectangular arrangement of scalars / usually presented in the form /

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

The rows of such a matrix A are $(a_{11}\ a_{12} \cdots a_{1n})$ $(a_{21}\ a_{22} \cdots a_{2n}) \cdots (a_{m1}\ a_{m2} \cdots a_{mn})$ are horizontal lists of scalars

The columns of A are vertical lists of scalars

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \cdots \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Simply we denote the matrix A by

$$A = [a_{ij}]_{m\times n}$$

where $a_{ij} = i^{th}$ row j^{th} column element of A.

Symmetric Matrices

A matrix 'A' is symmetric if

$$A^T = A$$

ie, If each $a_{ij} = a_{ji}$.

A matrix 'A' is skew matrix

$A^T = -A$ if each $a_{ij} = -a_{ij}$

Eg:

1. $A = \begin{bmatrix} 2 & -3 & 5 \\ -3 & 6 & 7 \\ 5 & 7 & -8 \end{bmatrix}$

$A^T = \begin{bmatrix} 2 & -3 & 5 \\ -3 & 6 & 7 \\ 5 & 7 & -8 \end{bmatrix}$

$\because A^T = A$

So the matrix A is symmetrix

$-A = \begin{bmatrix} -2 & 3 & -5 \\ 3 & -6 & -7 \\ -5 & -7 & 8 \end{bmatrix}$

$\because A^T \neq -A$

$\therefore$ A is not skew symmetrix

2. $B = \begin{bmatrix} 0 & 3 & -4 \\ -3 & 0 & 5 \\ -4 & -5 & 0 \end{bmatrix}$

$B^T = \begin{bmatrix} 0 & -3 & -4 \\ 3 & 0 & 5 \\ -4 & -5 & 0 \end{bmatrix}$

Since $B^T \neq B$ so the matrix is not symmetrix

$-B = \begin{bmatrix} 0 & -3 & 4 \\ 3 & 0 & 5 \\ -4 & -5 & 0 \end{bmatrix}$

Since $B^T = -B$

So the matrix is skew symmetric

3. $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $C^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

$-C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Since the given matrix C is not a square matrix so its neither symmetric nor skew symmetric

Orthogonal matrices

A real matrix 'A' is orthogonal if

$A^T = A^{-1}$

pre-multiply by A on both sides

$$A \cdot A^T = A \cdot A^{-1}$$
$$= I$$

$$\Rightarrow A \cdot A^T = I$$

Check whether the matrix A is orthogonal

$$A = \begin{bmatrix} 1/9 & 8/9 & -4/9 \\ 4/9 & -1/9 & -7/9 \\ 8/9 & 1/9 & 4/9 \end{bmatrix}$$

$$A = \begin{bmatrix} 1/9 & 8/9 & -4/9 \\ 4/9 & -1/9 & -7/9 \\ 8/9 & 1/9 & 4/9 \end{bmatrix}$$

$$A = \frac{1}{9} \begin{bmatrix} 1 & 8 & -4 \\ 4 & -1 & -7 \\ 8 & 1 & 4 \end{bmatrix}$$

$$A^T = \frac{1}{9}\begin{bmatrix} 1 & 4 & 8 \\ 8 & -4 & 1 \\ -4 & -7 & 4 \end{bmatrix}$$

$$A \cdot A^T = \frac{1}{9}\begin{bmatrix} 1 & 8 & -4 \\ 4 & -4 & -7 \\ 8 & 1 & 4 \end{bmatrix} \times \frac{1}{9}\begin{bmatrix} 1 & 4 & 8 \\ 8 & -4 & 1 \\ -4 & -7 & 4 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 1+64+16 & 4-32+28 & 8+8-16 \\ 4-32+28 & 16+16+49 & 32-4-28 \\ 8+8-16 & 32-4-21 & 64+1+16 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ A is orthoganal matrix

Rank of a matrix

The rank of a matrix is the order of the largest square sub-matrix whose determinant is non zero.

1. Find the rank of $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 5 \\ 3 & 6 & 6 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 0 & 4 & 5 \\ 3 & 6 & 6 \end{vmatrix} = 3(24-30) - 2(0-15) + 1(0-12)$$

$= 3(-6) + 30 + 1(12) = 0$

$\begin{vmatrix} 4 & 5 \\ 6 & 6 \end{vmatrix} = (24 - 30) = -6 \neq 0$

The rank of a matrix is 2.

2. $A = \begin{bmatrix} -2 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 4 & 7 \end{bmatrix}$

$|A| = \begin{vmatrix} -2 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 4 & 7 \end{vmatrix} = |A| = \begin{vmatrix} -2 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{vmatrix}$

$= -2(4-3) - 2(0-1) + 3(0-1)$

$= -2 + 2 - 3 = -3 \neq 0$

$\therefore$ The rank of the matrix is 3

Properties of orthogonal

* If A and B are orthogonal matrices then $A \times B$ is also orthogonal

* If A is orthogonal then A^T and A^{-1} are also orthogonal.

* If A is orthogonal then $|A| = \pm 1$. If $|A|$ is $+1$ then A is proper matrix. If $|A|$ is -1 then A is improper matrix.

* If λ is an Eigen value of an orthogonal matrix A, then $\frac{1}{\lambda}$ is also an Eigen value of A.

3. $A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 & 0 \\ -3 & 1 & 1 \\ -1 & -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 & 0 \\ 3 & 1 & 1 \\ -1 & -1 & 1 \end{vmatrix} + 0 - 1\begin{vmatrix} 2 & -1 & 1 \\ 3 & -3 & 1 \\ -1 & -1 & 1 \end{vmatrix}$$

$$= 1\{-1(1+1) - 1(-3+1) + 0\} + 1\{2(2) - 1(3+1) + 0\} - 1(2(3+1) + 1(-3+1) + 1(3-3))$$

$$= 1(-2+2) + 1(4-4) - 1(8-2+0)$$

$$= 0 + 0 + 6 = 6 \neq 0$$

$\therefore$ The rank of matrix is 4.

Quadratic forms:

A homogeneous polynomial of a second degree in any no. of variable is called the quadratic forms.

Examples for quadratic forms in two variables.

1) $x^2 + 3xy + y^2$
2) $x^2 + y^2 + z^2 + 2xyz + 4xy + 5yz + 3xz + x + y + z$

quadratic forms in three variables.

General form of a quadratic form:

The general form of a quadratic form in 'n' variables $x_1, x_2, \ldots, x_n$ is given by

$$Q = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_i x_j \quad \text{where } a_{ij} = a_{ji}$$

The matrix is symmetric.

It can always be written as $X^T A X$, where $A = [a_{ij}]$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Example:

Consider $Q = x_1^2 - 4x_1 x_2 + 3x_2^2$

This can be written as $Q = [x_1 \; x_2] \begin{array}{c} x^2 \quad xy \\ \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix} \\ yx \quad y^2 \end{array} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= X^T A X$ where $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix}$

To findout the symmetric matrix A for a quadratic form, a coefficient a_{ij} are the co-efficient x_i^2 in a quadratic form and the half of the co-efficient of $x_i x_j$ is placed in each of the position a_{ij} and a_{ji} of A.

1. Write the matrix form of the quadratic form
$x^2 - y^2 - z^2 + 4xy + 6xz - 2yz$

The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & -1 \\ 3 & -1 & -1 \end{bmatrix}$

The matrix form of the given quadratic form

$Q = X^T A X$

$Q = [x \; y \; z] \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & -1 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

2. Write the matrix form of the quadratic form
$x_1^2 + x_2^2 + x_3^2 + 3x_1 x_2 + 3x_2 x_3 + 3x_1 x_3$

The matrix form $\begin{bmatrix} 1 & 3/2 & 3/2 \\ 3/2 & 1 & 3/2 \\ 3/2 & 3/2 & 1 \end{bmatrix}$

∴ The matrix form of given quadratic form.

$$[x_1\ x_2\ x_3]\begin{bmatrix}1 & 3/2 & 3/2\\ 3/2 & 1 & 3/2\\ 3/2 & 3/2 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Show that $\begin{bmatrix}-1/3 & 2/3 & 2/3\\ 2/3 & -1/3 & 2/3\\ 2/3 & 2/3 & -1/3\end{bmatrix}$ is orthogonal.

Let $A = \begin{bmatrix}-1/3 & 2/3 & 2/3\\ 2/3 & -1/3 & 2/3\\ 2/3 & 2/3 & -1/3\end{bmatrix}$

$$A^T = \begin{bmatrix}-1/3 & 2/3 & 2/3\\ 2/3 & -1/3 & 2/3\\ 2/3 & 2/3 & -1/3\end{bmatrix}$$

$$A \cdot A^T = \frac{1}{3}\begin{bmatrix}-1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{bmatrix}\frac{1}{3}\begin{bmatrix}-1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix}1+4+4 & -2-2+4 & -2+4-2\\ -2-2+4 & 4+1+4 & 4-2-2\\ -2+4-2 & 4-2-2 & 4+4+1\end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = I$$

$A \cdot A^T = I$. ∴ The given matrix is orthogonal.

Eigen values and Eigen vectors.

Characteristic Equation and characteristic roots

The characteristic Equation of matrix A is given by

$|A-\lambda I|=0$ / where $|A-\lambda I|$ is a polynomial of degrees 'n' in λ / By solving the characteristic equation/ we get 'n' roots for λ/ These roots are called characteristic roots or Lation roots or Eigen roots or Eigen values/ corresponding to each characteristic root λ,/ the ~~characteristic~~ equation $(a-\lambda I)x=0$ has non-zero solution, vector x/ The solution vector x is called the characteristic vector or Eigen vector of the given matrix A.

Determine characteristic roots and characteristic vectors are matrix $A=\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$

The characteristic of A is given by

$|A-\lambda I|=0$

$$\begin{vmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -7 & 2 & -3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\left[(1-\lambda)(-3-\lambda)-2\right]-2\left[\cancel{2\lambda}^{(-3-\lambda)}_{2}-(-7)(1-a)\right]+0$$

$$\left[4-(-7)(1-\lambda)\right]=0$$

$$(2-\lambda)\left[-3-\lambda+3\lambda+\lambda^2-2\right]-2\left[-6-2\lambda+7\right]=0$$

$$(2-\lambda)\left[\lambda^2+3\lambda-\lambda-5\right]-2\left[-6-2\lambda+7\right]=0$$

$$(2-\lambda)\left[\lambda^2+2\lambda-5\right]-2\left[-6-2\lambda+7\right]=0$$

$$\left[2\lambda^2+4\lambda-10\right]-\left[\lambda^3+2\lambda^2-5\lambda\right]+12+4\lambda-14$$

$$-\lambda^3+0\lambda^2+13\lambda-12=0$$

$$\lambda^3-13\lambda+12=0 \quad -(1)$$

The characteristic equation is $\lambda^3-13\lambda+12=0$, By solving the equation we get

$\lambda^3 - S_1\lambda^2 + S_2\lambda -$

The Eigen values $\lambda = 1, 3, -4$

$$\begin{array}{r|rrrr}
1 & 1 & 0 & -13 & 12 \\
 & 0 & 1 & 1 & -12 \\
\hline
3 & 1 & 1 & -12 & \boxed{0} \\
 & 0 & 3 & 12 & \\
\hline
-4 & 1 & 4 & \boxed{0} & \\
 & 0 & -4 & & \\
\hline
 & 1 & \boxed{0} & &
\end{array}$$

$S_1 =$

$S_2 =$

$S_3 = |A|$

Consider $(A-\lambda I)X = 0 \quad -(2)$

Put $\lambda = 1$ in (2)

$$\begin{pmatrix} 2-1 & 2 & 0 \\ 2 & 1-1 & 1 \\ -7 & 2 & -3-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ -7 & 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{array}{cccc}
x_2 & x_3 & x_1 & x_2 \\
0 & 1 & 2 & 0 \\
2 & -4 & -7 & 2
\end{array}$$

$x_1 \qquad x_2 \qquad x_3$

$x_1 = 0-2 = -2$

$x_2 = -7+8 = 1$

$x_3 = 4-0 = 4$

when $\lambda = 1$, Eigen values $x = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$

put $\lambda = 3$ in (2)

$$\begin{pmatrix} 2-3 & 2 & 0 \\ 2 & 1-3 & 1 \\ -7 & 2 & -3-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} -1 & 2 & 0 \\ 2 & -2 & 1 \\ -7 & 2 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{array}{cccc} x_2 & x_3 & x_1 & x_2 \\ -2 & 1 & 2 & -2 \\ 2 & -6 & -7 & 2 \end{array}$$

$x_1 = 12-2 = 10$

$x_2 = -7+12 = 5$

$x_3 = 4-14 = -10$

when $\lambda = 3$, Eigen values $x = \begin{bmatrix} 10 \\ 5 \\ -10 \end{bmatrix}$

put $\lambda = -4$ in (2)

$$\begin{bmatrix} 6 & 2 & 0 \\ 2 & 5 & 1 \\ -7 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{array}{cccc} x_2 & x_3 & x_1 & x_2 \\ 5 & 1 & 2 & 5 \\ 2 & 1 & -7 & 2 \end{array}$$

$x_1 = 5-2 = 3$

$x_2 = -7-2 = -9$

$x_3 = 4+35 = 39$

when $\lambda = -4$, Eigen values $x = \begin{bmatrix} 3 \\ -9 \\ 39 \end{bmatrix}$

Determine the characteristic root and characteristic vector for are $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

The characteristic root of the equation given by

$|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix} - \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ 1 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 2-\lambda\,[(1-\lambda)(1-\lambda)-4] - 1[1(1-\lambda)+2] - 1[-2-1(1-\lambda)] = 0$$

$$\Rightarrow \lambda^3 - 4\lambda^2 + \lambda + 6 = 0 \quad -①$$

$$\begin{array}{r|rrrr}
2 & 1 & -4 & 1 & 6 \\
 & 0 & 2 & -4 & -6 \\
\hline
3 & 1 & -2 & -3 & \boxed{0} \\
 & 0 & 3 & 3 & \\
\hline
-1 & 1 & 1 & \boxed{0} & \\
 & 0 & -1 & & \\
\hline
 & 1 & \boxed{0} & &
\end{array}$$

$\lambda = 2, 3, -1$

characteristic vectors

$$(A - \lambda I)x = 0 \quad -②$$

Put $\lambda = 2$ in eqn ②

$$\begin{pmatrix} 2-2 & 1 & -1 \\ 1 & 1-2 & -2 \\ 1 & -2 & 1-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & -2 \\ 1 & -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{array}{cccc} x_2 & x_3 & x_1 & x_2 \\ -1 & -2 & 1 & -1 \\ -2 & -1 & 1 & -2 \end{array}$$

$$x_1 = 1-4 = -3$$

$$x_2 = -2+1 = -1$$

$$x_3 = -2+1 = -1$$

when $\lambda = 2$

Eigen value $x = \begin{pmatrix} -3 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$

put $\lambda = 3$ in eqn ②

$$\begin{pmatrix} 2-3 & 1 & -1 \\ 1 & 1-3 & -2 \\ 1 & -2 & 1-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} -1 & 1 & -1 \\ 1 & -2 & -2 \\ 1 & -2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{array}{cccc} x_2 & x_3 & x_1 & x_2 \\ -2 & 2 & 1 & -2 \\ -2 & -2 & 1 & -2 \end{array}$$

$$x_1 = 4-4 = 0$$

$$x_2 = -2+2 = 0$$

$$x_3 = -2+2 = 0$$

when $\lambda = 3$

Eigen value $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

put $\lambda = -1$ in eqn ②

$$\begin{pmatrix} 2+1 & 1 & -1 \\ 1 & 1+1 & -2 \\ 1 & -2 & 1+1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & -2 \\ 1 & -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{array}{cccc} x_2 & x_3 & x_1 & x_2 \\ 2 & -2 & 1 & 2 \\ -2 & 2 & 1 & -2 \end{array}$$

$$x_1 = 4-4 = 0$$

$$x_2 = -2-2 = -4$$

$$x_3 = -2-2 = -4$$

when $\lambda = -1$ in eqn ②

Eigen value $x = \begin{pmatrix} 0 \\ -4 \\ -4 \end{pmatrix}$.