UNIT - II

Limit of a function.

Let f be a function defined for all points in some neighbourhood of a point 'a' except possibly at the point 'a' itself. we say that the function f tends to the limit l as x tends to 'a' or symbolically

$$\lim_{x \to a} f(x) = L$$

if to each given positive number ϵ, there corresponds a positive number δ such that

$$|f(x) - L| < \epsilon \quad \text{when } 0 < |x-a| < \delta$$

i.e., $f(x) \in \,]l-\epsilon, l+\epsilon[$ for all those values of x (except possibly a) which $\in \,]a-\delta, a+\delta[$

Example: Using the definition of limit, prove that

(i) $\lim\limits_{x \to a} \dfrac{x^2-a^2}{x-a} = 2a$

(ii) $\lim\limits_{x \to a} \dfrac{x^2-4}{x-2} = 4$

(iii) $\lim\limits_{x \to 0} x \sin \dfrac{1}{x} = c$

Solution:

(i) $\lim_{x \to a} \frac{x^2 - a^2}{x - a} = 2a$

$$f(x) = \frac{x^2 - a^2}{(x-a)}, \quad x \neq a$$

Let $\varepsilon > 0$ be given. Then we want to find a, $\delta > 0$ such that

$$|f(x) - 2a| < \varepsilon \text{ whenever } 0 < |x-a| <$$

Now,

$$|f(x) - 2a| = \left| \frac{x^2 - a^2}{x-a} - 2a \right|$$

$$= \left| \frac{x^2 - a^2 - 2a(x-a)}{x-a} \right|$$

$$= \left| \frac{x^2 - a^2 - 2ax + 2a^2}{x-a} \right|$$

$$= \left| \frac{x^2 + a^2 - 2ax}{x-a} \right|$$

$$= \left| \frac{(x-a)^2}{x-a} \right|$$

$$= |x-a|$$

We can choose $\delta = \varepsilon$, then we have

$$|f(x) - 2a| < \varepsilon \text{ whenever } 0 < |x-a| <$$

Hence $\lim_{x \to a} \frac{x^2 - a^2}{x - a} = 2a$

(ii) $\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$

$f(x) = \frac{x^2-2^2}{(x-2)}, x \neq 2;$

Let $\epsilon > 0$ be given

i.e., $f(x) = \frac{(x-2)(x+2)}{(x-2)} = x+2$

$\lim_{x\to 2} (x+2) = 4 \longrightarrow (1)$

By definition

$|f(x) - L| < \epsilon \quad \forall\ (a < |x-a| < \delta)$

$|x+2-4| < \epsilon \quad \forall\ (2 < |x-2| < \delta)$

$|x-2| < \epsilon \quad \forall\ (2 < |x-2| < \delta)$

Hence $\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$

(iii) $\lim_{x\to 0} x \sin \frac{1}{x} = 0$

$f(x) = x \sin \frac{1}{x}$

Let $\epsilon > 0$ be given, Then we want to find a $\delta > 0$ such that

$|f(x) - 0| < \epsilon$ whenever $0 < |x-a| < \delta$

Now, ~~$|f(x) - 0| < \epsilon$ whenever $0 < |x-a| < \delta$~~

$|f(x) - 0| = |x \sin \frac{1}{x}| = |x| |\sin \frac{1}{x}|$

$\forall\ (0 < |x-a| < \delta)$

$|x| |\sin \frac{1}{x} - 0| < \epsilon$

Theorem. uniqueness of limit

Statement: The limit of a function at a point, if it exists is unique i.e., if $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} f(x) = l'$ then $l = l'$

Proof:

$$\left.\begin{aligned} \text{Let } \lim_{x \to a} f(x) &= l \\ \text{and } \lim_{x \to a} f(x) &= l' \end{aligned}\right\} \longrightarrow (1)$$

We shall show that $l = l'$. Let $\varepsilon > 0$ be given

From equation (1), there exists $\delta_1 > 0$ & $\delta_2 > 0$ such that

$$(2) \leftarrow |f(x) - l| < \varepsilon/2, \text{ when } 0 < |x-a| < \delta_1$$

$$(3) \leftarrow |f(x) - l'| < \epsilon/2 \text{ when } 0 < |x-a| < \delta_2$$

Let $\delta = \min(\delta_1, \delta_2)$ Then by equations (2) and (3), we get

$$|f(x) - l| < \varepsilon/2, \text{ when } |x-a| < \delta$$

$$|f(x) - l'| < \varepsilon/2, \text{ when } |x-a| < \delta$$

$$|l - l'| = |l - f(x) + f(x) - l'|$$

$$\leq |l - f(x)| + |f(x) - l'|$$

$$< \epsilon/2 + \epsilon/2 = \epsilon$$

$$\therefore |l - l'| < \varepsilon, \text{ when } 0 < |x-a| < \delta$$

Properties of limit (similar to limit of sequence)

Theorem

Statement: If $\lim_{x \to a} f(x) = l$ and

$\lim_{x \to a} g(x) = m$ then

$$\boxed{\lim_{x \to a} [f(x) \pm g(x)] = l \pm m}$$

Proof: Let $\epsilon > 0$ be given. Then in view of the given limit, there exist $\delta_1 > 0$, $\delta_2 > 0$

such that,

$|f(x) - l| < \epsilon/2$, when $0 < |x-a| < \delta_1$

$|g(x) - m| < \epsilon/2$, when $0 < |x-a| < \delta_2$

Let $\delta = \min(\delta_1, \delta_2)$, then we have

$|f(x) - l| < \epsilon/2$, when $0 < |x-a| < \delta$

$|g(x) - m| < \epsilon/2$, when $0 < |x-a| < \delta$

Now,

$$|\{f(x) \pm g(x)\} - (l \pm m)| = |(f(x) - l) \pm (g(x) - m)|$$
$$\leq |f(x) - l| + |g(x) - m|$$
$$< \epsilon/2 + \epsilon/2$$
$$= \epsilon$$

when $0 < |x-a| < \delta$

$\therefore |\{f(x) \pm g(x)\} - (l \pm m)| < \epsilon$,

whenever $0 < |x-a| < \delta$

Hence,

Theorem:

Statement:

If $\lim_{x\to a} f(x) = l$ and

$\lim_{x\to a} g(x) = m$ then

$$\boxed{\lim_{x\to a}[f(x) \cdot g(x)] = lm}$$

~~Statement~~ Proof:

~~If $\lim f(x)$~~ We have

$$|f(x)g(x) - lm| = |f(x) \cdot g(x) - lg(x) + lg(x) - lm|$$

$$= |g(x)[f(x) - l] + l[g(x) - m]|$$

$$\leq |g(x)||f(x) - l| + |l||g(x) - m| \rightarrow (1)$$

Since $\lim_{x\to a} f(x) = l$ and

$\lim_{x\to a} g(x) = m$, for $0 < \epsilon' < 1$,

there exist some $\delta > 0$, such that

$$(2) \begin{cases} |f(x) - l| < \epsilon' \text{ and} \\ |g(x) - m| < \epsilon' \text{ when } 0 < |x-a| < \delta \end{cases}$$

Now, $|g(x)| = |m + g(x) - m| < \epsilon'$,

when $0 < |x-a| < \delta$

$|g(x)| < |m| + \epsilon'$ when $0 < |x-a| < \delta$

From equations ①, ② and ③ we get,

$$|f(x)g(x) - lm| < \frac{(|m| + \epsilon')\epsilon' + |l|\epsilon'}{}$$

when $0 < |x-a| < \delta$

$$< (|m| + |l| + \epsilon)\epsilon'$$

when $0 < |x-a| < \delta$

$(\because \epsilon' < 1)$ Let us choose ϵ', such that

$$\epsilon' < \frac{\epsilon}{(|m| + |l| + \epsilon)}, \text{ where } \epsilon > 0$$

$\therefore |f(x)g(x) - lm| < \epsilon$, when $0 < |x-a| < \delta$

Hence, $\lim_{x\to a}[f(x)g(x)] = lm$
$= \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

Theorem:

If $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$ and $m \neq 0$

then $\boxed{\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{l}{m}}$

Proof:

We have, $\left|\frac{f(x)}{g(x)} - \frac{l}{m}\right| = \left|\frac{mf(x) - lg(x)}{g(x)\cdot m}\right|$

$$= \frac{|mf(x) - lg(x)|}{|g(x)|\cdot|m|}$$

$$= \frac{|mf(x) - lm + lm - lg(x)|}{|g(x)|\cdot|m|}$$

$$\leq \frac{|m(f(x) - l) + l(g(x) - m)|}{|g(x)|\cdot|m|} \quad \text{①}$$

such that

$$|g(x)| > \frac{|m|}{2}, \text{ when } 0<|x-a|<\delta,$$

$$② \boxed{\frac{1}{|g(x)|} < \frac{2}{|m|}}, \text{ when } 0<|x-a|<\delta,$$

Since $\lim_{x\to a} g(x) = l$, there exists some $\delta_2 > 0$

such that,

$$③ \boxed{|f(x) - l| < \epsilon \frac{|m|}{4}}, \text{ when } 0<|x-a|<\delta_2$$

since $\lim_{x\to a} g(x) = m$, there exists some $\delta_3 > 0$

such that,

$$④ \boxed{|g(x) - m| < \epsilon \frac{|m|^2}{4|l|}}, \text{ when } 0<|x-a|<\delta_3$$

Let $\delta = \min(\delta_1, \delta_2, \delta_3)$, Then from equations

①, ②, ③ and ④, we get

$$\left|\frac{f(x)}{g(x)} - \frac{l}{m}\right| \leq \frac{|m||f(x)-l| + |l||g(x)-m|}{|m|\,|g(x)|}$$

$$< \frac{|m|\left[\epsilon \frac{|m|}{4}\right] + \cancel{|l|}\left[\epsilon \frac{|m|^2}{4\cancel{|l|}}\right]}{|m| \cdot \frac{|m|}{2}}$$

$$< \frac{|m|^2 \epsilon \frac{1}{4} + |m|^2 \epsilon \frac{1}{4}}{\frac{|m|^2}{2}}$$

$$< \frac{2}{|m|^2}\left[|m|^2 \epsilon \frac{1}{4} + |m|^2 \epsilon \frac{1}{4}\right]$$

Hence $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{l}{m}$, $m \neq 0$

Problems

using the properties of limit, find the indicated limits

a) $\lim_{x\to 2} \frac{x^2+2x+3}{2x^3+1}$ Ans $\frac{11}{17}$

b) $\lim_{x\to 2}\left(\frac{1}{x+1} - \frac{1}{x-1}\right)$ Ans $-\frac{2}{3}$

c) $\lim_{x\to 1} \frac{x-1}{x^3+x^2-2x}$ Ans $\frac{1}{3}$

d) $\lim_{x\to 1} \frac{x^8-1}{x^4-1}$ Ans 2

e) $\lim_{x\to 2} \frac{9-x^2}{x+1}$ and $\lim_{x\to 2} (9-x^2)(x+1)$

solution:

$$\lim_{x\to 2} (9-x^2) = \lim_{x\to 2} 9 - \lim_{x\to 2} x^2$$

$$= 9 - 2^2 = 5$$

$$\lim_{x\to 2} (x+1) = \lim_{x\to 2} x + \lim_{x\to 2} 1$$

$$= 2+1 = 3$$

Hence $\lim_{x\to 2} \frac{9-x^2}{x+1} = \frac{5}{3}$

Continuity:

A function is said to be continuous at a particular point if the following three conditions are satisfied.

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$

Note:

A function is said to be continuous if you can trace its graph without lifting the pen from the paper. But a function is said to be discontinuous when it has any gap in between

Problem ①

Let a function be defined as

$$f(x) = \begin{cases} 5 - 2x & \text{for } x < 1 \\ 3 & \text{for } x = 3 \\ x + 2 & \text{for } x > 1 \end{cases}$$

Is this function continuous for all x?

Solution: Since for $x < 1$ and $x > 1$, the function $f(x)$ is defined by straight lines (that can be drawn continuously on a graph), the function will be continuous $\forall\, x \neq$

$\to f(1) = 3$ (given)

$\to$ left hand limit

$$= \lim_{x \to 1^-} f(x)$$

$$= \lim_{x \to 1^-} (5 - 2x)$$

$$= 5 - 2 \times 1$$

$$= 3$$

→ Right hand limit

$$= \lim_{x \to 1^+} f(x)$$

$$= \lim_{x \to 1^+} (x+2)$$

$$= 1+2$$

$$= 3$$

$$\Rightarrow \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 3 = f(1)$$

Thus all the three conditions are satisfied and the function $f(x)$ is found out to be continuous at $x = 1$

$\therefore$ $f(x)$ is continuous for all x.

Problem ②

Determine the constants a and b so that the function f defined below is continuous everywhere

$$f(x) = \begin{cases} 2x+1, & \text{if } x \leq 1 \\ ax^2 + b, & \boxed{\text{if } 1 < x < 3} \\ 5x + 2a & \text{if } x \geq 3 \end{cases}$$

Solution: since f is continuous everywhere, it must be continuous at $\boxed{x = 1}$ and $x = 3$

$\rightarrow f(1) = 3$

$\rightarrow$ Left hand limit

continuity at $x = 1$

$$f(x) = \begin{cases} 2x+1, & \text{if } x \le 1 \\ ax^2+b, & \text{if } 1 < x < 3 \end{cases}$$

$$\left.\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} (2x+1) = 3 \\ \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} (ax^2+b) = a+b \end{aligned}\right] \rightarrow ①$$

continuity at $x = 3$ (Right hand limit)

$$f(x) = \begin{cases} ax^2+b, & \text{if } 1 < x < 3 \\ 5x+2a, & \text{if } x \ge 3 \end{cases}$$

$$\left.\begin{aligned} \lim_{x \to 3^+} f(x) &= \lim_{x \to 3^+} (ax^2+b) = 9a+b \\ \lim_{x \to 3^+} f(x) &= \lim_{x \to 3^+} (5x+2a) = 15+2a \end{aligned}\right] \rightarrow ②$$

From ① and ② i.e., $9a+b = 15+2a$

$a+b = 3 \rightarrow ③$ i.e., $7a+b = 15 \rightarrow ④$

i.e.,
$$\begin{aligned} a+b &= 3 \\ 7a+b &= 15 \\ \hline -6a &= -12 \end{aligned}$$

$$\boxed{a = 2}$$

Substitute $a = 2$ in $a+b = 3$ then,

$$\boxed{b = 1}$$

Problem 3: Let the function defined as

$$f(x) = \begin{cases} x^2 & ; \ 0 \le x < 1 \\ x & : \ 1 \le x < 2 \\ \frac{1}{4} x^2 & : \ 2 \le x < 2 \end{cases}$$

Show that (i) It is continuous at $x = 1$ and

(ii) Discontinuous at $x = 2$

Prove that

Solution:

continuous at $x = 1$

$\rightarrow f(1) = 1$ exists

Left hand limit

$$f(x) = \begin{cases} x^2 ; & 0 \le x < 1 \\ x ; & 1 \le x < 2 \end{cases}$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2) = 1$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x) = 1$$

$f(x)$ is continuous at $x = 1$

Discontinuous at $x = 2$

$\rightarrow f(2) = 2$

Right hand limit

$$f(x) = \begin{cases} x ; & 1 \le x < 2 \\ \frac{1}{4} x^2 ; & 2 \le x < 2 \end{cases}$$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x = 2$ as for $1 \le x < 2$

Since $f(x) \neq f\left(\frac{1}{4}x^2\right)$, the given function $f(x)$ is discontinuous at $x = 2$

Problem (A)

Show that the following functions are continuous at $x = 0$

(i) $f(x) = \begin{cases} x \cdot \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(ii) $f(x) = \begin{cases} x \cdot \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Solution: (i) $f(x) = \begin{cases} x \cdot \sin\frac{1}{x}; & x \neq 0 \\ 0 & x = 0 \end{cases}$

Here, $f(0) = 0$, Also, we have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right), \ h > 0$$

$$= 0,$$

$$f(0-0) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} (-h) \sin\frac{1}{-h}$$

$$= 0$$

Thus $f(0+0) = f(0-0) = f(0) = 0$

So f is continuous at $x = 0$

Derivability of a function:

Let f be a real function with an interval $[a, b]$ as its domain

Derivability at an interior point

Let c be a interior point $[a, b]$ so that $a < c < b$

consider $\frac{f(x) - f(c)}{x - c}$, $x \neq c$

The function f is said to be derivable at c, if

$$\boxed{\lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c} \right]}$$

exists and the limit, called the derivative of f at c, is denoted by the symbol $f'(c)$ or by $Df(c)$

Right-hand derivative

$$\lim_{x \to (c+0)} \frac{f(x) - f(c)}{x - c} \quad \text{or} \quad \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

if it exists, is called the right-hand derivative of f at c and is denoted by $f'(c+0)$ or $f'(c+)$ or $Rf'(c)$

Left-hand derivative

$$\lim_{x \to (c-0)} \frac{f(x) - f(c)}{x - c} \quad \text{or} \quad \lim_{h \to 0^-} \frac{f(c-h) - f(c)}{h}$$

if it exists, is called the Left-hand derivative of f at c and is denoted by $f'(c-0)$ or $f'(c-)$ or $Lf'(c)$

A Necessary condition for the existence of a finite Derivative.

Theorem: Continuity is a necessary but not a sufficient condition for the existence of a finite derivative.

Proof: Let f be derivable at the point c.

so that $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exist

we have,

$$f(x)-f(c) = \left[\frac{f(x)-f(c)}{x-c}\right] x-c \quad ; \; x \neq c$$

$$\Rightarrow \lim_{x\to c}[f(x)-f(c)] = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \lim_{x\to c}(x-c)$$

$$\Rightarrow \lim_{x\to c}[f(x)-f(c)] = f'(c) \cdot 0 = 0$$

$$\Rightarrow \lim_{x\to c} f(x) = f(c)$$

$\Rightarrow f$ is continuous at c.

Thus continuity is a necessary condition for differentiability but it is not a sufficient condition for the existence of a finite derivative

Example:

Derivability of a function

Let f be a real function with an interval $[a, b]$ as its domain

Derivability at an interior point

Let c be a interior point $[a, b]$ so that $a < c < b$

consider $\frac{f(x) - f(c)}{x - c}$, $x \neq c$

The function f is said to be derivable at c, if

$$\boxed{\lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c} \right]}$$

exist and the limit, called the derivative of f at c, is denoted by the symbol $f'(c)$ or by $Df(c)$

Right-hand derivative

$$\lim_{x \to (c+0)} \frac{f(x) - f(c)}{x - c} \text{ or } \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

if it exists, is called the right-hand derivative of f at c and is denoted by $f'(c+0)$ or $f'(c+)$ or $Rf'(c)$

Left-hand derivative

$$\lim_{x \to (c-0)} \frac{f(x) - f(c)}{x - c} \text{ or } \lim_{h \to 0^-} \frac{f(c-h) - f(c)}{-h}$$

if it exists, is called the left-hand derivative of f at c and is denoted by $f'(c-0)$ or $f'(c-)$ or $Lf'(c)$

A Necessary condition for the existence of a finite Derivative.

Theorem: Continuity is a necessary but not a sufficient condition for the existence of a finite derivative.

Proof: Let f be derivable at the point c, so that $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exist

we have,

$$f(x) - f(c) = \left[\frac{f(x) - f(c)}{x-c} \right] x-c \quad ; \quad x \neq c$$

$$\Rightarrow \lim_{x \to c} [f(x) - f(c)] = \lim_{x \to c} \frac{f(x) - f(c)}{x-c} \lim_{x \to c}(x-c)$$

$$\Rightarrow \lim_{x \to c} [f(x) - f(c)] = f'(c) \cdot 0 = 0$$

$$\Rightarrow \lim_{x \to c} f(x) = f(c)$$

$\Rightarrow f$ is continuous at c.

Thus continuity is a necessary condition for differentiability but it is not a sufficient condition for the existence of a finite derivative

~~example~~:

Algebra of derivatives (3)

Theorem I If f and g be two functions which are defined on $[a, b]$ and derivable at any point $c \in [a, b]$, then their sum $f+g$ is also derivable at $x = c$ and $\boxed{(f+g)'(c) = f'(c) + g'(c)}$

Proof: Since f and g are derivable at c,

$$\therefore \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) \text{ and } \lim_{x \to c} \frac{g(x) - g(c)}{x - c} = g'(c) \quad \text{(1)}$$

consider, $\lim_{x \to c} \frac{(f+g)(x) - (f+g)(c)}{x - c}$

$$= \lim_{x \to c} \frac{[f(x) + g(x)] - [f(c) + g(c)]}{x - c}$$

$$= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} + \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

$$= f'(c) + g'(c), \text{ using (1)}$$

Hence, $f+g$ is derivable at c and

$$(f+g)'(c) = f'(c) + g'(c)$$

Theorem II: If f and g are two derivable functions at $x=c$, then fg is also derivable at $x=c$ and

$$\boxed{(fg)'(c) = f(c)g'(c) + g(c)f'(c)}$$

Proof:

Since f and g are derivable at $x=c$

$$\therefore \quad f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \longrightarrow (1)$$

$$g'(c) = \lim_{x\to c} \frac{g(x)-g(c)}{x-c}$$

Since f and g are derivable at $x=c$, f and g are continuous at $x=c$, Thus

$$\left.\begin{aligned} \lim_{x\to c} f(x) &= f(c) \\ \lim_{x\to c} g(x) &= g(c) \end{aligned}\right] \longrightarrow (2)$$

Now, $\lim\limits_{x\to c} \dfrac{(fg)(x)-(fg)(c)}{x-c}$

$$= \lim_{x\to c} \frac{f(x)g(x) - f(c)g(c)}{x-c}$$

$$= \lim_{x\to c} \frac{f(x)g(x) - f(c)g(c) + f(x)g(c) - f(x)g(c)}{x-c}$$

$$= \lim_{x\to c} \frac{f(x)[g(x)-g(c)] + g(c)[f(x)-f(c)]}{x-c}$$

$=$

$$= \lim_{x\to c} f(x) \lim_{x\to c} \frac{g(x)-g(c)}{x-c} + g(c) \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

$$= f(c)g'(c) + g(c)f'(c), \text{ using (1) and (2)}$$

Hence fg is derivable at c and

$$(fg)'c = f(c)g'(c) + g(c)f'(c)$$

Example 1: Show that the function

$$f(x) = \begin{cases} x\sin(1/x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is continuous at $x=0$, but not differentiable at $x=0$

Solution: Test for continuity at $x=0$

$$L.H.L = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} (0-h)\sin\left(\frac{1}{0-h}\right)$$

$$= \lim_{h\to 0} h \sin\frac{1}{h}$$

$$= 0 \times k, \text{ where } -1<k<1$$

$$= 0$$

$$R.H.L = \lim_{h\to 0} f(0+h) = \lim (0+h)\sin\frac{1}{(0+h)}$$

$$= \lim_{h\to 0} h\sin\frac{1}{h}$$

$$= 0$$

Also $f(0) = 0$, Thus we have

$$L.H.L = R.H.L = f(0)$$

$\Rightarrow f(x)$ is continuous at $x=0$

Test for differentiability at $x=0$ ⑥

$$L.H.D = \lim_{h\to 0^-} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0^-} \frac{h\sin(1/h)}{-h}$$

$$= -\lim_{h\to 0^-} \sin\left(\frac{1}{h}\right)$$

which does not exist because as $h\to 0$, $\sin\left(\frac{1}{h}\right)$ oscillates between -1 and 1, and does not tend to a unique and definite limit

$$R.H.D = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0^+} \frac{h\sin(1/h)}{h}$$

$$= \lim_{h\to 0^+} \sin\frac{1}{h}$$

which does not exist because $h\to 0$, $\sin\left(\frac{1}{h}\right)$ oscillates between -1 and 1 and does not tend to a unique and definite limit

Since neither the L.H.D nor the R.H.D exists at $x=0$, it follows that $f(x)$ is not differentiable at $x=0$.

Example 2: If $f(x) = x^2 \sin(\frac{1}{x})$ when $x \neq 0$ and $f(0) = 0$, show that f is derivable for every value of x but the derivative is not continuous for $x = 0$ (7)

Solution: Test for differentiability at $x = 0$

Let $h > 0$ we have

$$R.H.D = Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h}$$

$$= \lim_{h \to 0} h \sin \frac{1}{h} = 0 \times k = 0$$

where $-1 \leq k \leq 1$

$$L.H.D = Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{(-h)^2 \sin(-1/h)}{-h}$$

$$= \lim_{h \to 0} h \sin \frac{1}{h}$$

$$= 0 \times k = 0, \text{ where } -1 \leq k \leq 1$$

Since $Rf'(0) = Lf'(0)$, so $f(x)$ is derivable at $x = 0$ and $f'(0) = 0$

When $x \neq 0$, we easily see that $f(x)$ is also differentiable for such non-zero values of x. Thus $f(x)$ is derivable for every value of x.

To test for continuity of $f'(x)$ at $x=0$

$$f(x) = x^2 \sin\left(\frac{1}{x}\right)$$

$$f'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \text{ at } x \neq 0$$

$$f'(0) = 0$$

$$R.H.L = f'(0+0) = \lim_{h\to 0} f'(0+h) = \lim_{h\to 0} f'(h)$$

$$= \lim_{h\to 0} \left\{2h\sin\left(\frac{1}{h}\right) - \cos\left(\frac{1}{h}\right)\right\}$$

$$= 2\lim_{h\to 0} h \times \lim_{h\to 0} \sin\frac{1}{h} - \lim \cos\frac{1}{h}$$

$$= 2\times 0\times K - \lim_{h\to 0} \cos\frac{1}{h}$$

$$= -\lim_{h\to 0} \cos\frac{1}{h}, \text{ where } -1 \leq K \leq 1$$

As $h\to 0$, $\cos\left(\frac{1}{h}\right)$ oscillates between -1 and 1 and so $\cos\left(\frac{1}{h}\right)$ does not tend to unique and definite limit. Hence, it follows that R.H.L does not exist

Similarly L.H.L also does not exist

Hence $f'(x)$ is not continuous at $x=0$

Example 3: Let

$$f(x) = x \cdot \frac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{1/x} + e^{-1/x}}, \quad x \neq 0;$$

$f(0) = 0$. Show that f is continuous but not differentiable at $x = 0$

Solution:

we have $f(0) = 0$, Let $h > 0$, Also here

$$\text{R.H.L} = f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} h \cdot \cancel{\frac{e^{1/h} - e^{-1/h}}{e^{1/x} + e^{-1/x}}}$$

$$= \lim_{h \to 0} h \cdot \frac{e^{\frac{1}{h}} - e^{-\frac{1}{h}}}{e^{\frac{1}{h}} + e^{-\frac{1}{h}}}$$

$$= \lim_{h \to 0} h \cdot \frac{e^{\frac{1}{h}}(1 - e^{-\frac{2}{h}})}{e^{\frac{1}{h}}(1 + e^{-2/h})}$$

$$= \lim_{h \to 0} h \cdot \frac{1 - e^{-2/h}}{1 + e^{-2/h}} \qquad \lim_{h \to 0} e^{-2/h} = 0$$

$$= 0 \times \frac{1-0}{1+0} = 0$$

$$\text{L.H.L} = f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} (-h) \frac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}}$$

$$= \lim_{h \to 0} (-h) \frac{e^{-2/h} - 1}{e^{-2/h} + 1}$$

$$= 0 \times \frac{0-1}{0+1} = 0$$

To find differentiability

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}$$

$$= \lim_{h\to 0} \frac{h\,(e^{1/h} - e^{-1/h})/(e^{1/h} + e^{-1/h})}{h}$$

$$= \lim_{h\to 0} \frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}}$$

$$= \lim_{h\to 0} \frac{1 - e^{-2/h}}{1 + e^{-2/h}} = 1$$

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h\to 0} \frac{f(-h)}{-h}$$

$$= \lim_{h\to 0} \frac{e^{-2/h} - 1}{e^{-2/h} + 1} = -1$$

Thus $Rf'(0) \neq Lf'(0)$ and so $f(x)$ is not differentiable at $x = 0$

Uniform continuity:

A function f defined on an interval I is said to be uniformly continuous on I, if to each $\epsilon > 0$, there corresponds $\delta > 0$ such that

$$\boxed{|f(x) - f(y)| < \epsilon \quad \because |x-y| < \delta}$$

where x, y are arbitrary points of I

Theorem: Every uniformly continuous function on an interval is continuous on that interval, but the converse is not true.

Proof: Let a function f be uniformly continuous on an interval I, then given $\epsilon > 0$, there exists some of $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \text{ whenever } |x-y| < \delta,$$
$$\text{where } x, y \in I \longrightarrow ①$$

Let c be any point of I

Taking $y = c$ in ①, we have

$$|f(x) - f(c)| < \epsilon \text{ whenever } |x-c| < \delta$$

$\Rightarrow f$ is continuous at the point c

Since c is any point of I, it follows that f is continuous at every point of I.

Hence f is continuous on I.

Converse of the above theorem need not be true, i.e., every continuous function need not be uniformly continuous.

Consider the function f defined on $]0,1[$ as follows

$$f(x) = \frac{1}{x}, \quad \forall x \in]0,1[$$

Let c be any point of $]0,1[$. Then

$$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{1}{c} = \frac{1}{c} = f(c)$$

$\Rightarrow f(x)$ is continuous at $x = c$

Since c is any point $]0,1[$, it follows f is continuous on $]0,1[$

Now P.T f is not uniformly continuous in $]0,1[$. for any $\delta > 0$, find a positive integer n such that $\frac{1}{n} < \delta$

Let $x = \frac{1}{n}$ and $y = \frac{1}{2n}$. Then $x, y \in]0,1[$.

Also we have

$$|x - y| = \left|\frac{1}{n} - \frac{1}{2n}\right| = \left|\frac{1}{2n}\right| = \frac{1}{2n} < \frac{1}{n} < \delta$$

and $|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right|$

$$= |n - 2n| = |-n| = n > \frac{1}{2}$$

[$\because$ n is positive integer, So $n \geq 1$]

Hence if we take $\epsilon = \frac{1}{2} > 0$, then whatever $\delta > 0$ there exists $x, y \in]0,1[$ such that

$$|f(x) - f(y)| > \epsilon \text{ whenever } |x - y| < \delta$$

Hence $f(x) = \frac{1}{x}$ is not uniformly continuous in $]0,1[$

Thus $f(x) = \frac{1}{x}$ is continuous in $]0,1[$ but not uniformly continuous.

Theorem: If a function f is continuous on on a closed interval $[a, b]$ then it is uniformly continuous on $[a, b]$

Proof: Let, if possible, f be not uniformly continuous on $[a, b]$. Then there exists an $\epsilon > 0$ such that whatever $\delta > 0$ we take, we can find $x, y \in [a, b]$, for which

$$|f(x) - f(y)| \geq \epsilon \text{ when } |x - y| < \delta$$

In particular, for each positive integer n, we can find real numbers x_n, y_n in $[a, b]$ such that, $|f(x) - f(y_n)| \geq \epsilon$ when $|x_n - y_n| < \frac{1}{n}$ $\longrightarrow$ (1)

Since $a \leq x_n \leq b$ and $a \leq y_n \leq b$ $\forall n \in N$, the sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ are bounded and so by Bolzano - Weierstrass theorem, each has atleast one limit point, say α and β respectively. Hence α and β are limit points of $[a, b]$. Since a closed interval is a closed set and a closed set contains all its limiting points so $\alpha, \beta \in [a, b]$

Again since α is a limit of $\langle x_n \rangle$, there exist a convergent sequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that $x_{n_k} \to \alpha$ as $k \to \infty$ $\longrightarrow$ (2)

similarly since β is a limit point of $\langle y_n \rangle$ there exists convergent sequence $\langle y_{n_k} \rangle$ of $\langle y_n \rangle$ such that

$$y_{n_k} \to \beta \text{ as } k \to \infty \longrightarrow (3)$$

Now from (1), for all k we have

$$|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon$$

$$\text{when } |x_{n_k} - y_{n_k}| < \frac{1}{n_k} \longrightarrow (4)$$

from (2) and (3) inequalities, we have (4) as

$$\lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} y_{n_k}$$

so that $\alpha = \beta$ by (2) and (3)

But from the first of the inequalities (4), we see that in case the sequences $\langle f(x_{n_k}) \rangle$ and $\langle f(y_{n_k}) \rangle$ converge, the limits to which they converge and different.

Thus we have shown that there exist two sequences $\langle x_{n_k} \rangle$ and $\langle y_{n_k} \rangle$ both of which converge to α but $\langle f(x_{n_k}) \rangle$ and $\langle f(y_{n_k}) \rangle$ do not converge to the same limit

$\therefore$ f is not continuous at α, for otherwise the two sequences $\langle f(x_{n_k}) \rangle$ and $\langle f(y_{n_k}) \rangle$ would converge to same point $f(\alpha)$

Thus we arrive at a contradiction and hence the hypothesis that f is not uniformly continuous on $[a,b]$ is false.

Hence f must be uniformly continuous on $[a,b]$

Example 1: Is the function $f(x) = \frac{x}{(x+1)}$ uniformly continuous for $x \in [0,2]$

Solution: Let x, y be two arbitrary points in $[0,2]$. Then $x \geq 0$, $y \geq 0$

$$\Rightarrow x+1 \geq 1 \text{ and } y+1 \geq 1$$

$$\Rightarrow (x+1)(y+1) \geq 1 \longrightarrow ①$$

Now,

$$|f(x) - f(y)| = \left| \frac{x}{x+1} - \frac{y}{y+1} \right|$$

$$= \frac{|x-y|}{(x+1)(y+1)} \leq |x-y| \text{ by } ①$$

Let $\epsilon > 0$ be given, Taking $\delta = \epsilon$ we get

$$|f(x) - f(y)| < \epsilon \text{ whenever } |x-y| < \delta \quad \forall\, x, y \in [0,2]$$

Hence f is uniformly continuous in $[0,2]$

Example 2: Determine the value of δ of the function $f(x) = x^2 + 3x$, $x \in [-1, 1]$ is uniform continuity

Solution:

Given $f(x) = x^2 + 3x$

$f(y) = y^2 + 3y$

$$|f(x) - f(y)| = |x^2 + 3x - (y^2 + 3y)|$$

$$= |(x^2 + y^2) + 3(x - y)|$$

$$= |(x+y)(x-y) + 3(x-y)|$$

$$= |(x-y)[(x+y) + 3]|$$

$$< |x-y| [|x| + |y| + 3]$$

$$< |x-y| [|-1| + |1| + 3]$$

$$< |x-y| 5$$

$\Rightarrow |f(x) - f(y)| < |x-y| 5$

$\Longrightarrow 5|x-y| < \epsilon$

$\Longrightarrow |x-y| < \epsilon/5$

$\Rightarrow |x-y| < \delta$

The given function $f(x)$ is uniformly continuous over $[-1, 1]$

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Example (3) Find the value of δ, for which the following function is uniformly continuous

$$f(x) = x^3 + 3x^2 - 2x + 7, \quad x \in [-2, 2]$$

solution:.

$$f(x) = x^3 + 3x^2 - 2x + 7$$

$$f(y) = y^3 + 3y^2 - 2y + 7$$

$$|f(x) - f(y)| = |x^3 + 3x^2 - 2x + \cancel{7} - y^3 - 3y^2 + 2y - \cancel{7}|$$

$$= |x^3 - y^3 + 3x^2 - 3y^2 - 2x + 2y|$$

$$= |(x-y)(x+y)^2 + 3(x-y)^2 - 2(x-y)|$$

$$= ||x-y|\{(x+y)^2 + 3(x+y) - 2\}|$$

$$\leq |x-y|\{|x|^2 + |y|^2 + 2|x||y| + 3|x| + 3|y| - 2\}$$

$$\leq |x-y|\{|-2|^2 + |2|^2 + 2|-2||2| + 3|-2| + 3|2| - 2\}$$

$$\leq |x-y|\{|4| + |4| + 2|-2||2| + 3|-2| + 3|2| - 2\}$$

4 + 4 + 8 + 6 + 6 − 2

$$\leq |x-y|\{4 + 4 + 8 + 6 + 6 - 2\}$$

$$\leq |x-y|\{28 - 2\}$$

$$\leq |x-y|(26)$$

$$< |x-y| < \varepsilon/26$$, it is a finite number

$$\Rightarrow |x-y| < \delta$$, it is also a finite number

The given function $f(x)$ is uniformly continuous over $[-2, 2]$.

Pointwise convergence of series of functions:

Definition:

Let D be a subset of $\mathbb{R}$ and let $\{f_n\}$ be a sequence of functions defined on D. We say that $\{f_n\}$ converges pointwise on D if

$\lim_{n\to\infty} f_n(x)$ exists for each point x in D

This means that $\lim_{n\to\infty} f_n(x)$ is a real number that depends only on x.

If $\{f_n\}$ is pointwise convergent then the function defined by $f(x) = \lim_{n\to\infty} f_n(x)$, for every x in D, is called pointwise limit of the sequence $\{f_n\}$

Example 1:

Let $\{f_n\}$ be the sequence of functions on $\mathbb{R}$ defined by $f_n(x) = nx$.

This sequence does not converge pointwise on $\mathbb{R}$ because $\lim_{n\to\infty} f_n(x) = \infty$ for any $x > 0$

Example 2:

Consider the sequence $\{f_n\}$ of functions defined by

$$f_n(x) = \frac{nx + x^2}{n^2} \quad \text{for all } x \text{ in } \mathbb{R}$$

Show that $\{f_n\}$ converges pointwise.

Solution: For any real number x, we have,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{n^2} + \lim_{n\to\infty} \frac{x^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{x}{n} + \lim_{n\to\infty} \frac{x^2}{n^2}$$

$$= x \lim_{n\to\infty}\left(\frac{1}{n}\right) + x^2 \lim_{n\to\infty}\left(\frac{1}{n^2}\right)$$

$$= 0 + 0$$

$$= 0$$

Thus $\{f_n\}$ converges pointwise to the zero function on $\mathbb{R}$.

Example 3:

Consider the sequence $\{f_n\}$ of functions defined by $f_n(x) = n^2 x^n$ for $0 \leq x \leq 1$. Determine whether $\{f_n\}$ is pointwise convergent.

Solution: First of all, observe that $f_n(0)=0$ for every n in $\mathbb{N}$. So the sequence $\{f_n(0)\}$ is constant and converges to zero.

Now suppose $0 < x < 1$

$x^5 =$

then $n^2 x^n = n^2 \boxed{e^{n \ln(x)}}$ log

But $\ln(x) < 0$ when $0 < x < 1$

it follows that,

$$\lim_{n \to \infty} f_n(x) = 0 \quad \text{for } 0 < x < 1$$

Finally, $f_n(1) = n^2$ for all n.

So, $\lim\limits_{n \to \infty} f_n(1) = \infty$.

$\therefore \{f_n\}$ is not pointwise convergent on $[0,1]$