Soft Computing

UNIT-V:

Fuzzy systems: Crisp logic: Laws of Propositional logic-Inference in Propositional logic. Predicate logic: Interpretations of Predicate logic formula – Inference in Predicate logic. Fuzzy logic: Fuzzy quantifiers – Fuzzy inference, Fuzzy Rule based system – Defuzzification.

Text Book:

S.Rajasekaran & G.A.Vijayalakshmi Pai, “Neural Networks, Fuzzy Logic, And Genetic Algorithms Synthesis And Applications, PHI, 2005.

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7.1 CRISP LOGIC

Consider the statements "Water boils at 90°C" and "Sky is blue". An agreement or disagreement with these statements is indicated by a "True" or "False" value accorded to the statements. While the first statement takes on a value *false*, the second takes on a value *true*.

Thus, a statement which is either 'True' or 'False' but not both is called a *proposition*. A proposition is indicated by upper case letters such as *P*, *Q*, *R* and so on.

Example: *P*: Water boils at 90°C.
Q: Sky is blue.

are propositions.

A simple proposition is also known as an *atom*. Propositions alone are insufficient to represent phenomena in the real world. In order to represent complex information, one has to build a sequence of propositions linked using *connectives* or *operators*. Propositional logic recognizes five major operators as shown in Table 7.1.

Table 7.1 Propositional logic connectives

Symbol	Connective	Usage	Description
$\wedge$	and	$P \wedge Q$	P and Q are true.
$\vee$	or	$P \vee Q$	Either P or Q is true.
$\neg$ or $\sim$	not	$\sim P$ or $\neg P$	P is not true.
$\Rightarrow$	implication	$P \Rightarrow Q$	P implies Q is true.
$=$	equality	$P = Q$	P and Q are equal (in truth values) is true.

Observe that ∧, ∨, ⇒, and = are 'binary' operators requiring two propositions while ~ is a 'unary' operator requiring a single proposition. ∧ and ∨ operations are referred to as *conjunction* and *disjunction* respectively. In the case of ⇒ operator, the proposition occurring before the '⇒' symbol is called as the *antecedent* and the one occurring after is called as the *consequent*.

The semantics or meaning of the logical connectives are explained using a *truth table*. A truth table comprises rows known as *interpretations*, each of which evaluates the logical formula for the given set of truth values. Table 7.2 illustrates the truth table for the five connectives.

Table 7.2 Truth table for the connectives ∧, ∨, ~, ⇒ , =

P	Q	$P \wedge Q$	$P \vee Q$	$\sim P$	$P \Rightarrow Q$	$P = Q$
T	T	T	T	F	T	T
T	F	F	T	F	F	F
F	F	F	F	T	T	T
F	T	F	T	T	T	F

T : True, F : False

A logical formula comprising n propositions will have 2^n interpretations in its truth table. A formula which has all its interpretations recording true is known as a *tautology* and the one which records false for all its interpretations is known as *contradiction*.

Example 7.1

Obtain a truth table for the formula $(P \vee Q) \Rightarrow (\sim P)$. Is it a tautology?

Solution

The truth table for the given formula is

P	Q	$P \vee Q$	$\sim P$	$P \vee Q \Rightarrow \sim P$
T	F	T	F	F
F	T	T	T	T
T	T	T	F	F
F	F	F	T	T

No, it is not a tautology since all interpretations do not record 'True' in its last column.

Example 7.2

Is $((P \Rightarrow Q) \wedge (Q \Rightarrow P) = (P = Q)$ a tautology?

Solution

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	A: $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$	B: $P = Q$	$A = B$
T	F	F	T	F	F	T
F	T	T	F	F	F	T
T	T	T	T	T	T	T
F	F	T	T	T	T	T

Yes, the given formula is a tautology.

Example 7.3

Show that $(P \Rightarrow Q) = (\sim P \vee Q)$

Solution

The truth table for the given formula is

P	Q	$A: P \Rightarrow Q$	$\sim P$	$B: \sim P \vee Q$	$A = B$
T	T	T	F	T	T
T	F	F	F	F	T
F	F	T	T	T	T
T	T	T	T	T	T

Since the last column yields 'True' for all interpretations, it is a tautology.

The logical formula presented in Example 7.3 is of practical importance since $(P \Rightarrow Q)$ is shown to be equivalent to $(\sim P \vee Q)$, a formula devoid of '$\Rightarrow$' connective. This equivalence can therefore be utilised to eliminate '$\Rightarrow$' in logical formulae.

It is useful to view the '$\Rightarrow$' operator from a set oriented perspective. If X is the universe of discourse and A, B are sets defined in X, then propositions P and Q could be defined based on an element $x \in X$ belonging to A or B. That is,

$$P: x \in A$$

$$Q: x \in B \tag{7.1}$$

Here, P, Q are true if $x \in A$ and $x \in B$ respectively, and $\sim P$, $\sim Q$ are true if $x \notin A$ and $x \notin B$ respectively. In such a background, $P \Rightarrow Q$ which is equivalent to $(\sim P \vee Q)$ could be interpreted as

$$(P \Rightarrow Q) : x \notin A \quad \text{or} \quad x \in B \tag{7.2}$$

However, if the '$\Rightarrow$' connective deals with two different universes of discourse, that is, $A \subset X$ and $B \subset Y$ where X and Y are two universes of discourse then the '$\Rightarrow$' connective is represented by the relation R such that

$$R = (A \times B) \cup (\bar{A} \times Y) \tag{7.3}$$

In such a case, $P \Rightarrow Q$ is linguistically referred to as IF A THEN B. The compound proposition $(P \Rightarrow Q) \vee (\sim P \Rightarrow S)$ linguistically referred to as IF A THEN B ELSE C is equivalent to

$$\text{IF } A \text{ THEN } B \ (P \Rightarrow Q)$$

$$\text{IF } \sim A \text{ THEN } C \ (\sim P \Rightarrow S) \tag{7.4}$$

where P, Q, and S are defined by sets A, B, C, $A \subset X$, and B, $C \subset Y$.

7.1.1 Laws of Propositional Logic

Crisp sets as discussed in Section 6.2.2. exhibit properties which help in their simplification.

Similarly, propositional logic also supports the following laws which can be effectively used for their simplification. Given P, Q, R to be the propositions,

(i) *Commutativity*

$$(P \vee Q) = (Q \vee P)$$
$$(P \wedge Q) = (Q \wedge P)$$

(ii) *Associativity*

$$(P \vee Q) \vee R = P \vee (Q \vee R)$$
$$(P \wedge Q) \wedge R = P \wedge (Q \wedge R)$$

(iii) *Distributivity*

$$(P \vee Q) \wedge R = (P \wedge R) \vee (Q \wedge R)$$
$$(P \wedge Q) \vee R = (P \vee R) \wedge (Q \vee R)$$

(iv) *Identity*

$$P \vee \text{false} = P$$
$$P \wedge \text{True} = P$$
$$P \wedge \text{False} = \text{False}$$
$$P \vee \text{True} = \text{True}$$

(v) *Negation*

$$P \wedge \sim P = \text{False}$$
$$P \vee \sim P = \text{True}$$

(vi) *Idempotence*

$$P \vee P = P$$
$$P \wedge P = P$$

(vii) *Absorption*

$$P \wedge (P \vee Q) = P$$
$$P \vee (P \wedge Q) = P$$

(viii) *De Morgan's laws*

$$\sim(P \vee Q) = (\sim P \wedge \sim Q)$$
$$\sim(P \wedge Q) = (\sim P \vee \sim Q)$$

(ix) *Involution*

$$\sim(\sim P) = P$$

Each of these laws can be tested to be a tautology using truth tables.

Example 7.4

Verify De Morgan's laws.

(a) $\sim(P \vee Q) = (\sim P \wedge \sim Q)$

(b) $\sim(P \wedge Q) = (\sim P \vee \sim Q)$

Solution

(a)

P	Q	$P \vee Q$	$A: \sim(P \vee Q)$	$\sim P$	$\sim Q$	$B: \sim P \wedge \sim Q$	$A = B$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

Therefore, $\sim(P \vee Q) = (\sim P \wedge \sim Q)$

(b)

P	Q	$P \wedge Q$	$A: \sim(P \wedge Q)$	$\sim P$	$\sim Q$	$B: \sim P \vee \sim Q$	$A = B$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	T	T	T
F	F	F	T	T	F	T	T

Therefore $\sim(P \wedge Q) = (\sim P \vee \sim Q)$

Example 7.5

Simplify $(\sim(P \wedge Q) \Rightarrow R) \wedge P \wedge Q$

Solution

Consider $(\sim(P \wedge Q) \Rightarrow R) \wedge P \wedge Q$

$= (\sim \sim(P \wedge Q) \vee R) \wedge P \wedge Q$

(by eliminating '$\Rightarrow$' using $(P \Rightarrow Q) = (\sim P \vee Q)$)

$= ((P \wedge Q) \vee R) \wedge P \wedge Q$ (by the law of involution)

$= (P \wedge Q)$ (by the law of absorption)

7.1.2 Inference in Propositional Logic

Inference is a technique by which, given a set of *facts* or *postulates* or *axioms* or *premises* F_1, $F_2, \ldots, F_n$, a *goal* G is to be derived. For example, from the facts "Where there is smoke there is fire", and "There is smoke in the hill", the statement "Then the hill is on fire" can be easily deduced.

In propositional logic, three rules are widely used for inferring facts, namely

(i) *Modus Ponens*

(ii) *Modus Tollens*, and

(iii) *Chain rule*

Modus ponens (mod pons)

Given $P \Rightarrow Q$ and P to be true, Q is true.

$$\frac{\begin{array}{c} P \Rightarrow Q \\ P \end{array}}{Q} \tag{7.14}$$

Here, the formulae above the line are the *premises* and the one below is the *goal* which can be inferred from the premises.

Modus tollens

Given $P \Rightarrow Q$ and $\sim Q$ to be true, $\sim P$ is true.

$$\frac{\begin{array}{c} P \Rightarrow Q \\ \sim Q \end{array}}{\sim P} \tag{7.15}$$

Chain rule

Given $P \Rightarrow Q$ and $Q \Rightarrow R$ to be true, $P \Rightarrow R$ is true.

$$\frac{\begin{array}{c} P \Rightarrow Q \\ Q \Rightarrow R \end{array}}{P \Rightarrow R} \tag{7.16}$$

Note that the chain rule is a representation of the *transitivity* relation with respect to the '$\Rightarrow$' connective.

Example 7.6

Given

(i) $C \vee D$

(ii) $\sim H \Rightarrow (A \wedge \sim B)$

(iii) $(C \vee D) \Rightarrow \sim H$

(iv) $(A \wedge \sim B) \Rightarrow (R \vee S)$

Can $(R \vee S)$ be inferred from the above?

Solution

From (i) and (iii) using the rule of Modus Ponens , $\sim H$ can be inferred.

(i) $C \vee D$

(iii) $(C \vee D) \Rightarrow \sim H$

$$\frac{\begin{array}{c} C \vee D \\ (C \vee D) \Rightarrow \sim H \end{array}}{\sim H} \quad \text{(v)}$$

From (ii) and (iv) using the chain rule, $\sim H \Rightarrow (R \vee S)$ can be inferred.

(ii) $\sim H \Rightarrow (A \wedge \sim B)$

(iv) $(A \wedge \sim B) \Rightarrow (R \vee S)$

$$\frac{\begin{array}{c} \sim H \Rightarrow (A \wedge \sim B) \\ (A \wedge \sim B) \Rightarrow (R \vee S) \end{array}}{\sim H \Rightarrow (R \vee S)} \quad \text{(vi)}$$

From (v) and (vi) using the rule of Modus Ponens $(R \vee S)$ can be inferred.

(vi) $\sim H \Rightarrow (R \vee S)$

(v) $\sim H$

$$\frac{\begin{array}{c} \sim H \Rightarrow (R \vee S) \\ \sim H \end{array}}{R \vee S}$$

Hence, the result.

7.2 PREDICATE LOGIC

In propositional logic, events are symbolised as propositions which acquire either 'True/False' values. However, there are situations in the real world where propositional logic falls short of its expectation. For example, consider the following statements:

P : All men are mortal.

Q : Socrates is a man.

From the given statements it is possible to infer that Socrates is mortal. However, from the propositions *P*, *Q* which symbolise these statements nothing can be made out. The reason being, propositional logic lacks the ability to symbolise *quantification*. Thus, in this example, the quantifier "All" which represents the entire class of men encompasses Socrates as well, who is declared to be a man, in proposition *Q*. Therefore, by virtue of the first proposition *P*, Socrates who is a man also becomes a mortal, giving rise to the deduction Socrates is mortal. However, the deduction is not directly perceivable owing to the shortcomings in propositional logic. Therefore, propositional logic needs to be augmented with more tools to enhance its logical abilities.

Predicate logic comprises the following apart from the connectives and propositions recognized by propositional logic:

(i) Constants

(ii) Variables

(ii) Predicates

(iv) Quantifiers

(v) Functions

Constants represent objects that do not change values.

Example Pencil, Ram, Shaft, 100°C.

Variables are symbols which represent values acquired by the objects as qualified by the quantifier with which they are associated with.

Example x, y, z.

Predicates are representative of associations between objects that are constants or variables and acquire truth values 'True' or 'False'. A *predicate* carries a name representing the association followed by its arguments representing the objects it is to associate.

Example

likes (Ram, tea) (Ram likes tea)

plays (Sita, x) (Sita plays anything)

Here, likes and plays are predicate names and Ram, tea and Sita, x are the associated objects. Also, the predicates acquire truth values. If Ram disliked tea, likes (Ram, tea) acquires the value *false* and if Sita played any sport, plays (Sita, x) would acquire the value *true* provided x is suitably qualified by a quantifier.

Quantifiers are symbols which indicate the two types of quantification, namely, *All* ($\forall$) and *Some* ($\exists$). '$\forall$' is termed *universal quantifier* and '$\exists$' is termed *existential quantifier*.

Example Let,

man (x) : x is a man.
mortal (x) : x is mortal.
mushroom (x) : x is a mushroom.
poisonous (x) : x is poisonous.

Then, the statements

All men are mortal.
Some mushrooms are poisonous.

are represented as

$$\forall x \text{ (man } (x) \Rightarrow \text{mortal } (x))$$

$$\exists x \text{ (mushroom } (x) \wedge \text{poisonous } (x))$$

Here, a useful rule to follow is that a universal quantifier goes with implication and an existential quantifier with conjunction. Also, it is possible for logical formula to be quantified by multiple quantifiers.

Example Every ship has a captain.

$\forall x \exists y$ (ship $(x) \Rightarrow$ captain (x, y))

where, ship (x) : x is a ship

captain (x, y) : y is the captain of x.

Functions are similar to predicates in form and in their representation of association between objects but unlike predicates which acquire truth values alone, functions acquire values other than truth values. Thus, functions only serve as object descriptors.

Example

plus (2, 3) (2 plus 3 which is 5)

mother (Krishna) (Krishna's mother)

Observe that plus () and mother () indirectly describe "5" and "Krishna's mother" respectively.

Example 7.7

Write predicate logic statements for

(i) Ram likes all kinds of food

(ii) Sita likes anything which Ram likes.

(iii) Raj likes those which Sita and Ram both like.

(iv) Ali likes some of which Ram likes.

Solution

Let food (x) : x is food.

likes (x, y) : x likes y

Then the above statements are translated as

(i) $\forall x$ food $(x) \Rightarrow$ likes (Ram, x))

(ii) $\forall x$ (likes (Ram, x) $\Rightarrow$ likes (Sita, x))

(iii) $\forall x$ (likes (Sita, x) $\wedge$ likes (Ram, x)) $\Rightarrow$ likes (Raj, x))

(iv) $\exists x$ (likes (Ram, x) $\wedge$ likes (Ali, x))

The application of the rule of universal quantifier and rule of existential quantifier can be observed in the translations given above.

7.2.1 Interpretations of Predicate Logic Formula

For a formula in propositional logic, depending on the truth values acquired by the propositions, the truth table interprets the formula. But in the case of predicate logic, depending on the truth values acquired by the predicates, the nature of the quantifiers, and the values taken by the constants and functions over a domain D, the formula is interpreted.

Example

Interpret the formulae

(i) $\forall x\, p(x)$

(ii) $\exists x\, p(x)$

where the domain $D = \{1, 2\}$ and

$p(1)$	$p(2)$
True	False

Solution

(i) $\forall x\, p(x)$ is true only if $p(x)$ is true for all values of x in the domain D, otherwise it is false. Here, for $x = 1$ and $x = 2$, the two possible values for x chosen from D, namely $p(1)$ = true and $p(2)$ = false respectively, yields (i) to be false since $p(x)$ is not true for $x = 2$. Hence, $\forall x\, p(x)$ is false.

(ii) $\exists x\, p(x)$ is true only if there is atleast one value of x for which $p(x)$ is true. Here, for $x = 1$, $p(x)$ is true resulting in (ii) to be true. Hence, $\exists x\, p(x)$ is true.

Example 7.8

Interpret $\forall x\, \exists y\, P(x, y)$ for $D = \{1, 2\}$ and

$P(1, 1)$	$P(1, 2)$	$P(2, 1)$	$P(2, 2)$
True	False	False	True

Solution

For $x = 1$, there exists a y, $(y = 1)$ for which $P(x, y)$, i.e. $(P(1,1))$ is true.
For $x = 2$, there exists a y, $(y = 2)$ for which $P(x, y)$ $(P(2, 2))$ is true.
Thus, for all values of x there exists a y for which $P(x, y)$ is true.
Hence, $\forall x\, \exists y\, P(x, y)$ is true.

7.2.2 Inference in Predicate Logic

The rules of inference such as Modus Ponens, Modus Tollens and Chain rule, and the laws of propositional logic are applicable for inferring predicate logic but not before the quantifiers have been appropriately eliminated (refer Chang & Lee, 1973).

Example

Given (i) All men are mortal.

(ii) Confucius is a man.

Prove: Confucius is mortal.

Translating the above into predicate logic statements

(i) $\forall x$ (man $(x) \Rightarrow$ mortal (x))

(ii) man (Confucius)

(iii) mortal (Confucius)

Since (i) is a tautology qualified by the universal quantifier for x = Confucius, the statement is true, i.e.

$$\text{man (Confucius)} \Rightarrow \text{mortal (Confucius)}$$

$$\Rightarrow \sim\text{man (Confucius)} \vee \text{mortal (Confucius)}$$

But from (ii), man (Confucius) is true.

Hence (iv) simplifies to

$$\text{False} \vee \text{mortal (Confucius)}$$

$$= \text{mortal (Confucius)}$$

Hence, Confucius is mortal has been proved.

Example 7.9

Given (i) Every soldier is strong-willed.

(ii) All who are strong-willed and sincere will succeed in their career.

(iii) Indira is a soldier.

(iv) Indira is sincere.

Prove: Will Indira succeed in her career?

Solution

Let soldier (x) : x is a soldier.

strong-willed (x) : x is a strong-willed.

sincere (x) : x is sincere.

succeed_career (x) : x succeeds in career.

Now (i) to (iv) are translated as

$$\forall x\ (\text{soldier}\ (x) \Rightarrow \text{strong-willed}\ (x)) \quad \text{(i)}$$

$$\forall x\ ((\text{strong-willed}\ (x) \wedge \text{sincere}\ (x)) \Rightarrow \text{succeed\_career}\ (x)) \quad \text{(ii)}$$

$$\text{soldier (Indira)} \quad \text{(iii)}$$

$$\text{sincere (Indira)} \quad \text{(iv)}$$

To show whether Indira will succeed in her career, we need to show

$$\text{succeed\_career(Indira) is true.} \quad \text{(v)}$$

Since (i) and (ii) are quantified by $\forall$, they should be true for x = Indira.
Substituting x = Indira in (i) results in (soldier (Indira) $\Rightarrow$ strong-willed (Indira),

i.e. $\quad$ ~soldier (Indira) $\vee$ strong-willed (Indira) $\quad$ (vi)

Since from (iii) soldier (Indira) is true, (vi) simplifies to

strong-willed (Indira) $\quad$ (vii)

Substituting x = Indira in (ii),

(strong-willed (Indira) $\wedge$ sincere (Indira)) $\Rightarrow$ succeed_career (Indira)

i.e. $\quad$ ~(strong-willed (Indira) $\wedge$ sincere (Indira)) $\vee$ succeed_career (Indira)

(Θ $P \Rightarrow Q = {\sim}P \vee Q$))

i.e. $\quad$ ~(strong-willed (Indira) $\vee$ ~sincere (Indira)) $\vee$ succeed_career (Indira)

(De Morgan's law) (viii)

From (vii), strong-willed (Indira) is true and from (iv) sincere (Indira) is true. Substituting these in (viii),

False $\vee$ False $\vee$ succeed_career (Indira)

i.e. $\quad$ succeed_career (Indira) $\quad$ (using law of identity)

Hence, Indira will succeed in her career is true.

7.3 FUZZY LOGIC

In crisp logic, the truth values acquired by propositions or predicates are 2-valued, namely *True*, *False* which may be treated numerically equivalent to (0, 1). However, in fuzzy logic, truth values are multivalued such as *absolutely true, partly true, absolutely false, very true*, and so on and are numerically equivalent to (0–1).

Fuzzy propositions

A *fuzzy proposition* is a statement which acquires a fuzzy truth value. Thus, given $\tilde{P}$ to be a fuzzy proposition, $T(\tilde{P})$ represents the truth value (0–1) attached to $\tilde{P}$. In its simplest form, fuzzy propositions are associated with fuzzy sets. The fuzzy membership value associated with the fuzzy set $\tilde{A}$ for $\tilde{P}$ is treated as the fuzzy truth value $T(\tilde{P})$.

i.e. $T(\tilde{P}) = \mu_{\tilde{A}}(x)$ where $0 \le \mu_{\tilde{A}}(x) \le 1$ (7.17)

Example

$\tilde{P}$: Ram is honest.

$T(\tilde{P}) = 0.8$, if $\tilde{P}$ is partly true.

$T(\tilde{P}) = 1$, if $\tilde{P}$ is absolutely true.

Fuzzy connectives

Fuzzy logic similar to crisp logic supports the following connectives:

(i) *Negation* : $-$

(ii) *Disjunction* : $\vee$

(iii) *Conjunction* : $\wedge$

(iv) *Implication* : $\Rightarrow$

Table 7.3 illustrates the definition of the connectives. Here $\tilde{P}$, $\tilde{Q}$ are fuzzy propositions and $T(\tilde{P})$, $T(\tilde{Q})$, are their truth values.

Table 7.3 Fuzzy connectives

Symbol	Connective	Usage	Definition
$-$	Negation	$\bar{\tilde{P}}$	$1 - T(\tilde{P})$
$\vee$	Disjunction	$\tilde{P} \vee \tilde{Q}$	$\max(T(\tilde{P}), T(\tilde{Q}))$
$\wedge$	Conjunction	$\tilde{P} \wedge \tilde{Q}$	$\min(T(\tilde{P}), T(\tilde{Q}))$
$\Rightarrow$	Implication	$\tilde{P} \Rightarrow \tilde{Q}$	$\sim \tilde{P} \vee \tilde{Q} = \max(1 - T(\tilde{P}), T(\tilde{Q}))$

$\tilde{P}$ and $\tilde{Q}$ related by the '$\Rightarrow$' operator are known as antecedent and consequent respectively. Also, just as in crisp logic, here too, '$\Rightarrow$' represents the IF-THEN statement as

IF x is $\tilde{A}$ THEN y is $\tilde{B}$, and is equivalent to

$$\tilde{R} = (\tilde{A} \times \tilde{B}) \cup (\bar{\tilde{A}} \times \tilde{Y}) \tag{7.18}$$

The membership function of $\tilde{R}$ is given by

$$\mu_{\tilde{R}}(x, y) = \max(\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)), 1 - \mu_{\tilde{A}}(x)) \tag{7.19}$$

Also, for the compound implication IF x is $\tilde{A}$ THEN y is $\tilde{B}$ ELSE y is $\tilde{C}$ the relation R is equivalent to

$$\tilde{R} = (\tilde{A} \times \tilde{B}) \cup (\bar{\tilde{A}} \times C) \tag{7.20}$$

The membership function of $\tilde{R}$ is given by

$$\mu_{\tilde{R}}(x, y) = \max(\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)), \min(1 - \mu_{\tilde{A}}(x), \mu_{\tilde{C}}(y))) \tag{7.21}$$

Example

$\tilde{P}$: Mary is efficient, $T(\tilde{P}) = 0.8$

$\tilde{Q}$: Ram is efficient, $T(\tilde{Q}) = 0.65$

(i) $\bar{\tilde{P}}$: Mary is not efficient.

$$T(\bar{\tilde{P}}) = 1 - T(\tilde{P}) = 1 - 0.8 = 0.2$$

(ii) $\tilde{P} \wedge \tilde{Q}$: Mary is efficient and so is Ram.

$$\begin{aligned} T(\tilde{P} \wedge \tilde{Q}) &= \min (T(\tilde{P}), T(\tilde{Q})) \\ &= \min (0.8\ ,\ 0.65) \\ &= 0.65 \end{aligned}$$

(iii) $T(\tilde{P} \vee \tilde{Q})$: Either Mary or Ram is efficient.

$$\begin{aligned} T(\tilde{P} \vee \tilde{Q}) &= \max (T(\tilde{P}), T(\tilde{Q})) \\ &= \max (0.8, 0.65) \\ &= 0.8 \end{aligned}$$

(iv) $\tilde{P} \Rightarrow \tilde{Q}$: If Mary is efficient then so is Ram.

$$\begin{aligned} T(\tilde{P} \Rightarrow \tilde{Q}) &= \max (1 - T(\tilde{P}), T(\tilde{Q})) \\ &= \max (0.2, 0.65) \\ &= 0.65 \end{aligned}$$

Example 7.10

Let $X = \{a, b, c, d\}$ $Y = \{1, 2, 3, 4\}$

and $\tilde{A} = \{(a, 0)(b, 0.8)(c, 0.6)(d, 1)\}$

$\tilde{B} = \{(1, 0.2)(2, 1)(3, 0.8)(4, 0)\}$

$\tilde{C} = \{(1, 0)(2, 0.4)(3, 1)(4, 0.8)\}$

Determine the implication relations

(i) IF x is $\tilde{A}$ THEN y is $\tilde{B}$.

(ii) IF x is $\tilde{A}$ THEN y is $\tilde{B}$ ELSE y is $\tilde{C}$.

Solution

To determine (i) compute

$$\tilde{R} = (\tilde{A} \times \tilde{B}) \cup (\bar{\tilde{A}} \times Y) \quad \text{where}$$

$$\mu_{\tilde{R}}(x, y) = \max(\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)), 1 - \mu_{\tilde{A}}(x))$$

$$\tilde{A} \times \tilde{B} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1 & 0.8 & 0 \end{bmatrix} \end{array}$$

$$\bar{\tilde{A}} \times Y = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Here, Y the universe of discourse could be viewed as {(1, 1) (2, 1) (3, 1) (4, 1)} a fuzzy set all of whose elements x have $\mu(x) = 1$.

Therefore,

$$\tilde{R} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.4 & 0.6 & 0.6 & 0.4 \\ 0.2 & 0.1 & 0.8 & 0 \end{bmatrix} \end{array}$$

which represents IF x is $\tilde{A}$ THEN y is $\tilde{B}$.

To determine (ii) compute

$$\tilde{R} = (\tilde{A} \times \tilde{B}) \cup (\bar{\tilde{A}} \times \tilde{C}) \text{ where}$$

$$\mu_{\tilde{R}}(x, y) = \max(\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)), \min(1 - \mu_{\tilde{A}}(x), \mu_{\tilde{C}}(y)))$$

$$\tilde{A} \times \tilde{B} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1 & 0.8 & 0 \end{bmatrix} \end{array}$$

$$\bar{\tilde{A}} \times \tilde{C} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0.4 & 1 & 0.8 \\ 0 & 0.2 & 0.2 & 0.2 \\ 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Therefore,

$$\tilde{R} = \max((\tilde{A} \times \tilde{B}), (\bar{\tilde{A}} \times \tilde{C})) \text{ gives}$$

$$\tilde{R} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0.4 & 1 & 0.8 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.2 & 0.6 & 0.6 & 0.4 \\ 0.2 & 1 & 0.8 & 0 \end{bmatrix} \end{array}$$

The above $\tilde{R}$ represents IF x is $\tilde{A}$ THEN y is $\tilde{B}$ ELSE y is $\tilde{C}$.

7.3.1 Fuzzy Quantifiers

Just as in crisp logic where predicates are quantified by quantifiers, fuzzy logic propositions are also quantified by *fuzzy quantifiers*. There are two classes of fuzzy quantifiers such as

(i) Absolute quantifiers and

(ii) Relative quantifiers

While absolute quantifiers are defined over $\Re$, relative quantifiers are defined over [0–1].

Example

Absolute quantifier	Relative quantifier
round about 250	almost
much greater than 6	about
some where around 20	most

7.3.2 Fuzzy Inference

Fuzzy inference also referred to as *approximate reasoning* refers to computational procedures used for evaluating linguistic descriptions. The two important inferring procedures are

(i) Generalized Modus Ponens (GMP)

(ii) Generalized Modus Tollens (GMT)

GMP is formally stated as

$$\begin{array}{ll} \text{IF} & x \text{ is } \tilde{A} \text{ THEN } y \text{ is } \tilde{B} \\ & \qquad x \text{ is } \tilde{A}' \\ \hline & \qquad y \text{ is } \tilde{B}' \end{array} \tag{7.22}$$

Here, $\tilde{A}$, $\tilde{B}$, $\tilde{A}'$ and $\tilde{B}'$ are fuzzy terms. Every fuzzy linguistic statement above the line is analytically known and what is below is analytically unknown.

To compute the membership function of $\tilde{B}'$, the max-min composition of fuzzy set A' with $\tilde{R}(x, y)$ which is the known implication relation (IF-THEN relation) is used. That is,

$$\tilde{B}' = \tilde{A}' \circ \tilde{R}(x, y) \tag{7.23}$$

In terms of membership function,

$$\mu_{\tilde{B}'}(y) = \max(\min(\mu_{\tilde{A}'}(x), \mu_{\tilde{R}}(x, y))) \tag{7.24}$$

where $\mu_{\tilde{A}'}(x)$ is the membership function of $\tilde{A}'$, $\mu_{\tilde{R}}(x, y)$ is the membership function of the implication relation and $\mu_{\tilde{B}'}(y)$ is the membership function of $\tilde{B}'$.

On the other hand, GMT has the form

$$\begin{array}{ll} \text{IF} & x \text{ is } \tilde{A} \text{ THEN } y \text{ is } \tilde{B} \\ & \quad y \text{ is } \tilde{B}' \\ \hline & \quad x \text{ is } \tilde{A}' \end{array}$$

The membership of $\tilde{A}'$ is computed on similar lines as

$$\tilde{A}' = \tilde{B}' \circ \tilde{R}(x, y)$$

In terms of membership function,

$$\mu_{\tilde{A}'}(x) = \max(\min(\mu_{\tilde{B}'}(y), \mu_{\tilde{R}}(x, y))) \tag{7.25}$$

Example

Apply the fuzzy Modus Ponens rule to deduce Rotation is quite slow given

(i) If the temperature is high then the rotation is slow.

(ii) The temperature is very high.

Let H (High), VH (Very High), $\tilde{S}$ (Slow) and QS (Quite Slow) indicate the associated fuzzy sets as follows:

For $X = \{30, 40, 50, 60, 70, 80, 90, 100\}$, the set of temperatures and $Y = \{10, 20, 30, 40, 50, 60\}$, the set of rotations per minute.

$$\tilde{H} = \{(70, 1)\ (80, 1)\ (90, 0.3)\}$$

$$\tilde{VH} = \{(90, 0.9)\ (100, 1)\}$$

$$\tilde{QS} = \{(10, 1)\ (20, 0.8)\}$$

$$\tilde{S} = \{(30, 0.8)\ (40, 1)\ (50, 0.6)\}$$

To derive $\tilde{R}(x, y)$ representing the implication relation (i), we need to compute

$$\tilde{R}(x, y) = \max(\tilde{H} \times \tilde{S}, \bar{\tilde{H}} \times Y)$$

$\tilde{H} \times \tilde{S} =$

	10	20	30	40	50	60
30	0	0	0	0	0	0
40	0	0	0	0	0	0
50	0	0	0	0	0	0
60	0	0	0	0	0	0
70	0	0.8	0.8	1	0.6	0
80	0	0	0.8	1	0.6	0
90	0	[illegible]	0.3	0.3	0.3	0
100	0	0	0	0	0	0

$$\bar{\bar{H}} \times Y = \begin{array}{c} \\ 30 \\ 40 \\ 50 \\ 60 \\ 70 \\ 80 \\ 90 \\ 100 \end{array} \begin{array}{c} \begin{array}{cccccc} 10 & 20 & 30 & 40 & 50 & 60 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$\tilde{R}(x, y) = \begin{array}{c} \\ 30 \\ 40 \\ 50 \\ 60 \\ 70 \\ 80 \\ 90 \\ 100 \end{array} \begin{array}{c} \begin{array}{cccccc} 10 & 20 & 30 & 40 & 50 & 60 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0.8 & 0.8 & 1 & 0.6 & 0 \\ 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

To deduce Rotation is quite slow we make use of the composition rule

$$\tilde{QS} = V\tilde{H} \circ \tilde{R}(x, y)$$

$$= [0\ 0\ 0\ 0\ 0\ 0\ 0.9\ 1] \times \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0.8 & 0.8 & 1 & 0.6 & 0 \\ 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= [1\ 1\ 1\ 1\ 1\ 1]$$

7.4 FUZZY RULE BASED SYSTEM

Fuzzy linguistic descriptions are formal representations of systems made through fuzzy IF-THEN rules. They encode knowledge about a system in statements of the form—
IF (a set of conditions) are satisfied THEN (a set of consequents) can be inferred.
Fuzzy IF-THEN rules are coded in the form—

$$\text{IF } (x_1 \text{ is } \tilde{A}_1, x_2 \text{ is } \tilde{A}_2, \ldots, x_n \text{ is } \tilde{A}_n) \text{ THEN } (y_1 \text{ is } \tilde{B}_1, y_2 \text{ is } \tilde{B}_2, \ldots, y_n \text{ is } \tilde{B}_n).$$

where linguistic variables x_i, y_j take the values of fuzzy sets A_i and B_j respectively.

Example

If there is heavy rain and strong winds
then there must be severe flood warning.

Here, heavy, strong, and severe are fuzzy sets qualifying the variables rain, wind, and flood warning respectively.

A collection of rules referring to a particular system is known as a *fuzzy rule base.* If the conclusion C to be drawn from a rule base R is the conjunction of all the individual consequents C_i of each rule, then

$$C = C_1 \cap C_2 \cap \ldots \cap C_n \tag{7.26}$$

where

$$\mu_C(y) = \min(\mu_{C_1}(y), \mu_{C_2}(y), \ldots, \mu_{C_n}(y)), \forall y \in Y \tag{7.27}$$

where Y is the universe of discourse.

On the other hand, if the conclusion C to be drawn from a rule base R is the disjunction of the individual consequents of each rule, then

$$C = C_1 \cup C_2 \cup C_3 \ldots \cup C_n \tag{7.28}$$

where

$$\mu_C(y) = \max(\mu_{C_1}(y), \mu_{C_2}(y), \ldots, \mu_{C_n}(y)), \forall y \in Y \tag{7.29}$$

7.5 DEFUZZIFICATION

In many situations, for a system whose output is fuzzy, it is easier to take a crisp decision if the output is represented as a single scalar quantity. This conversion of a fuzzy set to single crisp value is called *defuzzification* and is the reverse process of *fuzzification*.

Several methods are available in the literature (Hellendoorn and Thomas, 1993) of which we illustrate a few of the widely used methods, namely *centroid method*, *centre of sums*, and *mean of maxima*.

Centroid method

Also known as the *centre of gravity* or the *centre of area* method, it obtains the centre of area (x^*) occupied by the fuzzy set. It is given by the expression

$$x^* = \frac{\int \mu(x)\, x\, dx}{\int \mu(x)\, dx} \tag{7.30}$$

for a continuous membership function, and

$$x^* = \frac{\sum_{i=1}^{n} x_i \cdot \mu(x_i)}{\sum_{i=1}^{n} \mu(x_i)} \tag{7.31}$$

for a discrete membership function.

Here, n represents the number of elements in the sample, x_i's are the elements, and $\mu(x_i)$ is its membership function.

Centre of sums (COS) method

In the centroid method, the overlapping area is counted once whereas in *centre of sums*, the overlapping area is counted twice. COS builds the resultant membership function by taking the algebraic sum of outputs from each of the contributing fuzzy sets $\tilde{A}_1, \tilde{A}_2, \ldots$, etc. The defuzzified value x^* is given by

$$x^* = \frac{\sum_{i=1}^{N} x_i \cdot \sum_{k=1}^{n} \mu_{\tilde{A}_k}(x_i)}{\sum_{i=1}^{N} \sum_{k=1}^{n} \mu_{\tilde{A}_k}(x_i)} \tag{7.32}$$

COS is actually the most commonly used defuzzification method. It can be implemented easily and leads to rather fast inference cycles.

Mean of maxima (MOM) defuzzification

One simple way of defuzzifying the output is to take the crisp value with the highest degree of membership. In cases with more than one element havin^ the maximum value, the mean value of the maxima is taken. The equation of the defuzzified value x^* is given by

$$x^* = \frac{\sum_{x_i \in M} x_i}{|M|} \tag{7.33}$$

where $M = \{x_i \,|\, \mu(x_i)$ is equal to the *height* of fuzzy set$\}$
$|M|$ is the cardinality of the set M. In the continuous case, M could be defined as

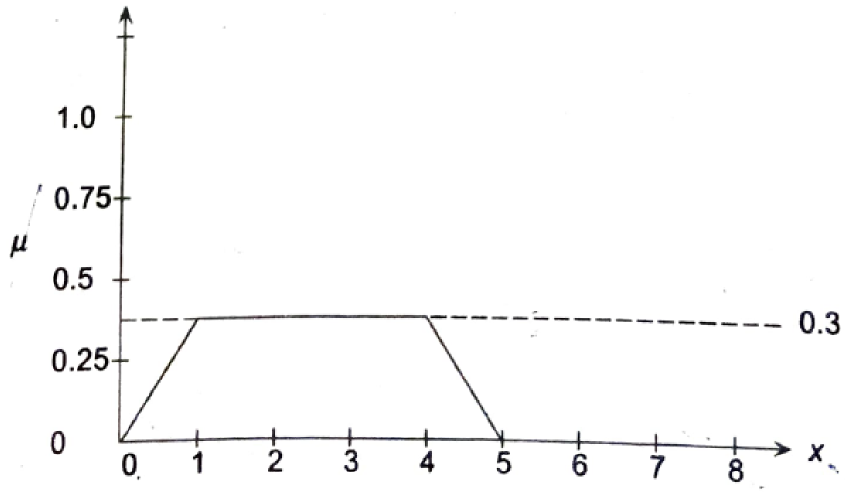
$$M = \{x \in [-c, c] \,|\, \mu(x) \text{ is equal to the height of the fuzzy set}\} \tag{7.34}$$

In such a case, the *mean of maxima* is the arithmetic average of mean values of all intervals contained in M including zero length intervals.

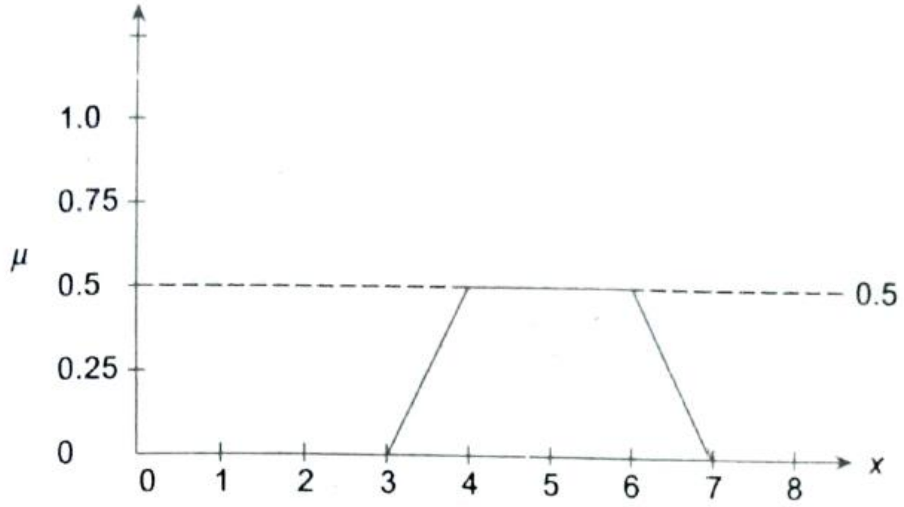
The *height* of a fuzzy set A, i.e. $h(A)$ is the largest membership grade obtained by any element in that set.

Example

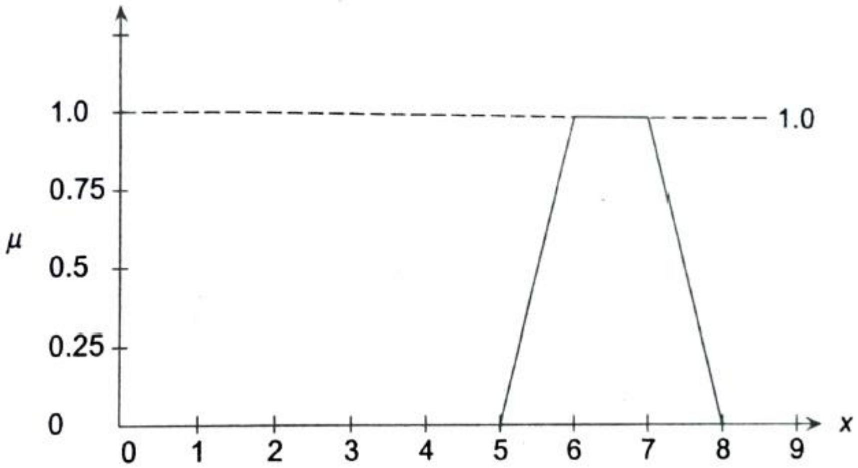
$\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$ are three fuzzy sets as shown in Fig. 7.1(a), (b), and (c). Figure 7.2 illustrates the aggregate of the fuzzy sets.



(a) $\tilde{A}_1$



(b) $\tilde{A}_2$



(c) $\tilde{A}_3$

Fig. 7.1 Fuzzy sets $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$.

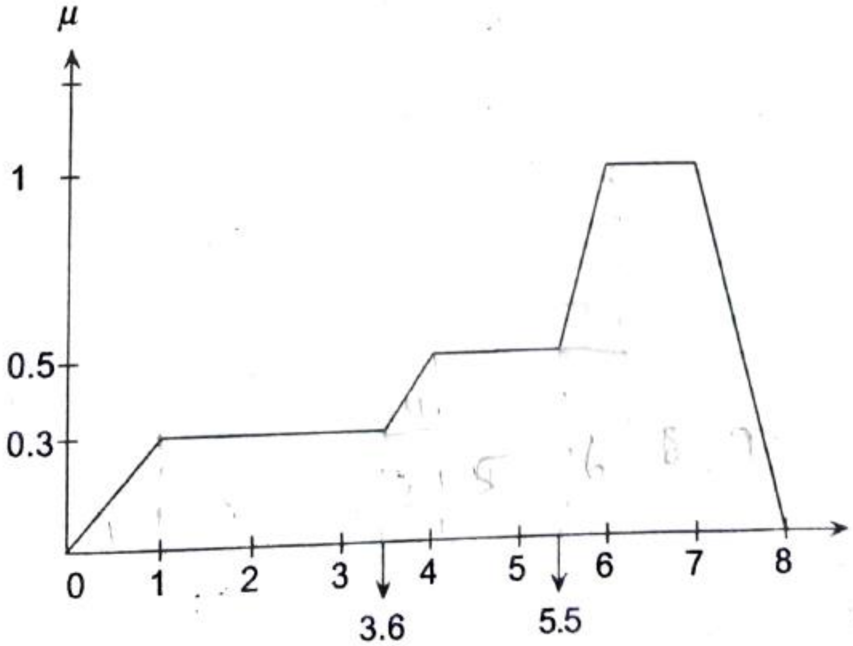


Fig. 7.2 Aggregated fuzzy set of $\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$.

The defuzzification using (i) centroid method, (ii) centre of sums method, and (iii) mean of maxima method is illustrated as follows.

Centroid method

To compute x^*, the centroid, we view the aggregated fuzzy sets as shown in Figs. 7.2 and 7.3. Note that in Fig. 7.3 the aggregated output has been divided into areas for better understanding.

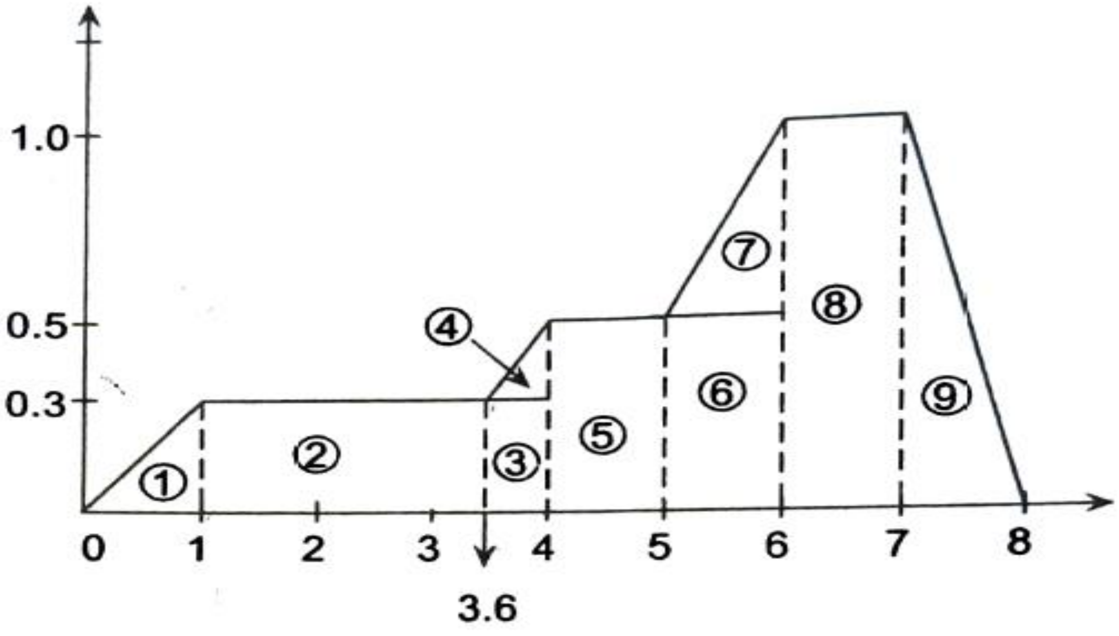


Fig. 7.3 Aggregated fuzzy set of $\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$ viewed as area segments.

Table 7.4 illustrates the computations for obtaining x^*.

Table 7.4 Computation of x^*

Area segment no.	Area (A)	$\bar{x}$	$A\bar{x}$
1	$\frac{1}{2} \times 0.3 \times 1 = 0.15$	0.67	0.1005
2	$2.6 \times 0.3 = 0.78$	2.3	1.748
3	$0.3 \times 0.4 = 0.12$	3.8	0.456
4	$\frac{1}{2} \times 0.4 \times 0.2 = 0.04$	3.8667	0.1546
5	$1.5 \times 0.5 = 0.75$	4.75	3.5625
6	$1.5 \times 0.5 = 0.75$	5.75	1.4375
7	$\frac{1}{2} \times 0.5 \times 0.5 = 0.125$	5.833	0.729
8	$1 \times 1 = 1$	6.5	6.5
9	$\frac{1}{2} \times 1 \times 1 = 0.5$	7.33	3.665

In Table 7.4, Area (A) shows the area of the segments of the aggregated fuzzy set and $\bar{x}$ shows the corresponding centroid. Now,

$$x^* = \frac{\sum A\bar{x}}{\sum \bar{A}}$$

i.e.

$$x^* = 18.353/3.695$$

$$= 4.9$$

Centre of sums method

Here, unlike centroid method the overlapping area is counted not once but twice. Making use of the aggregated fuzzy set shown in Fig.7.2, the centre of sums, x^* is given by

$$x^* = \frac{\frac{1}{2} \times 0.3 \times (3+5) \times 2.5 + \frac{1}{2} \times 0.5 \times (4+2) \times 5 + \frac{1}{2} \times 1 \times (3+1) \times 6.5}{\frac{1}{2} \times 0.3 \times (3+5) + \frac{1}{2} \times 0.5 \times (4+2) + \frac{1}{2} \times 1 \times (3+5)}$$

$$= 2.3$$

Here, the areas covered by the fuzzy sets $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$ (Refer Figs. 7.1(a), (b), and (c)) are given by

$$\frac{1}{2} \times 0.3 \times (3+5), \quad \frac{1}{2} \times 0.5 \times (4+2), \text{ and } \frac{1}{2} \times 1 \times (3+1) \text{ respectively.}$$

Mean of maxima method

Since the aggregated fuzzy set shown in Fig. 7.2 is a continuous set, x^* the mean of maxima is computed as $x^* = 6.5$.

Here, $M = \{X \in [6, 7] \mid \mu(x) = 1\}$ and the height of the aggregated fuzzy set is 1.

Figure 7.4 shows the defuzzified outputs using the above three methods.

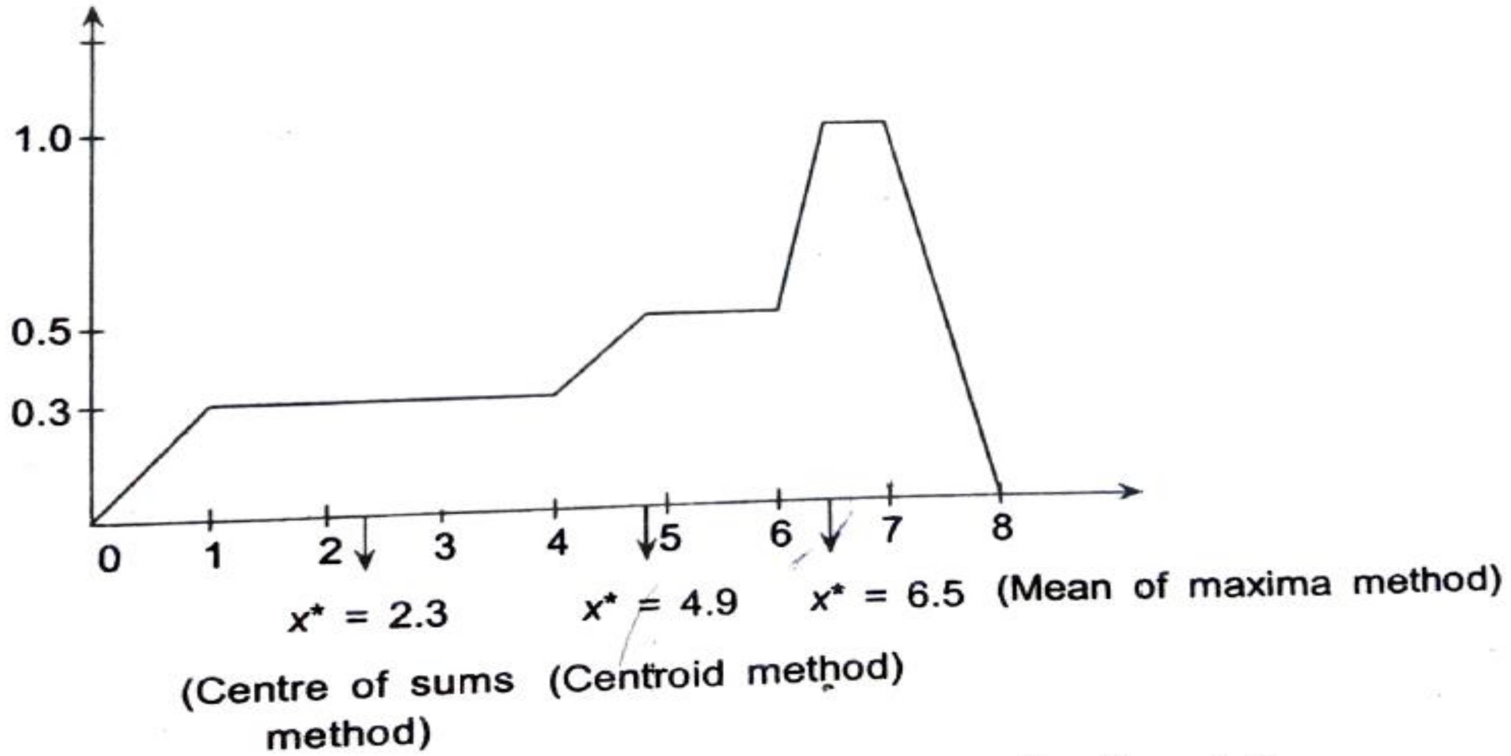


Fig. 7.4 Defuzzified outputs of the aggregate of $\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$.