SOFT COMPUTING

UNIT-IV:

Fuzzy Set Theory: Fuzzy versus crisp, Crisp sets: Operation on Crisp sets- Properties of Crisp Sets-Partition and Covering. Fuzzy sets: Membership Function – Basic fuzzy set Operations- properties of fuzzy sets. Crisp relations: Cartesian product-Other Crisp Relations-Operations on Relations. Fuzzy relations: Fuzzy Cartesian product- Operations on Fuzzy Relations.

TEXT BOOK

S.Rajasekaran & G.A.Vijayalakshmi Pai, “Neural Networks, Fuzzy logic, and Genetic Algorithms Synthesis and Applications, PHI, 2005.

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Fuzzy Set Theory

- It was Lotfi A. Zadeh who propounded the fuzzy set theory in his seminal paper (Zadeh, 1965).

Fuzzy Versus CRISP

- Consider the query, "is water colourless?" The answr to this a definite Yes/True, or No/False, as warranted by the situation.
- If "Yes"/"true" is accorded a value of 1 and "no"/"false" is accorded a value of 0, this statement results in 0/1 type of situation.
- Such a logic which demands a binary (0/1) type of handing is termed crisp in the domin of fuzzy set theory.
- Thus, statements such as "Temperature is 32'C, "the running time of the program is 4 seconds" are examples of crisp situations.

Fuzzy versus Crisp

- On the other hand, consider the statement, "Is Ram honest?" The answer to this query need not be a definite "yes" or "no".
- Considering the degree to which one knows Ram, a variety of answer spanning a range, such as "extremely honest", "honest at time", "very honest" could be generated.
- If, for instance, "Extremely honest" were to be accorded a value of 1, at the high end of the spectrum of values, "extremely dishonest" a value of 0 at the low end of the spectrum, then, "honest at times" and "very hones" could be assigned values of 0.4 and 0.85 respectively.
- The situation is therefore so fluid that it can accept value between 0 and 1, in contrast to the earlier one which was either a 0 or 1.
- Such a situation is termed fuzzy figure 6.1 shows a simple diagram to illustrate fuzzy and crisp situations.

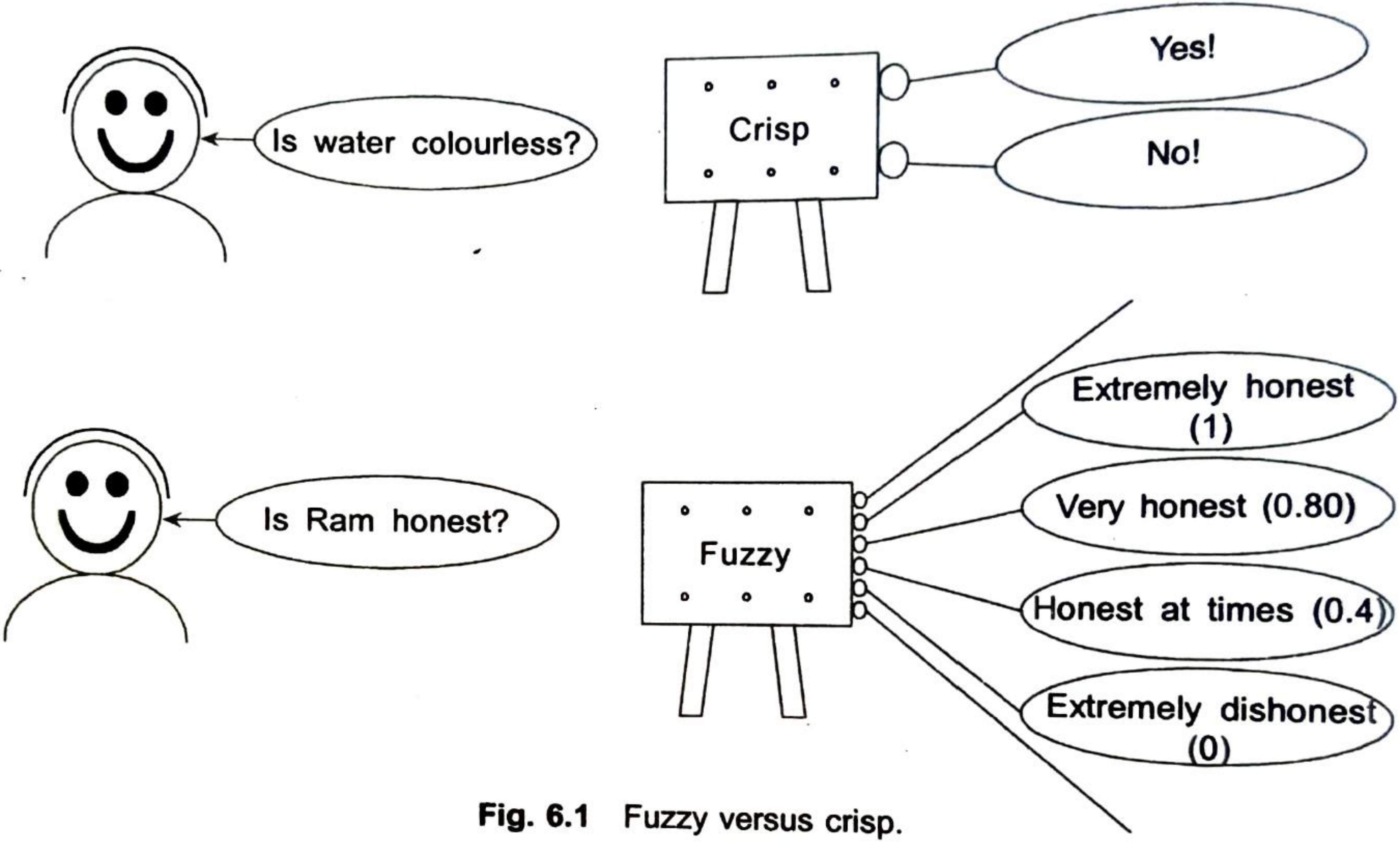


Fig. 6.1 Fuzzy versus crisp.

- Crisp set theory was propounded by Grorge Cantor is fundamental to the study of fuzzy sets.

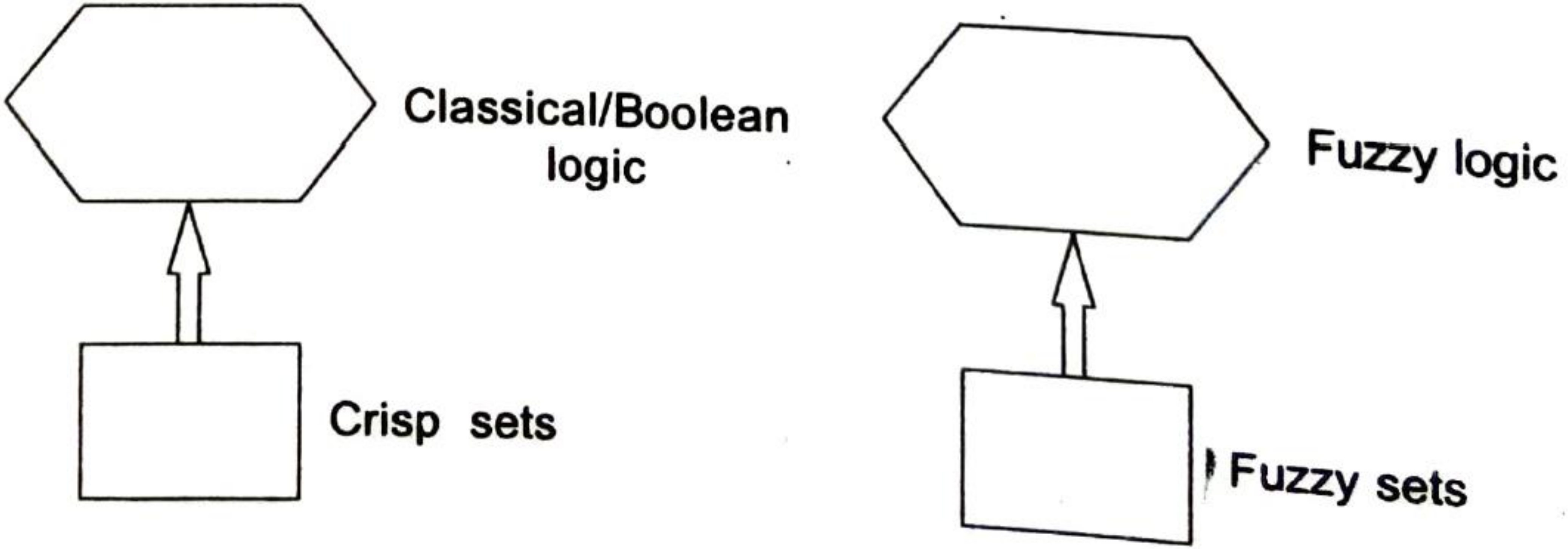


Fig. 6.2 Crisp sets and fuzzy sets.

6.2 CRISP SETS

Universe of discourse

The *universe of discourse* or *universal set* is the set which, with reference to a particular context, contains all possible elements having the same characteristics and from which sets can be formed. The universal set is denoted by E.

Example

(i) The universal set of all numbers in Euclidean space.

(ii) The universal set of all students in a university.

Set

A set is a *well defined* collection of objects. Here, well defined means the object either belongs to or does not belong to the set (observe the "crispness" in the definition).

A set in certain contexts may be associated with its universal set from which it is derived.

Given a set A whose objects are $a_1, a_2, a_3, \ldots, a_n$, we write A as $A = \{a_1, a_2, \ldots, a_n\}$. Here, $a_1, a_2, \ldots, a_n$ are called the *members* of the set. Such a form of representing a set is known as *list form*.

Example

$$A = \{\text{Gandhi, Bose, Nehru}\}$$

$$B = \{\text{Swan, Peacock, Dove}\}$$

A set may also be defined based on the properties the members have to satisfy. In such a case, a set A is defined as

$$A = \{x \mid P(x)\} \tag{6.1}$$

Here, $P(x)$ stands for the property P to be satisfied by the member x. This is read as 'A is the set of all X such that $P(x)$ is satisfied'.

Example

$$A = \{x \mid x \text{ is an odd number}\}$$

$$B = \{y \mid y > 0 \text{ and } y \bmod 5 = 0\}$$

Venn diagram

Venn diagrams are pictorial representations to denote a set. Given a set A defined over a universal set E, the Venn diagram for A and E is as shown in Fig. 6.3.

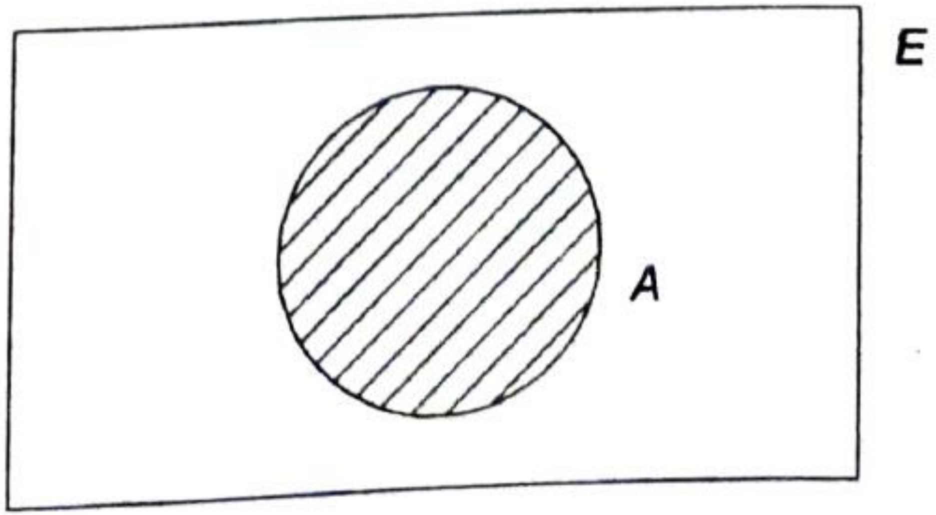


Fig. 6.3 Venn diagram of a set *A*.

Example

In Fig. 6.3, if E represents the set of university students then A may represent the set of female students.

Membership

An element x is said to be a member of a set A if x belongs to the set A. The membership is indicated by '$\in$' and is pronounced "belongs to". Thus, $x \in A$ means x belongs to A and $x \notin A$ means *x does not belong to A.*

Example

Given $A = \{4, 5, 6, 7, 8, 10\}$, for $x = 3$ and $y = 4$, we have $x \notin A$ and $y \in A$

Here, observe that each element either belongs to or does not belong to a set. The concept of membership is definite and therefore crisp (1—belongs to, 0—does not belong to). In contrast, as we shall see later, a fuzzy set accommodates membership values which are not only 0 or 1 but *anything* between 0 and 1.

Cardinality

The number of elements in a set is called its cardinality. Cardinality of a set A is denoted as $n(A)$ or $|A|$ or $\#A$.

Example

If $A = \{4, 5, 6, 7\}$ then $|A| = 4$

Family of sets

A set whose members are sets themselves, is referred to as a family of sets.

Example

$A = \{\{1, 3, 5\}, \{2, 4, 6\}, \{5, 10\}\}$ is a set whose members are the sets $\{1, 3, 5\}$, $\{2, 4, 6\}$, and $\{5, 10\}$.

Null Set/Empty Set

A set is said to be a *null set* or *empty set* if it has no members. A null set is indicated as ∅ or { } and indicates an impossible event. Also, $|\varnothing| = 0$.

Example

The set of all prime ministers who are below 15 years of age.

Singleton Set

A set with a single element is called a *singleton set*. A singleton set has cardinality of 1.

Example

If $A = \{a\}$, then $|A| = 1$

Subset

Given sets A and B defined over E the universal set, A is said to be a *subset* of B if A is fully contained in B, that is, every element of A is in B.

Denoted as $A \subset B$, we say that A is a subset of B, or A is a *proper subset* of B. On the other hand, if A is contained in or equivalent to that of B then we denote the subset relation as $A \subseteq B$. In such a case, A is called the *improper* subset of B.

Superset

Given sets A and B on E the universal set, A is said to be a *superset* of B if every element of B is contained in A.

Denoted as $A \supset B$, we say A is a superset of B or A contains B. If A contains B and is equivalent to B, then we denote it as $A \supseteq B$.

Example

Let $A = \{3, 4\}$ $B = \{3, 4, 5\}$ and $C = \{4, 5, 3\}$

Here, $A \subset B$, and $B \supset A$

$C \subseteq B$, and $B \supseteq C$

Power set

A *power set* of a set A is the set of all possible subsets that are derivable from A including null set.

A power set is indicated as $P(A)$ and has cardinality of $|P(A)| = 2^{|A|}$

Example

Let $A = \{3, 4, 6, 7\}$

$P(A) = \{\{3\}, \{4\}, \{6\}, \{7\}, \{3, 4\}, \{4, 6\}, \{6, 7\}, \{3, 7\}, \{3, 6\}, \{4, 7\}, \{3, 4, 6\}, \{4, 6, 7\}, \{3, 6, 7\}, \{3, 4, 7\}, \{3, 4, 6, 7\}, \emptyset\}$

Here, $|A| = 4$ and $|P(A)| = 2^4 = 16$.

6.2.1 Operations on Crisp Sets

Union (∪)

The union of two sets A and B ($A \cup B$) is the set of all elements that belong to A or B or both.

$$A \cup B = \{x/x \in A \text{ or } x \in B\} \quad (6.2)$$

Example

Given $A = \{a, b, c, 1, 2\}$ and $B = \{1, 2, 3, a, c\}$, we get $A \cup B = \{a, b, c, 1, 2, 3\}$ Figure 6.4 illustrates the Venn diagram representation for $A \cup B$

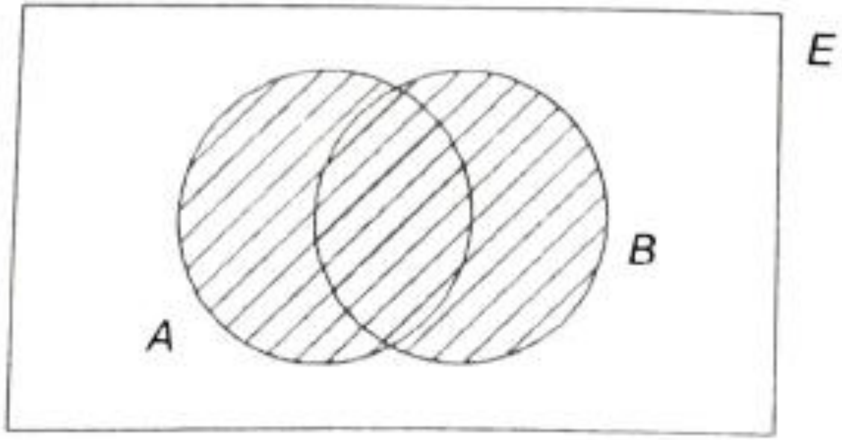


Fig. 6.4 Venn diagram for $A \cup B$.

Intersection ($\cap$)

The intersection of two sets A and B ($A \cap B$) is the set of all elements that belong to A and B

$$A \cap B = \{x / x \in A \text{ and } x \in B\} \tag{6.3}$$

Any two sets which have $A \cap B = \varnothing$ are called *Disjoint Sets.*

Example

Given $A = \{a, b, c, 1, 2\}$ and $B = \{1, 2, 3, a, c\}$, we get $A \cap B = \{a, c, 1, 2\}$

Figure 6.5 illustrates the Venn diagram for $A \cap B$

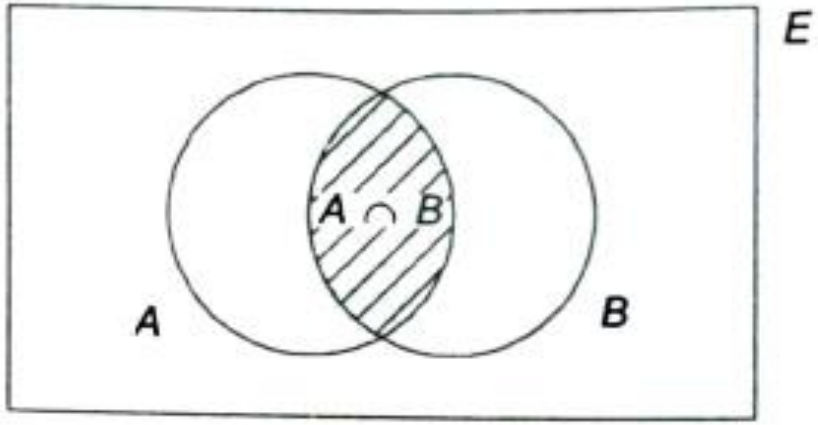


Fig. 6.5 Venn diagram for $A \cap B$.

Complement **(^{c})**

The complement of a set $A\,(\bar{A}\,|\,A^c)$ is the set of all elements which are in E but not in A.

$$A^c = \{x/x \notin A,\ x \in E\} \tag{6.4}$$

Example

Given $X = \{1, 2, 3, 4, 5, 6, 7\}$ and $A = \{5, 4, 3\}$, we get $A^c = \{1, 2, 6, 7\}$

Figure 6.6 illustrates the Venn diagram for A^c.

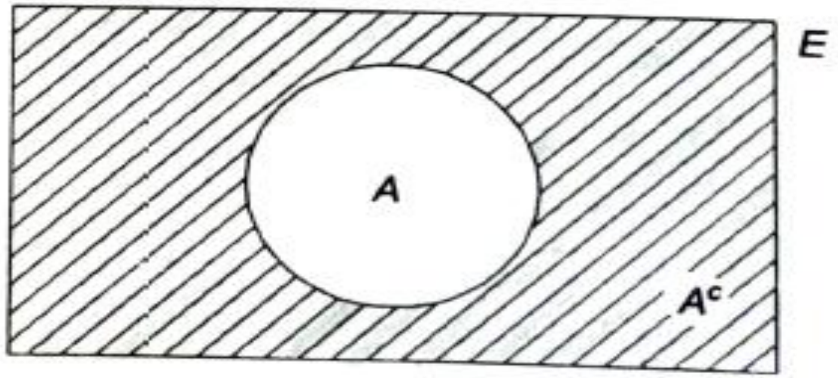


Fig. 6.6 Venn diagram for A^c.

Difference **(–)**

The difference of the set A and B is $A - B$, the set of all elements which are in A but not in B.

$$A - B = \{x \mid x \in A \text{ and } x \notin B\} \tag{6.5}$$

Example

Given $A = \{a, b, c, d, e\}$ and $B = \{b, d\}$, we get $A - B = \{a, c, e\}$

Figure 6.7 illustrates the Venn diagram for $A - B$.

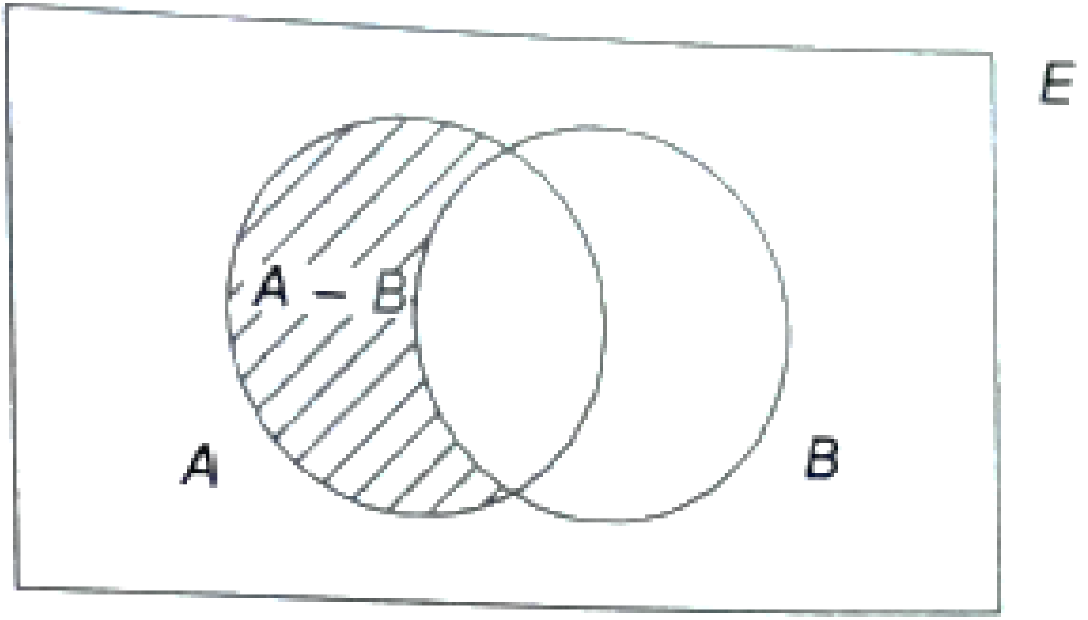


Fig. 6.7 Venn diagram for $A - B$.

6.2.2 Properties of Crisp Sets

The following properties of sets are important for further manipulation of sets.

1 *Commutativity:*

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A \qquad (6.6)$$

2 *Associativity:*

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C) \qquad (6.7)$$

3 *Distributivity:*

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \qquad (6.8)$$

4 *Idempotence:*

$$A \cup A = A$$
$$A \cap A = A \qquad (6.9)$$

5. *Identity:*

$$A \cup \varnothing = A$$
$$A \cap E = A$$
$$A \cap \varnothing = \varnothing$$
$$A \cup E = E \qquad (6.10)$$

6 *Law of Absorption:*

$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A \qquad (6.11)$$

7. *Transitivity:* If $A \subseteq B$, $B \subseteq C$ then $A \subseteq C$ (6.12)

8. *Involution:* $(A^c)^c = A$ (6.13)

9. *Law of the Excluded Middle:* $A \cup A^c = E$ (6.14)

10. *Law of Contradiction:* $A \cap A^c = \varnothing$ (6.15)

11. *De Morgan's laws:*

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c \qquad (6.16)$$

All the properties could be verified by means of Venn diagrams.

Example 6.1

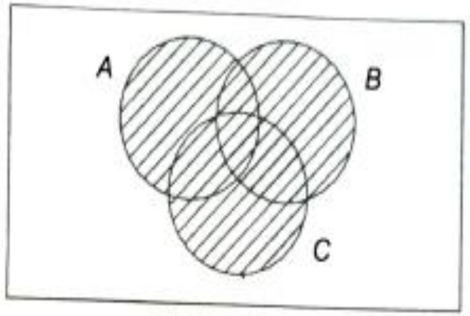
Given three sets A, B, and C. Prove De Morgan's laws using Venn diagrams.

Solution

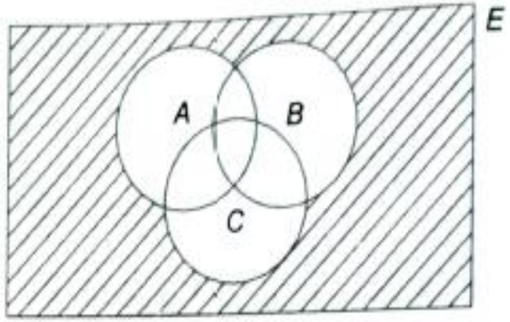
To prove De Morgan's laws, we need to show that

(i) $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$

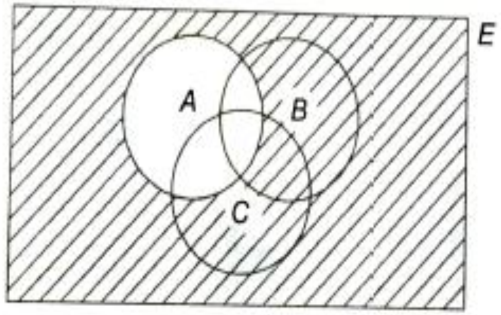
(ii) $(A \cap B \cap C)^c = A^c \cup B^c \cup C^c$



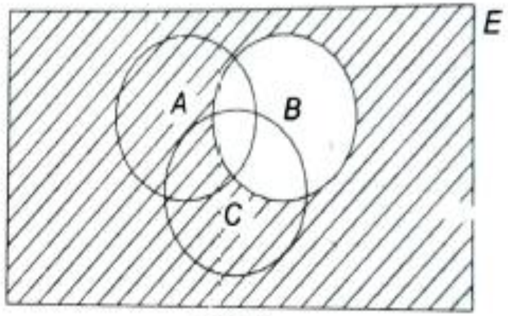
(a) $A \cup B \cup C$



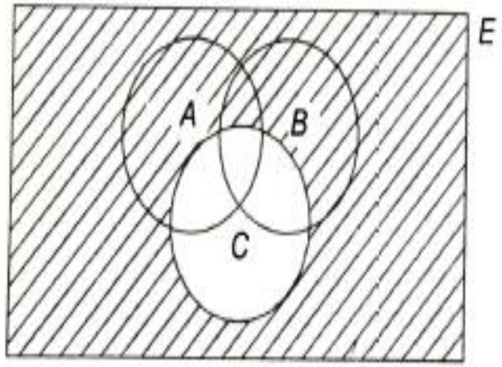
(b) $(A \cup B \cup C)^c$



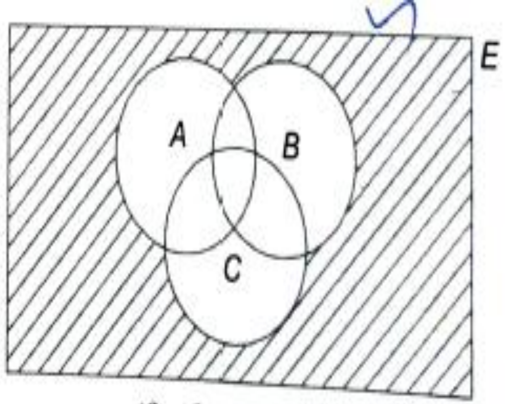
(c) A^c



(d) B^c

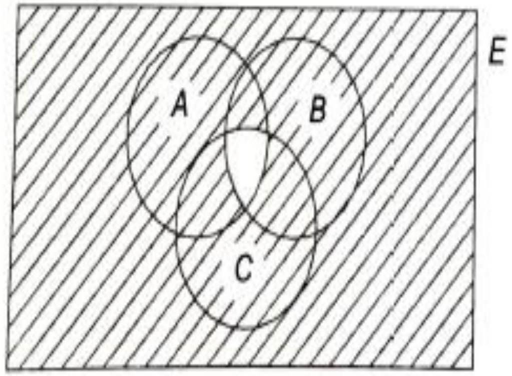


(e) C^c

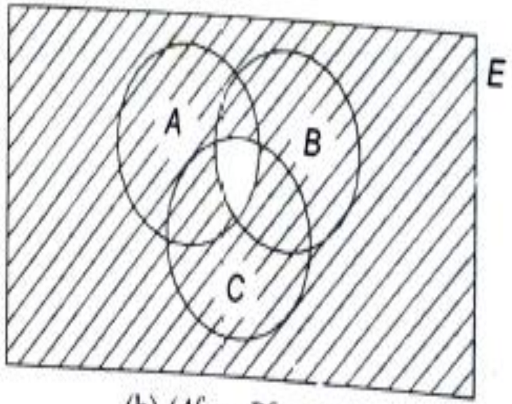


(f) $A^c \cap B^c \cap C^c$

(i) Here, it can be seen that $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$.



(g) $(A \cap B \cap C)^c$



(h) $(A^c \cup B^c \cup C^c)$

(ii) Figures (g) to (h) show that $(A \cap B \cap C)^c = A^c \cup B^c \cup C^c$.

Example 6.2

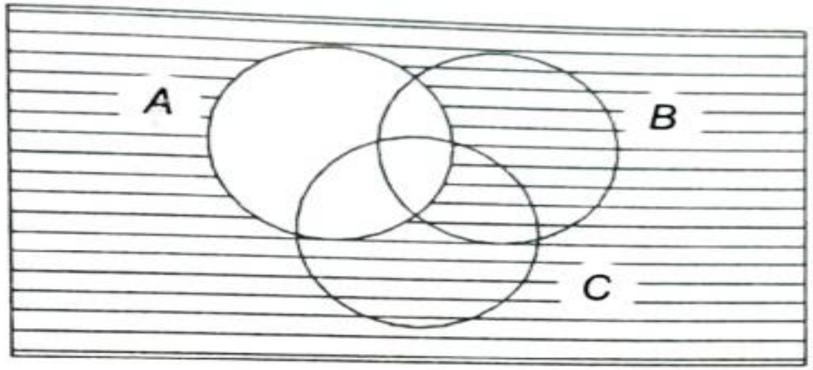
Let the sets A, B, C, and E be given as follows:

E = all students enrolled in the university cricket club.

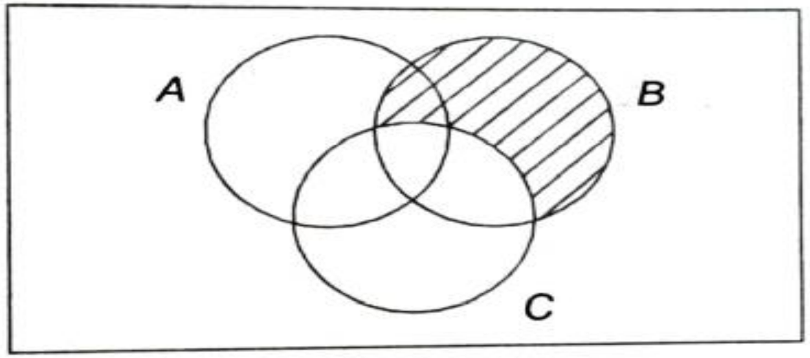
A = male students, B = bowlers, and C = batsmen.

Draw individual Venn diagrams to illustrate (a) female students (b) bowlers who are not batsmen (c) female students who can both bowl and bat.

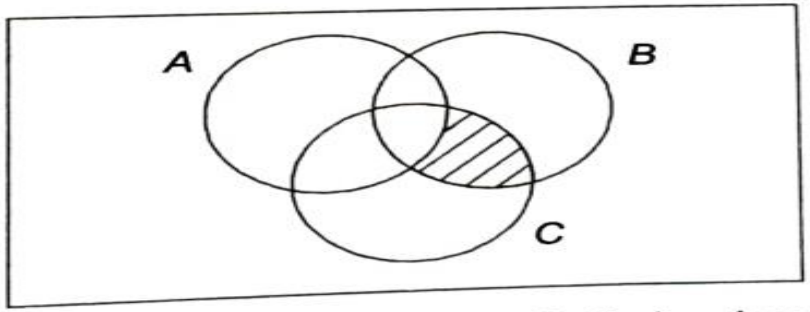
Solution



(a) Female students



(b) Bowlers who are not batsmen



(c) Female students who can both bowl and bat

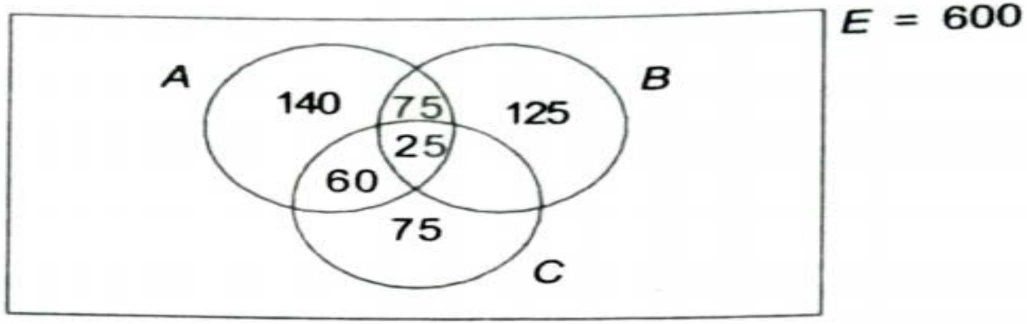
Example 6.3

In Example 6.2, assume that $|E| = 600$, $|A| = 300$, $|B| = 225$, $|C| = 160$. Also, let the number of male students who are bowlers $(A \cap B)$ be 100, 25 of whom are batsmen too $(A \cap B \cap C)$, and the total number of male students who are batsmen $(A \cap C)$ be 85.

Determine the number of students who are: (i) Females, (ii) Not bowlers, (iii) Not batsmen, (iv) Females and who can bowl but not bat.

Solution

From the given data, the Venn diagram obtained is as follows:



(i) No. of female students $|A^c| = |E| - |A| = 600 - 300 = 300$

(ii) No. of students who are not bowlers $|B^c| = |E| - |B| = 600 - 225 = 375$

(iii) No. of students who are not batsmen $|C^c| = |E| - |C| = 600 - 160 = 440$

(iv) No. of female students who can bowl $|A^c \cap B| = 125$ (from the Venn diagram)

6.2.3 Partition and Covering

Partition

A *partition* on A is defined to be a set of non-empty subsets A_i, each of which is pairwise disjoint and whose union yields the original set A.

Partition on A indicated as $\Pi(A)$, is therefore

$$\text{(i)} \quad A_i \cap A_j = \varnothing \quad \text{for each pair } (i, j) \in I,\ i \neq j \qquad (6.17)$$

$$\text{(ii)} \quad \bigcup_{i \in I} A_i = A$$

The members A_i of the partition are known as blocks (refer Fig. 6.8).

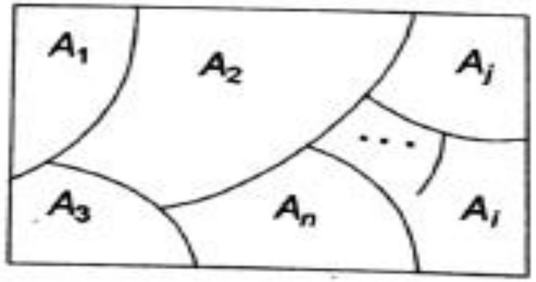


Fig. 6.8 Partition of set A.

Example

Given A = {a, b, c, d, e}, A_1 = {a, b}, A_2 = {c, d} and A_3 = {e}, which gives

$$A_1 \cap A_2 = \varnothing,\ A_1 \cap A_3 = \varnothing,\ A_2 \cap A_3 = \varnothing$$

Also,

$$A_1 \cup A_2 \cup A_3 = A = \{a, b, c, d, e\}$$

Hence, $\{A_1, A_2, A_3\}$, is a partition on A.

Covering

A *covering* on A is defined to be a set of non-empty subsets A_i whose union yields the original set A. The non-empty subsets need not be disjoint (Refer Fig. 6.9).

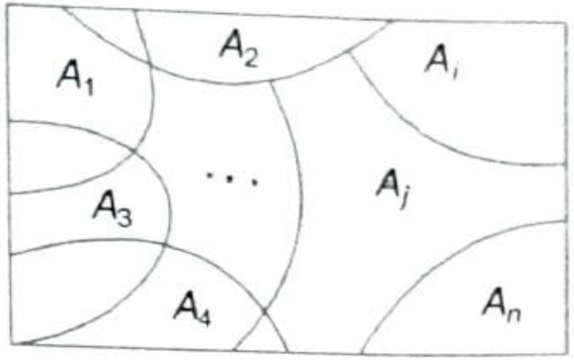


Fig. 6.9 Covering of set *A*.

Example

Given $A = \{a, b, c, d, e\}$, $A_1 = \{a, b\}$, $A_2 = \{b, c, d\}$, and $A_3 = \{d, e\}$. This gives

$$A_1 \cap A_2 = \{b\}$$

$$A_1 \cap A_3 = \varnothing$$

$$A_2 \cap A_3 = \{d\}$$

Also,
$$A_1 \cup A_2 \cup A_3 = \{a, b, c, d, e\} = A$$

Hence, $\{A_1, A_2, A_3\}$ is a covering on A.

Rule of Addition

Given a partition on A where A_i, $i = 1, 2, \ldots, n$ are its non-empty subsets then,

$$|A| = \left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| \tag{6.18}$$

Example

Given $A = \{a, b, c, d, e\}$, $A_1 = \{a, b\}$, $A_2 = \{c, d\}$, $A_3 = \{e\}$, $|A| = 5$, and $\Sigma_{i=1}^{3}|A_i| = 2 + 2 + 1 = 5$

Rule of Inclusion and Exclusion

Rule of addition is not applicable on the covering of set A, especially if the subsets are not pairwise disjoint. In such a case, the rule of inclusion and exclusion is applied.

Example

Given A to be a covering of n sets $A_1, A_2, \ldots, A_n$,

for $n = 2$,.

$$|A| = |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| \tag{6.19}$$

for $n = 3$,

$$|A| = |A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3| \tag{6.20}$$

Generalizing,

$$|A| = |\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n}|A_i| - \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n}|A_i \cap A_j|$$

$$+ \sum_{\substack{i=1 \\ i \neq j \neq k}}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} |A_i \cap A_j \cap A_k| \ldots (-i)^{n+1}\left|\bigcap_{i=1}^{n} A_i\right| \tag{6.21}$$

6.3 FUZZY SETS

Fuzzy sets support a flexible sense of membership of elements to a set. While in crisp set theory, an element either belongs to or does not belong to a set, in fuzzy set theory many degrees of membership (between 0 and 1) are allowed. Thus, a membership function $\mu_A^{(x)}$ is associated with a fuzzy set $\tilde{A}$ such that the function maps every element of the universe of discourse X (or the reference set) to the interval [0, 1].

Formally, the mapping is written as $\mu_{\tilde{A}}(x) : X \to [0, 1]$

A *fuzzy set* is defined as follows:

If X is a universe of discourse and x is a particular element of X, then a fuzzy set A defined on X may be written as a collection of ordered pairs

$$A = \{(x, \mu_{\tilde{A}}(x)), x \in X\} \tag{6.23}$$

where each pair $(x, \mu_{\tilde{A}}(x))$ is called a singleton. In crisp sets, $\mu_{\tilde{A}}(x)$ is dropped.

An alternative definition which indicates a fuzzy set as a union of all $\mu_{\tilde{A}}(x)/x$ singletons is given by

$$A = \sum_{x_i \in X} \mu_{\tilde{A}}(x_i)/x_i \quad \text{in the discrete case} \tag{6.24}$$

and

$$A = \int_X \mu_{\tilde{A}}(x)/x \quad \text{in the continuous case} \tag{6.25}$$

Here, the summation and integration signs indicate the union of all $\mu_{\tilde{A}}(x)/x$ singletons.

Example

Let $X = \{g_1, g_2, g_3, g_4, g_5\}$ be the reference set of students. Let $\tilde{A}$ be the fuzzy set of "smart" students, where "smart" is a fuzzy linguistic term.

$$\tilde{A} = \{(g_1, 0.4)\,(g_2, 0.5)\,(g_3, 1)\,(g_4, 0.9)\,(g_5, 0.8)\}$$

Here $\tilde{A}$ indicates that the smartness of g_1 is 0.4, g_2 is 0.5 and so on when graded over a scale of 0–1.

Though fuzzy sets model vagueness, it needs to be realized that the definition of the sets varies according to the context in which it is used. Thus, the fuzzy linguistic term "tall" could have one kind of fuzzy set while referring to the height of a building and another kind of fuzzy set while referring to the height of human beings.

6.3.1 Membership Function

The membership function values need not always be described by discrete values. Quite often, these turn out to be as described by a continuous function.

The fuzzy membership function for the fuzzy linguistic term "cool" relating to temperature may turn out to be as illustrated in Fig. 6.10.

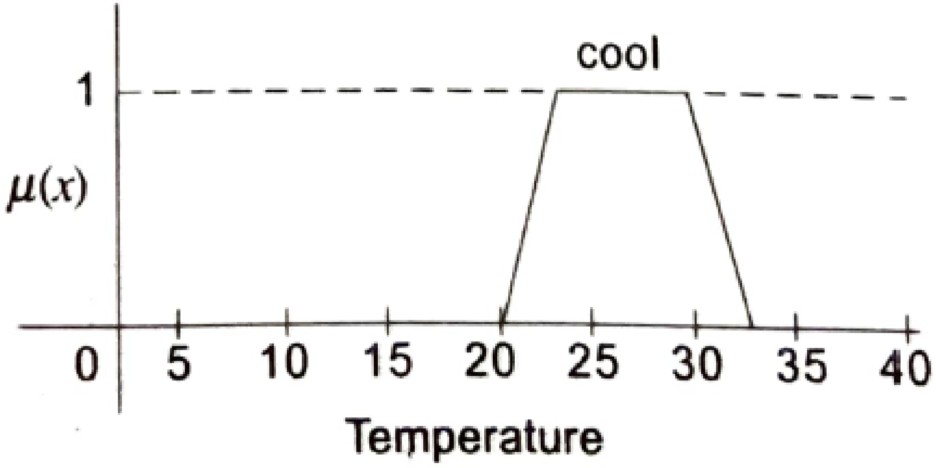


Fig. 6.10 Continuous membership function for "cool".

A membership function can also be given mathematically as

$$\mu_{\tilde{A}}(x) = \frac{1}{(1+x)^2}$$

The graph is as shown in Fig. 6.11.

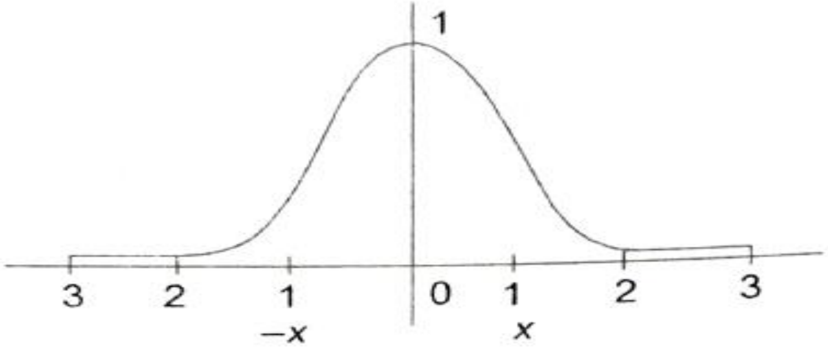


Fig. 6.11 Continuous membership function dictated by a mathematical function.

Different shapes of membership functions exist. The shapes could be triangular, trapezoidal, curved or their variations as shown in Fig. 6.12.

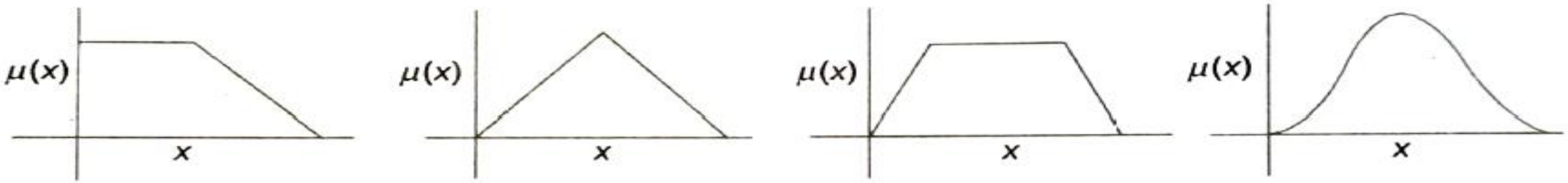


Fig. 6.12 Different shapes of membership function graphs.

Example

Consider the set of people in the following age groups

0–10	40–50
10–20	50–60
20–30	60–70
30–40	70 and above

The fuzzy sets "young", "middle-aged", and "old" are represented by the membership function graphs as illustrated in Fig. 6.13.

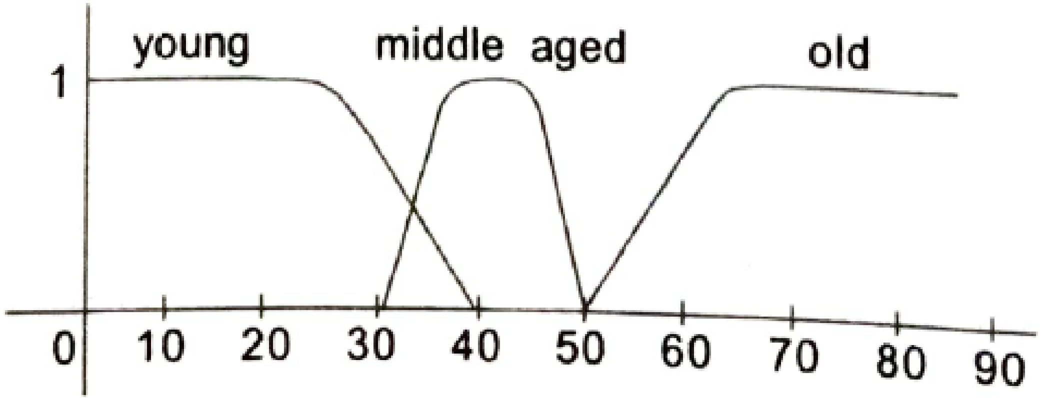


Fig. 6.13 Example of fuzzy sets expressing "young", "middle-aged", and "old".

6.3.2 Basic Fuzzy Set Operations

Given X to be the universe of discourse and $\tilde{A}$ and $\tilde{B}$ to be fuzzy sets with $\mu_A(x)$ and $\mu_B(x)$ as their respective membership functions, the basic fuzzy set operations are as follows:

Union

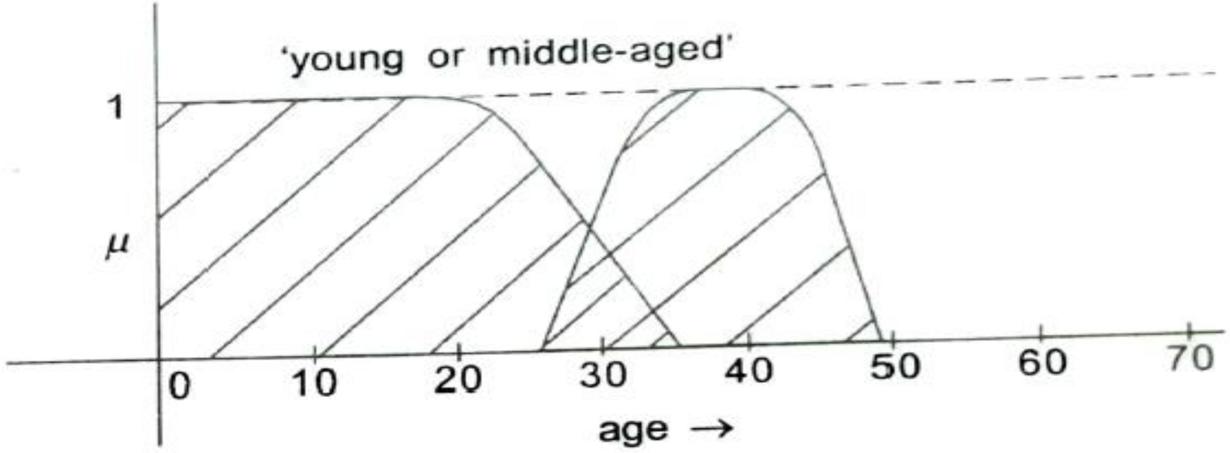
The union of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a new fuzzy set $\tilde{A} \cup \tilde{B}$ also on X with a membership function defined as

$$\mu_{\tilde{A} \cup \tilde{B}}(x) = \max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \quad (6.26)$$

Example

Let $\tilde{A}$ be the fuzzy set of young people and $\tilde{B}$ be the fuzzy set of middle-aged people as illustrated in Fig. 6.13. Now $\tilde{A} \cup \tilde{B}$, the fuzzy set of "young or middle-aged" will be given by

$\tilde{A} \cup \tilde{B}$:



In its discrete form, for x_1, x_2, x_3

if $\tilde{A} = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0)\}$ and $\tilde{B} = \{(x_1, 0.8), (x_2, 0.2), (x_3, 1)\}$

$$\tilde{A} \cup \tilde{B} = \{(x_1, 0.8), (x_2, 0.7), (x_3, 1)\}$$

since,

$$\mu_{\tilde{A} \cup \tilde{B}}(x_1) = \max(\mu_{\tilde{A}}(x_1), \mu_{\tilde{B}}(x_1))$$

$$= \max\ (0.5, 0.8)$$

$$= 0.8$$

$$\mu_{\tilde{A} \cup \tilde{B}}(x_2) = \max(\mu_{\tilde{A}}(x_2), \mu_{\tilde{B}}(x_2)) = \max(0.2, 0.7) = 0.7$$

$$\mu_{\tilde{A} \cup \tilde{B}}(x_3) = \max(\mu_{\tilde{A}}(x_3), \mu_{\tilde{B}}(x_3)) = \max(0, 1) = 1$$

Intersection

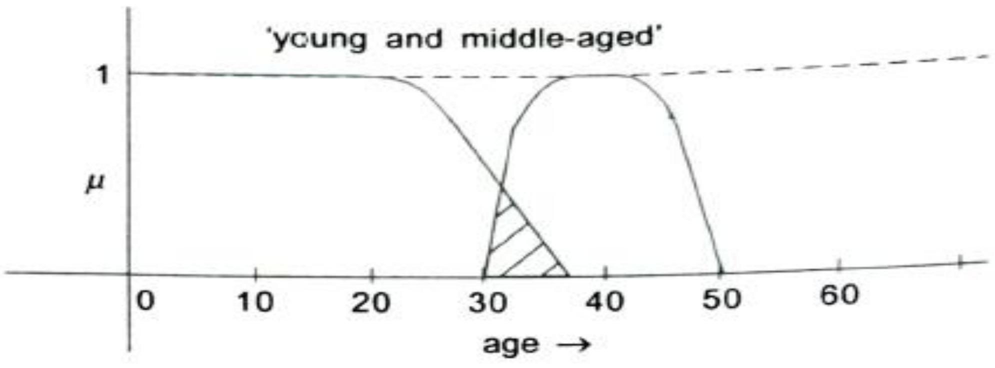
The intersection of fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a new fuzzy set $\tilde{A} \cap \tilde{B}$ with membership function defined as

$$\mu_{\tilde{A} \cap \tilde{B}}(x) = \min(u_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \tag{6.27}$$

Example

For $\tilde{A}$ and $\tilde{B}$ defined as "young" and "middle-aged" as illustrated in previous examples.

$\tilde{A} \cap \tilde{B}$:



In its discrete form, for x_1, x_2, x_3

if $\tilde{A} = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0)\}$ and $\tilde{B} = \{(x_1, 0.8), (x_2, 0.2), (x_3, 1)\}$

$$\tilde{A} \cap \tilde{B} = \{(x_1, 0.5), (x_2, 0.2), (x_3, 0)\}$$

since,

$$\mu_{\tilde{A} \cap \tilde{B}}(x_1) = \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{B}}(x_1))$$
$$= \min(0.5, 0.8)$$
$$= 0.5$$

$$\mu_{\tilde{A} \cap \tilde{B}}(x_2) = \min(\mu_{\tilde{A}}(x_2), \mu_{\tilde{B}}(x_2))$$
$$= \min(0.7, 0.2)$$
$$= 0.2$$

$$\mu_{\tilde{A} \cap \tilde{B}}(x_3) = \min(\mu_{\tilde{A}}(x_3), \mu_{\tilde{B}}(x_3))$$
$$= \min(0, 1)$$
$$= 0$$

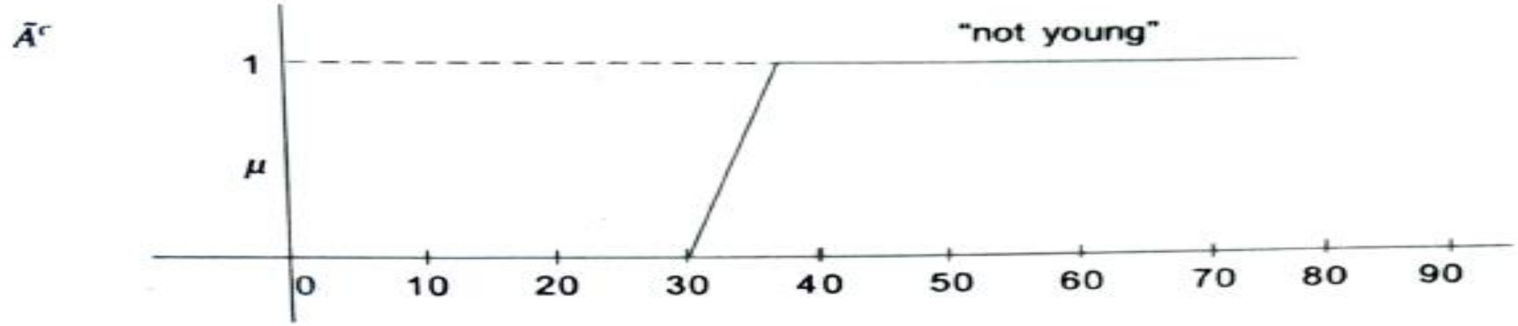
Complement

The complement of a fuzzy set $\tilde{A}$ is a new fuzzy set $\tilde{A}^c$ with a membership function

$$\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x) \tag{6.28}$$

Example

For the fuzzy set $\tilde{A}$ defined as "young" the complement "not young" is given by $\tilde{A}^c$. In its discrete form, for x_1, x_2, and x_3



if $\tilde{A} = \{(x_1, 0.5)\ (x_2, 0.7)\ (x_3, 0)\}$

then, $\tilde{A}^c = \{(x_1, 0.5)\ (x_2, 0.3)\ (x_3, 1)\}$

since,

$$\begin{aligned}
\mu_{\tilde{A}^c}(x_1) &= 1 - \mu_{\tilde{A}}(x_1) \\
&= 1 - 0.5 \\
&= 0.5 \\
\mu_{\tilde{A}^c}(x_2) &= 1 - \mu_{\tilde{A}}(x_2) \\
&= 1 - 0.7 \\
&= 0.3 \\
\mu_{\tilde{A}^c}(x_3) &= 1 - \mu_{\tilde{A}}(x_3) \\
&= 1 - 0 \\
&= 1
\end{aligned}$$

Other operations are,

Product of two fuzzy sets

The product of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a new fuzzy set $\tilde{A} \cdot \tilde{B}$ whose membership function is defined as

$$\mu_{\tilde{A}\cdot\tilde{B}}(x) = \mu_{\tilde{A}}(x)\,\mu_{\tilde{B}}(x) \tag{6.29}$$

Example

$$\tilde{A} = \{(x_1, 0.2), (x_2, 0.8), (x_3, 0.4)\}$$

$$\tilde{B} = \{(x_1, 0.4)\ (x_2, 0), (x_3, 0.1)\}$$

$$\tilde{A} \cdot \tilde{B} = \{(x_1, 0.08)\ (x_2, 0)\ (x_3, 0.04)\}$$

Since

$$\mu_{\tilde{A}\cdot\tilde{B}}(x_1) = \mu_{\tilde{A}}(x_1)\cdot\mu_{\tilde{A}}(x_2)$$

$$= 0.2 \cdot 0.4 = 0.08$$

$$\mu_{\tilde{A}\cdot\tilde{B}}(x_2) = \mu_{\tilde{A}}(x_2)\cdot\mu_{\tilde{A}}(x_2)$$

$$= 0.8 \cdot 0 = 0$$

$$\mu_{\tilde{A}\cdot\tilde{B}}(x_3) = \mu_{\tilde{A}}(x_3)\cdot\mu_{\tilde{A}}(x_3)$$

$$= 0.4 \cdot 0.1$$

$$= 0.04$$

Equality

Two fuzzy sets $\tilde{A}$ and $\tilde{B}$ are said to be equal ($\tilde{A} = \tilde{B}$) if $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$ (6.30)

Example

$$\tilde{A} = \{(x_1, 0.2)(x_2, 0.8)\}$$

$$\tilde{B} = \{(x_1, 0.6)(x_2, 0.8)\}$$

$$\tilde{C} = \{(x_1, 0.2)(x_2, 0.8)\}$$

$$\tilde{A} \neq \tilde{B}$$

since

$$\mu_{\tilde{A}}(x_1) \neq \mu_{\tilde{B}}(x_1) \quad \text{although}$$

$$\mu_{\tilde{A}}(x_2) = \mu_{\tilde{B}}(x_2)$$

but

$$\tilde{A} = \tilde{C}$$

since

$$\mu_{\tilde{A}}(x_1) = \mu_{\tilde{C}}(x_1) = 0.2$$

and

$$\mu_{\tilde{A}}(x_2) = \mu_{\tilde{C}}(x_2) = 0.8$$

Product of a fuzzy set with a crisp number

Multiplying a fuzzy set $\tilde{A}$ by a crisp number a results in a new fuzzy set product $a.\ \tilde{A}$ with the membership function

$$\mu_{a\cdot\tilde{A}}(x) = a\cdot\mu_{\tilde{A}}(x) \tag{6.31}$$

Example

$$\tilde{A} = \{(x_1, 0.4), (x_2, 0.6), (x_3, 0.8)\}$$

For $a = 0.3$

$$a\cdot\tilde{A} = \{(x_1, 0.12), (x_2, 0.18), (x_3, 0.24)\}$$

since,

$$\mu_{a\cdot\tilde{A}}(x_1) = a\cdot\mu_{\tilde{A}}(x_1)$$
$$= 0.3\cdot 0.4$$
$$= 0.12$$

$$\mu_{a\cdot\tilde{A}}(x_2) = a\cdot\mu_{\tilde{A}}(x_2)$$
$$= 0.3\cdot 0.6$$
$$= 0.18$$

$$\mu_{a\cdot\tilde{A}}(x_3) = a\cdot\mu_{\tilde{A}}(x_3)$$
$$= 0.3\cdot 0.8$$
$$= 0.24$$

Power of a fuzzy set

The α power of a fuzzy set $\tilde{A}$ is a new fuzzy set A^{α} whose membership function is given by

$$\mu_{\tilde{A}^{\alpha}}(x) = (\mu_{\tilde{A}}(x))^{\alpha} \tag{6.32}$$

Raising a fuzzy set to its second power is called *Concentration* (CON) and taking the square root is called *Dilation* (DIL).

Example

$$\tilde{A} = \{(x_1, 0.4), (x_2, 0.2), (x_3, 0.7)\}$$

For $\alpha = 2$

$$\mu_{\tilde{A}^2}(x) = (\mu_{\tilde{A}}(x))^2$$

Hence, $(\tilde{A})^2 = \{(x_1, 0.16), (x_2, 0.04), (x_3, 0.49)\}$

Since

$$\mu_{\tilde{A}^2}(x_1) = (\mu_{\tilde{A}}(x_1))^2 = (0.4)^2 = 0.16$$

$$\mu_{\tilde{A}^2}(x_2) = (\mu_{\tilde{A}}(x_2))^2 = (0.2)^2 = 0.04$$

$$\mu_{\tilde{A}^2}(x_3) = (\mu_{\tilde{A}}(x_3))^2 = (0.7)^2 = 0.49$$

Difference

The difference of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a new fuzzy set $\tilde{A} - \tilde{B}$ defined as

$$\tilde{A} - \tilde{B} = (\tilde{A} \cap \tilde{B}^c) \tag{6.33}$$

Example

$$\tilde{A} = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.6)\}; \ \tilde{B} = \{(x_1, 0.1), (x_2, 0.4), (x_3, 0.5)\}$$

$$\tilde{B}^c = \{(x_1, 0.9), (x_2, 0.6), (x_3, 0.5)\}$$

$$\tilde{A} - \tilde{B} = \tilde{A} \cap \tilde{B}^c$$

$$= \{(x_1, 0.2)(x_2, 0.5)(x_3, 0.5)\}$$

Disjunctive sum

The disjunctive sum of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a new fuzzy set $\tilde{A} \oplus \tilde{B}$ defined as

$$\tilde{A} \oplus \tilde{B} = (\tilde{A}^c \cap \tilde{B}) \cup (\tilde{A} \cap \tilde{B}^c) \tag{6.34}$$

Example

$$\tilde{A} = \{(x_1, 0.4)(x_2, 0.8)(x_3, 0.6)\}$$

$$\tilde{B} = \{(x_1, 0.2)(x_2, 0.6)(x_3, 0.9)\}$$

Now,

$$\tilde{A}^c = \{(x_1, 0.6)(x_2, 0.2)(x_3, 0.4)\}$$

$$\tilde{B}^c = \{(x_1, 0.8)(x_2, 0.4)(x_3, 0.1)\}$$

$$\tilde{A}^c \cap \tilde{B} = \{(x_1, 0.2)(x_2, 0.2)(x_3, 0.4)\}$$

$$\tilde{A} \cap \tilde{B}^c = \{(x_1, 0.4)(x_2, 0.4)(x_3, 0.1)\}$$

$$\tilde{A} \oplus \tilde{B} = \{(x_1, 0.4)(x_2, 0.4)(x_3, 0.4)\}$$

6.3.3 Properties of Fuzzy Sets

Fuzzy sets follow some of the properties satisfied by crisp sets. In fact, crisp sets can be thought of as special instances of fuzzy sets. Any fuzzy set $\tilde{A}$ is a subset of the reference set X. Also, the membership of any element belonging to the null set $\varnothing$ is 0 and the membership of any element belonging to the reference set is 1.

The properties satisfied by fuzzy sets are

Commutativity:

$$\tilde{A} \cup \tilde{B} = \tilde{B} \cup \tilde{A}$$

$$\tilde{A} \cap \tilde{B} = \tilde{B} \cap \tilde{A} \tag{6.35}$$

Associativity:

$$\tilde{A} \cup (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cup \tilde{C}$$

$$\tilde{A} \cap (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cap \tilde{C} \tag{6.36}$$

Distributivity:

$$\tilde{A} \cup (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cap (\tilde{A} \cup \tilde{C})$$

$$\tilde{A} \cap (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cup (\tilde{A} \cap \tilde{C}) \tag{6.37}$$

Idempotence:

$$\tilde{A} \cup \tilde{A} = \tilde{A}$$

$$\tilde{A} \cap \tilde{A} = \tilde{A} \tag{6.38}$$

Identity:

$$\tilde{A} \cup \varnothing = \tilde{A}$$

$$\tilde{A} \cup X = \tilde{A}$$

$$\tilde{A} \cap \varnothing = \varnothing$$

$$\tilde{A} \cup X = X \tag{6.39}$$

Transitivity: If $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$, then $\tilde{A} \subseteq \tilde{C}$ (6.40)

Involution:

$$(\tilde{A}^c)^c = \tilde{A} \tag{6.41}$$

De Morgan's laws:

$$(\tilde{A} \cap \tilde{B})^c = (\tilde{A}^c \cup \tilde{B}^c)$$

$$(\tilde{A} \cup \tilde{B})^c = (\tilde{A}^c \cap \tilde{B}^c)$$

(6.42)

6.4 CRISP RELATIONS

In this section, we review crisp relations as a prelude to fuzzy relations. The concept of relations between sets is built on the Cartesian product operator of sets.

6.4.1 Cartesian Product

The *Cartesian product* of two sets A and B denoted by $A \times B$ is the set of all ordered pairs such that the first element in the pair belongs to A and the second element belongs to B.

i.e.
$$A \times B = \{(a,b)/a \in A,\ b \in B\}$$

If $A \neq B$ and A and B are non-empty then $A \times B \neq B \times A$.

The Cartesian product could be extended to n number of sets

$$\mathop{\times}_{i=1}^{n} A_i = \{(a_1, a_2, a_3, \ldots, a_n)/a_i \in A_i \ \text{ for every } i = 1, 2, \ldots, n\} \tag{6.45}$$

Observe that
$$\left|\mathop{\times}_{i=1}^{n} A_i\right| = \prod_{i=1}^{n} |A_i| \tag{6.46}$$

Example

Given $A_1 = \{a, b\}, A_2 = \{1, 2\}, A_3 = \{\alpha\}$,

$$A_1 \times A_2 = \{(a, 1), (b, 1), (a, 2), (b, 2)\},\ |A_1 \times A_2| = 4, \quad \text{and} \quad |A_1| = |A_2| = 2$$

Here,
$$|A_1 \times A_2| = |A_1| \cdot |A_2|$$

Also,
$$A_1 \times A_2 \times A_3 = \{(a, 1, \alpha), (a, 2, \alpha), (b, 1, \alpha), (b, 2, \alpha)\}$$
$$|A_1 \times A_2 \times A_3| = 4 = |A_1| \cdot |A_2| \cdot |A_3|$$

6.4.2 Other Crisp Relations

An *n-ary* relation denoted as $R(X_1, X_2, \ldots, X_n)$ among crisp sets $X_1, X_2, \ldots, X_n$ is a subset of the Cartesian product $\overset{n}{\underset{i=1}{\times}} X_i$ and is indicative of an association or relation among the tuple elements.

For $n = 2$, the relation $R(X_1, X_2)$ is termed as a *binary* relation; for $n = 3$, the relation is termed *ternary*; for $n = 4$, *quarternary*; for $n = 5$, *quinary* and so on.

If the universe of discourse or sets are finite, the *n-ary* relation can be expressed as an *n-dimensional relation matrix*. Thus, for a binary relation $R(X, Y)$ where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$, the relation matrix R is a two dimensional matrix where X represents the rows, Y represents the columns and $R(i, j) = 1$ if $(x_i, y_j) \in R$ and $R(i, j) = 0$ if $(x_i, y_j) \notin R$.

Example

Given $X = \{1, 2, 3, 4\}$,

$$X \times X = \begin{Bmatrix} (1,1)(1,2)(1,3)(1,4)(2,1)(2,2)(2,3)(2,4) \\ (3,1)(3,2)(3,3)(3,4)(4,1)(4,2)(4,3)(4,4) \end{Bmatrix}$$

Let the relation R be defined as

$$R = \{(x, y)/y = x + 1,\ x, y \in X\}$$

$$R = \{(1, 2)(2, 3)(3, 4)\}$$

The relation matrix R is given by

$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

6.4.3 Operations on Relations

Given two relations R and S defined on $X \times Y$ and represented by relation matrices, the following operations are supported by R and S

Union: $R \cup S$

$$R \cup S(x, y) = \max(R(x, y), S(x, y)) \quad (6.47)$$

Intersection: $R \cap S$

$$R \cap S(x, y) = \min(R(x, y), S(x, y)) \quad (6.48)$$

Complement: $\bar{R}$

$$\bar{R}(x, y) = 1 - R(x, y) \quad (6.49)$$

Composition of relations: $R \circ S$

Given R to be a relation on X, Y and S to be a relation on Y, Z then $R \circ S$ is a composition of relation on X, Z defined as

$$R \circ S = \{(x, z)/(x, z) \in X \times Z,\ \exists\ y \in Y \text{ such that } (x, y) \in R \text{ and } (y, z) \in S\} \quad (6.50)$$

A common form of the composition relation is the *max-min composition*.

Max-min composition:

Given the relation matrices of the relation R and S, the max-min composition is defined as

For $$T = R \circ S$$

$$T(x, z) = \max_{y \in Y}(\min(R(x, y), S(y, z))) \quad (6.51)$$

Example

Let R, S be defined on the sets $\{1, 3, 5\} \times \{1, 3, 5\}$

Let $R: \{(x, y) \mid y = x + 2\}$, $S: \{(x, y) \mid x \le y\}$

$R = \{(1, 3)(3, 5)\}$, $S = \{(1, 3)(1, 5)(3, 5)\}$

The relation matrices are

$$R: \begin{array}{c} \\ 1 \\ 3 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \qquad S: \begin{array}{c} \\ 1 \\ 3 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Using max-min composition

$$R \circ S = \begin{array}{c} \\ 1 \\ 3 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

since

$$R \circ S(1, 1) = \max\{\min(0, 0), \min(1, 0), \min(0, 0)\}$$

$$= \max(0, 0, 0) = 0.$$

$$R \circ S(1, 3) = \max\{0, 0, 0\} = 0$$

$$R \circ S(1, 5) = \max\{0, 1, 0\} = 1.$$

Similarly,

$$R \circ S(3, 1) = 0.$$

$$R \circ S(3, 3) = R \circ S(3, 5) = R \circ S(5, 1) = R \circ S(5, 3) = R \circ S(5, 5) = 0$$

$R \circ S$ from the relation matrix is $\{(1, 5)\}$.

Also,

$$S \circ R = \begin{array}{c} \\ 1 \\ 3 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

6.5 FUZZY RELATIONS

Fuzzy relation is a fuzzy set defined on the Cartesian product of crisp sets $X_1, X_2, \ldots, X_n$ where the n-tuples $(x_1, x_2, \ldots, x_n)$ may have varying degrees of membership within the relation. The membership values indicate the strength of the relation between the tuples.

Example

Let R be the fuzzy relation between two sets X_1 and X_2 where X_1 is the set of diseases and X_2 is the set of symptoms.

X_1 = {typhoid, viral fever, common cold}

X_2 = {running nose, high temperature, shivering}

The fuzzy relation R may be defined as

	Running nose	*High temperature*	*Shivering*
Typhoid	0.1	0.9	0.8
Viral fever	0.2	0.9	0.7
Common cold	0.9	0.4	0.6

6.5.1 Fuzzy Cartesian Product

Let $\tilde{A}$ be a fuzzy set defined on the universe X and $\tilde{B}$ be a fuzzy set defined on the universe Y, the Cartesian product between the fuzzy sets $\tilde{A}$ and $\tilde{B}$ indicated as $\tilde{A} \times \tilde{B}$ and resulting in a fuzzy relation $\tilde{R}$ is given by

$$\tilde{R} = \tilde{A} \times \tilde{B} \subset X \times Y \tag{6.52}$$

where $\tilde{R}$ has its membership function given by

$$\begin{aligned} \mu_{\tilde{R}}(x, y) &= \mu_{\tilde{A} \times \tilde{B}}(x, y) \\ &= \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)) \end{aligned} \tag{6.53}$$

Example

Let $\tilde{A} = \{(x_1, 0.2), (x_2, 0.7), (x_3, 0.4)\}$ and $\tilde{B} = \{(y_1, 0.5), (y_2, 0.6)\}$ be two fuzzy sets defined on the universes of discourse $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ respectively. Then the fuzzy relation $\tilde{R}$ resulting out of the fuzzy Cartesian product $\tilde{A} \times \tilde{B}$ is given by

$$\tilde{R} = \tilde{A} \times \tilde{B} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

since,

$$\tilde{R}(x_1, y_1) = \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{B}}(y_1)) = \min(0.2, 0.5) = 0.2$$
$$\tilde{R}(x_1, y_2) = \min(0.2, 0.6) = 0.2$$
$$\tilde{R}(x_2, y_1) = \min(0.7, 0.5) = 0.5$$
$$\tilde{R}(x_2, y_2) = \min(0.7, 0.6) = 0.6$$
$$\tilde{R}(x_3, y_1) = \min(0.4, 0.5) = 0.4$$
$$\tilde{R}(x_3, y_2) = \min(0.4, 0.6) = 0.4$$

Min

6.5.2 Operations on Fuzzy Relations

Let $\tilde{R}$ and $\tilde{S}$ be fuzzy relations on $X \times Y$.

Union

$$\mu_{\tilde{R} \cup \tilde{S}}(x, y) = \max\left(\mu_{\tilde{R}}(x, y),\ \mu_{\tilde{S}}(x, y)\right) \tag{6.54}$$

Intersection

$$\mu_{\tilde{R} \cap \tilde{S}}(x, y) = \min\left(\mu_{\tilde{R}}(x, y),\ \mu_{\tilde{S}}(x, y)\right) \tag{6.55}$$

Complement

$$\mu_{\tilde{R}^c}(x, y) = 1 - \mu_{\tilde{R}}(x, y) \tag{6.56}$$

Composition of relations

The definition is similar to that of crisp relation. Suppose $\tilde{R}$ is a fuzzy relation defined on $X \times Y$, and $\tilde{S}$ is a fuzzy relation defined on $Y \times Z$, then $\tilde{R} \circ \tilde{S}$ is a fuzzy relation on $X \times Z$. The fuzzy max-min composition is defined as

$$\mu_{\tilde{R} \circ \tilde{S}}(x, z) = \max_{y \in Y}\left(\min\left(\mu_{\tilde{R}}(x, y),\ \mu_{\tilde{S}}(y, z)\right)\right) \tag{6.57}$$

$$\mu_{\tilde{R}\circ\tilde{S}}(x, z) = \max_{y \in Y} (\min(\mu_{\tilde{R}}(x, y), \ \mu_{\tilde{S}}(y, z)))$$

Example

$X = \{x_1, x_2, x_3\}$ $Y = \{y_1, y_2\}$ $Z = \{z_1, z_2, z_3\}$

Let $\tilde{R}$ be a fuzzy relation

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.9 \\ 0.8 & 0.6 \end{bmatrix} \end{array}$$

Let $\tilde{S}$ be a fuzzy relation

$$\begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.6 & 0.4 & 0.7 \\ 0.5 & 0.8 & 0.9 \end{bmatrix} \end{array}$$

Then $R \circ S$, by max-min composition yields,

$$R \circ S = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.4 & 0.5 \\ 0.5 & 0.8 & 0.9 \\ 0.6 & 0.6 & 0.7 \end{bmatrix} \end{array}$$

$$\mu_{\tilde{R}\circ\tilde{S}}(x_1, z_1) = \max(\min(0.5, 0.6), \min(0.1, 0.5))$$
$$= \max(0.5, 0.1)$$
$$= 0.5$$

$$\mu_{\tilde{R}\circ\tilde{S}}(x_1, z_2) = \max(\min(0.5, 0.4), \min(0.1, 0.8))$$
$$= \max(0.4, 0.1)$$
$$= 0.4$$

Similarly,

$$\mu_{\tilde{R}\circ\tilde{S}}(x_1, z_3) = \max(0.5, 0.1) = 0.5$$

$$\mu_{\tilde{R}\circ\tilde{S}}(x_2, z_1) = \max(0.2, 0.5) = 0.5$$

$$\mu_{\tilde{R}\circ\tilde{S}}(x_2, z_2) = \max(0.2, 0.8) = 0.8$$

$$\mu_{\tilde{R}\circ\tilde{S}}(x_2, z_3) = \max(0.2, 0.9) = 0.9$$

$$\mu_{\tilde{R}\circ\tilde{S}}(x_3, z_1) = \max(0.6, 0.5) = 0.6$$

$$\mu_{\tilde{R}\circ\tilde{S}}(x_3, z_2) = \max(0.4, 0.6) = 0.6$$

$$\mu_{\tilde{R}\circ\tilde{S}}(x_3, z_3) = \max(0.7, 0.6) = 0.7$$