18MIT23C - Network Security

Unit - II

Advanced Encryption Standard

AES Requirements

- private key symmetric block cipher
- 128-bit data, 128/192/256-bit keys
- stronger & faster than Triple-DES
- active life of 20-30 years (+ archival use)
- provide full specification & design details
- both C & Java implementations
- NIST have released all submissions & unclassified analyses

AES Evaluation Criteria

- initial criteria:
  - security – effort for practical cryptanalysis
  - cost – in terms of computational efficiency
  - algorithm & implementation characteristics
- final criteria
  - general security
  - ease of software & hardware implementation
  - implementation attacks
  - flexibility (in en/decrypt, keying, other factors)

AES Shortlist

- after testing and evaluation, shortlist in Aug-99:
  - MARS (IBM) - complex, fast, high security margin
  - RC6 (USA) - v. simple, v. fast, low security margin
  - Rijndael (Belgium) - clean, fast, good security margin
  - Serpent (Euro) - slow, clean, v. high security margin
  - Twofish (USA) - complex, v. fast, high security margin
- then subject to further analysis & comment
- saw contrast between algorithms with
  - few complex rounds verses many simple rounds
  - which refined existing ciphers verses new proposals

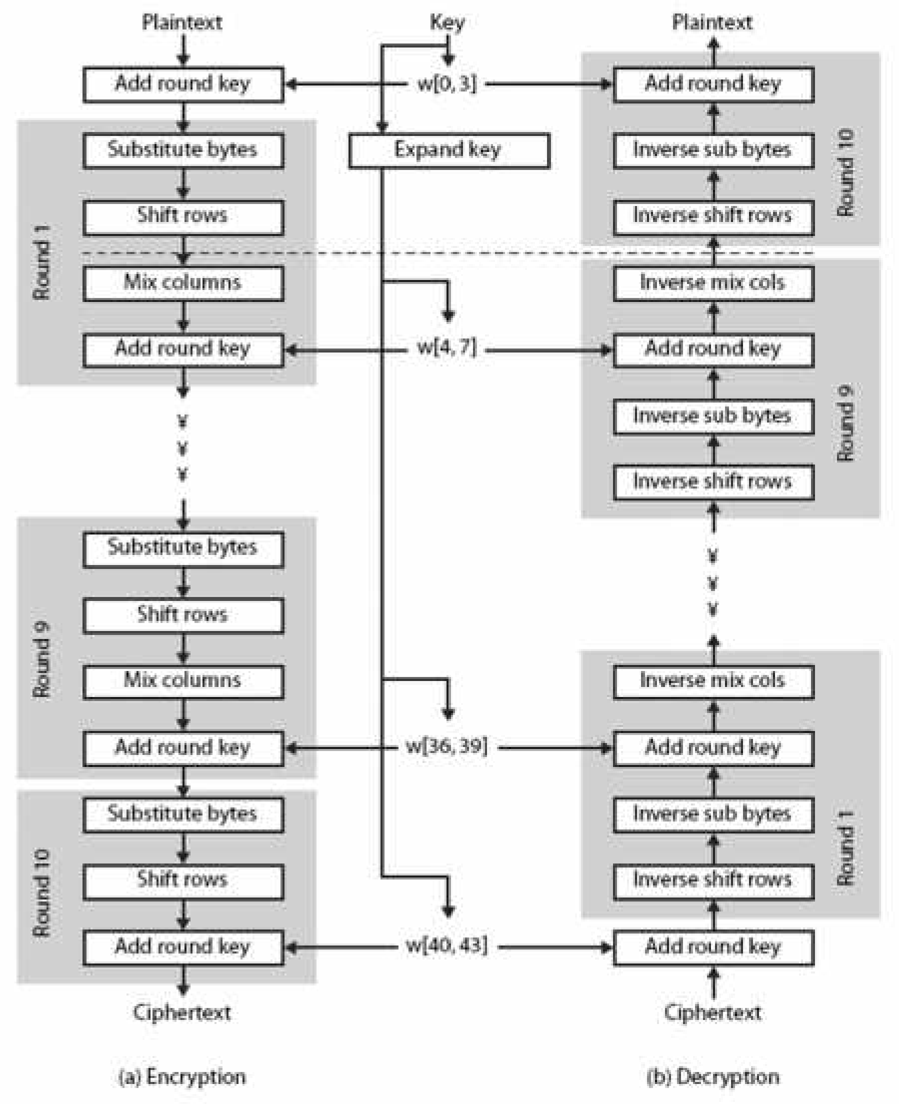
The AES Cipher - Rijndael

- designed by Rijmen-Daemen in Belgium
- has 128/192/256 bit keys, 128 bit data
- an **iterative** rather than **feistel** cipher
  - processes data as block of 4 columns of 4 bytes
  - operates on entire data block in every round
- designed to be:
  - resistant against known attacks
  - speed and code compactness on many CPUs
  - design simplicity

Rijndael

- data block of 4 columns of 4 bytes is state
- key is expanded to array of words
- has 9/11/13 rounds in which state undergoes:
  - byte substitution (1 S-box used on every byte)
  - shift rows (permute bytes between groups/columns)
  - mix columns (subs using matrix multipy of groups)
  - add round key (XOR state with key material)
  - view as alternating XOR key & scramble data bytes
- initial XOR key material & incomplete last round
- with fast XOR & table lookup implementation

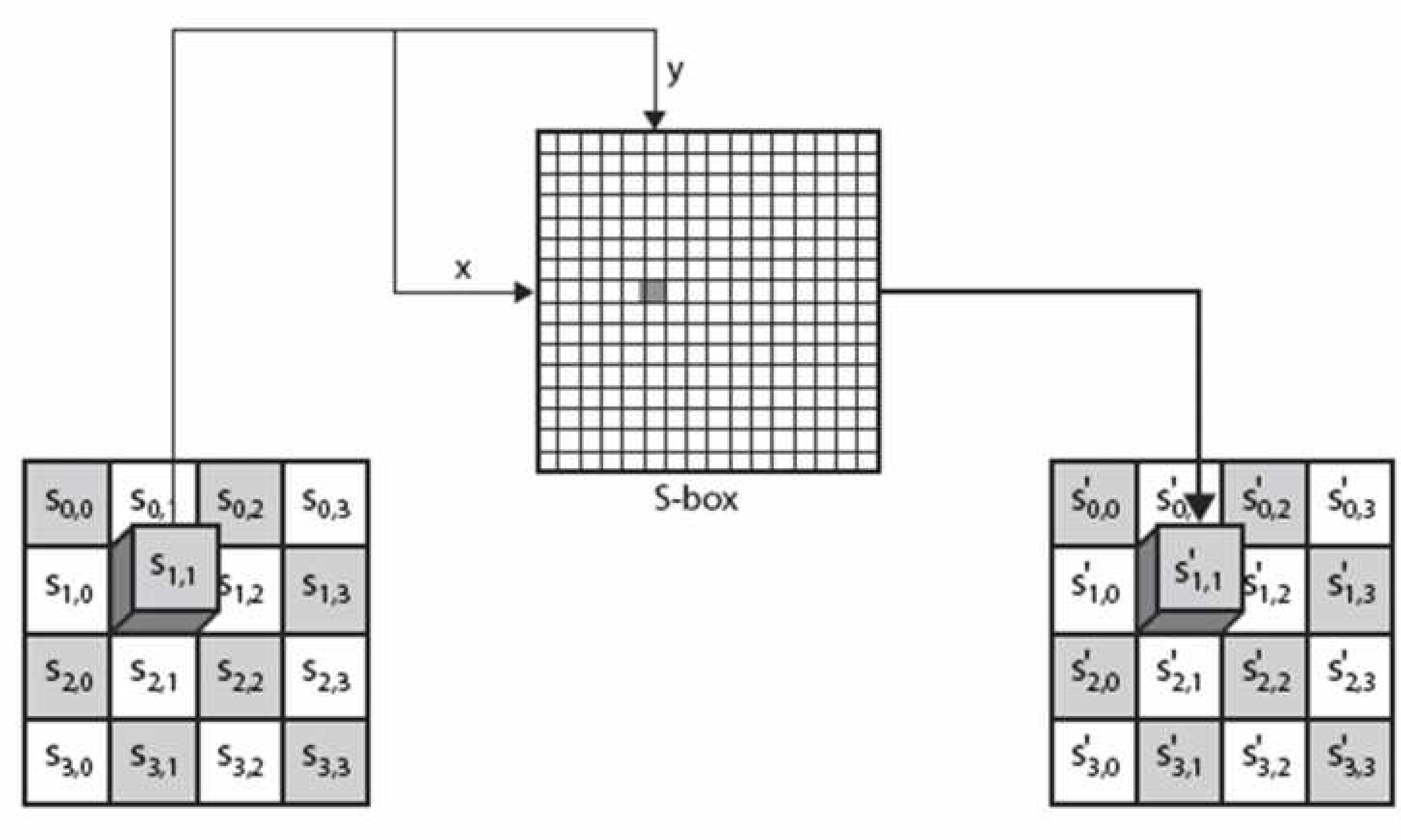
Rijndael


Plaintext
Key
Plaintext
Add round key
w[0, 3]
Add round key
Substitute bytes
Expand key
Inverse sub bytes
Round 10
Shift rows
Inverse shift rows
Round 1
Mix columns
Inverse mix cols
Add round key
w[4, 7]
Add round key
Inverse sub bytes
Round 9
Inverse shift rows
¥
¥
¥
Substitute bytes
¥
¥
¥
Shift rows
Round 9
Mix columns
Inverse mix cols
Add round key
w[36, 39]
Add round key
Substitute bytes
Inverse sub bytes
Round 10
Shift rows
Inverse shift rows
Round 1
Add round key
w[40, 43]
Add round key
Ciphertext
Ciphertext
(a) Encryption
(b) Decryption

Byte Substitution

- a simple substitution of each byte
- uses one table of 16x16 bytes containing a permutation of all 256 8-bit values
- each byte of state is replaced by byte indexed by row (left 4-bits) & column (right 4-bits)
  - eg. byte {95} is replaced by byte in row 9 column 5
  - which has value {2A}
- S-box constructed using defined transformation of values in GF(2^8)
- designed to be resistant to all known attacks

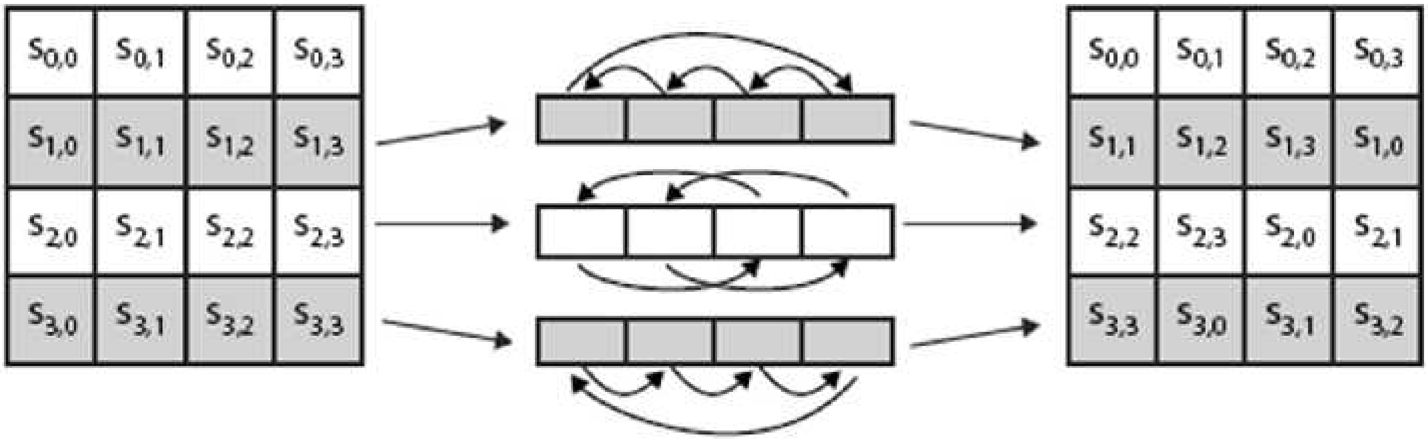
Byte Substitution



Shift Rows

- a circular byte shift in each each
  - 1st row is unchanged
  - 2nd row does 1 byte circular shift to left
  - 3rd row does 2 byte circular shift to left
  - 4th row does 3 byte circular shift to left
- decrypt inverts using shifts to right
- since state is processed by columns, this step permutes bytes between the columns

Shift Rows



$s_{0,0}$	$s_{0,1}$	$s_{0,2}$	$s_{0,3}$
$s_{1,0}$	$s_{1,1}$	$s_{1,2}$	$s_{1,3}$
$s_{2,0}$	$s_{2,1}$	$s_{2,2}$	$s_{2,3}$
$s_{3,0}$	$s_{3,1}$	$s_{3,2}$	$s_{3,3}$

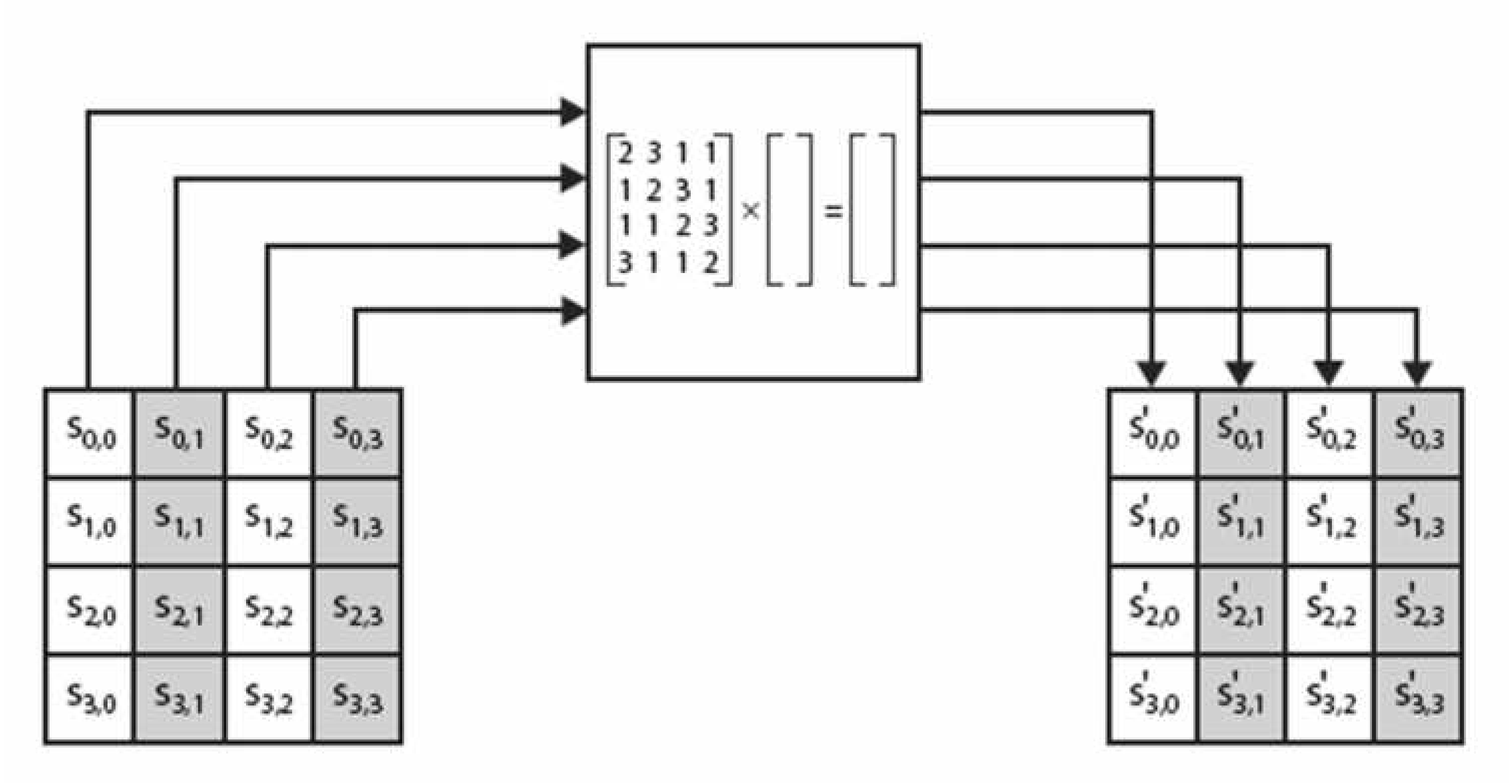
$s_{0,0}$	$s_{0,1}$	$s_{0,2}$	$s_{0,3}$
$s_{1,1}$	$s_{1,2}$	$s_{1,3}$	$s_{1,0}$
$s_{2,2}$	$s_{2,3}$	$s_{2,0}$	$s_{2,1}$
$s_{3,3}$	$s_{3,0}$	$s_{3,1}$	$s_{3,2}$

Mix Columns

- each column is processed separately
- each byte is replaced by a value dependent on all 4 bytes in the column
- effectively a matrix multiplication in GF(2^8) using prime poly m(x) =$x^8+x^4+x^3+x+1$

$$\begin{bmatrix} 02 & 03 & 01 & 01 \\ 01 & 02 & 03 & 01 \\ 01 & 01 & 02 & 03 \\ 03 & 01 & 01 & 02 \end{bmatrix} \begin{bmatrix} s_{0,0} & s_{0,1} & s_{0,2} & s_{0,3} \\ s_{1,0} & s_{1,1} & s_{1,2} & s_{1,3} \\ s_{2,0} & s_{2,1} & s_{2,2} & s_{2,3} \\ s_{3,0} & s_{3,1} & s_{3,2} & s_{3,3} \end{bmatrix} = \begin{bmatrix} s'_{0,0} & s'_{0,1} & s'_{0,2} & s'_{0,3} \\ s'_{1,0} & s'_{1,1} & s'_{1,2} & s'_{1,3} \\ s'_{2,0} & s'_{2,1} & s'_{2,2} & s'_{2,3} \\ s'_{3,0} & s'_{3,1} & s'_{3,2} & s'_{3,3} \end{bmatrix}$$

Mix Columns



$$\begin{bmatrix}2&3&1&1\\1&2&3&1\\1&1&2&3\\3&1&1&2\end{bmatrix} \times \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$s_{0,0}$	$s_{0,1}$	$s_{0,2}$	$s_{0,3}$
$s_{1,0}$	$s_{1,1}$	$s_{1,2}$	$s_{1,3}$
$s_{2,0}$	$s_{2,1}$	$s_{2,2}$	$s_{2,3}$
$s_{3,0}$	$s_{3,1}$	$s_{3,2}$	$s_{3,3}$

$s'_{0,0}$	$s'_{0,1}$	$s'_{0,2}$	$s'_{0,3}$
$s'_{1,0}$	$s'_{1,1}$	$s'_{1,2}$	$s'_{1,3}$
$s'_{2,0}$	$s'_{2,1}$	$s'_{2,2}$	$s'_{2,3}$
$s'_{3,0}$	$s'_{3,1}$	$s'_{3,2}$	$s'_{3,3}$

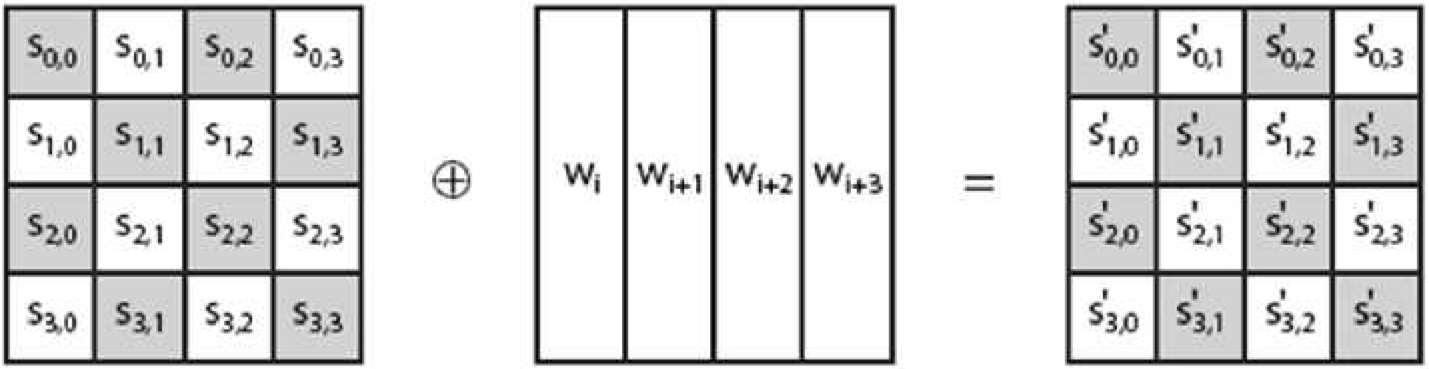
Mix Columns

- can express each col as 4 equations
  - to derive each new byte in col
- decryption requires use of inverse matrix
  - with larger coefficients, hence a little harder
- have an alternate characterisation
  - each column a 4-term polynomial
  - with coefficients in GF(2^8)
  - and polynomials multiplied modulo (x^4+1)

Add Round Key

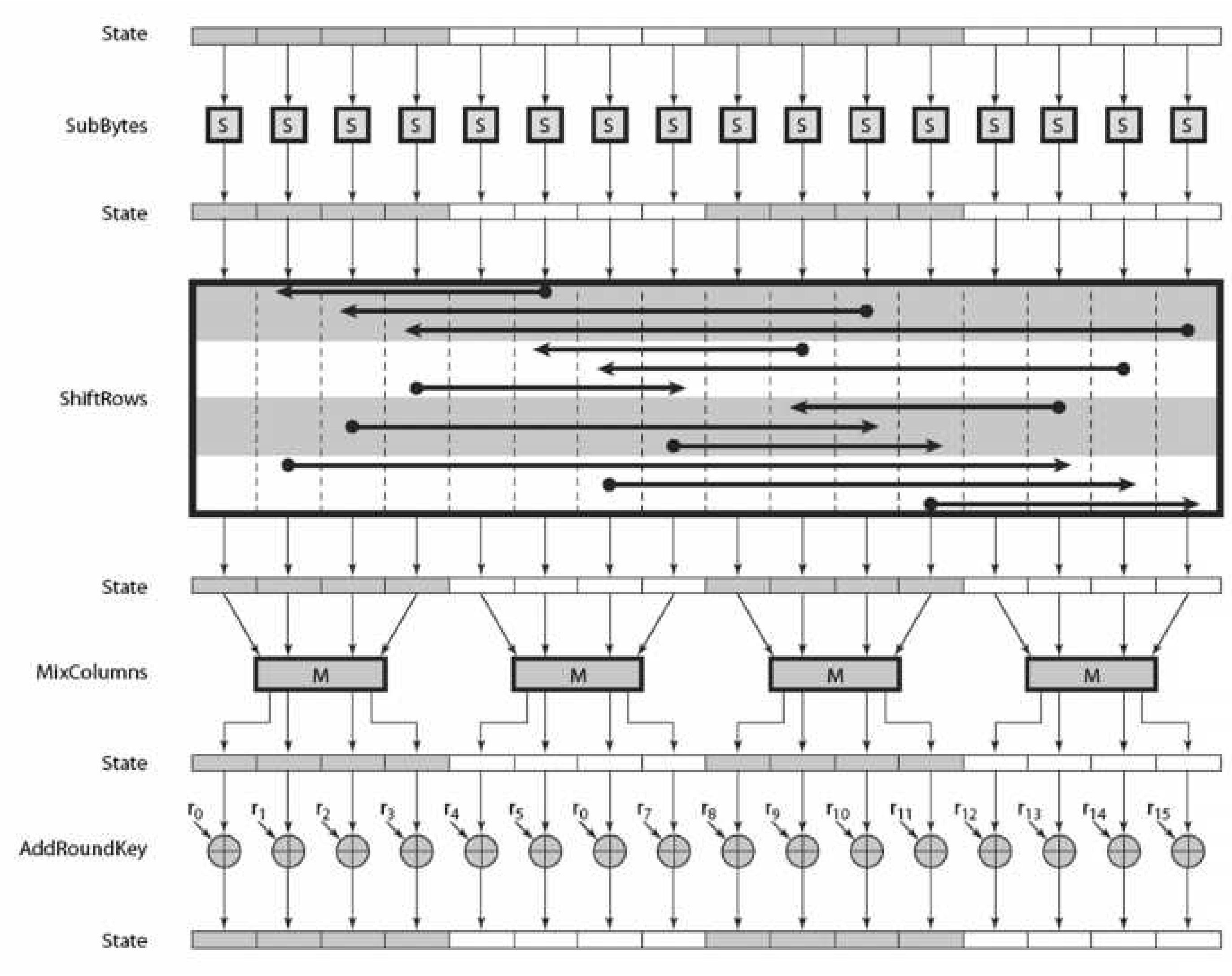
- XOR state with 128-bits of the round key
- again processed by column (though effectively a series of byte operations)
- inverse for decryption identical
  - since XOR own inverse, with reversed keys
- designed to be as simple as possible
  - a form of Vernam cipher on expanded key
  - requires other stages for complexity / security

Add Round Key



$$\begin{array}{|c|c|c|c|}\hline s_{0,0} & s_{0,1} & s_{0,2} & s_{0,3} \\ \hline s_{1,0} & s_{1,1} & s_{1,2} & s_{1,3} \\ \hline s_{2,0} & s_{2,1} & s_{2,2} & s_{2,3} \\ \hline s_{3,0} & s_{3,1} & s_{3,2} & s_{3,3} \\ \hline \end{array} \oplus \begin{array}{|c|c|c|c|}\hline w_i & w_{i+1} & w_{i+2} & w_{i+3} \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline s'_{0,0} & s'_{0,1} & s'_{0,2} & s'_{0,3} \\ \hline s'_{1,0} & s'_{1,1} & s'_{1,2} & s'_{1,3} \\ \hline s'_{2,0} & s'_{2,1} & s'_{2,2} & s'_{2,3} \\ \hline s'_{3,0} & s'_{3,1} & s'_{3,2} & s'_{3,3} \\ \hline \end{array}$$

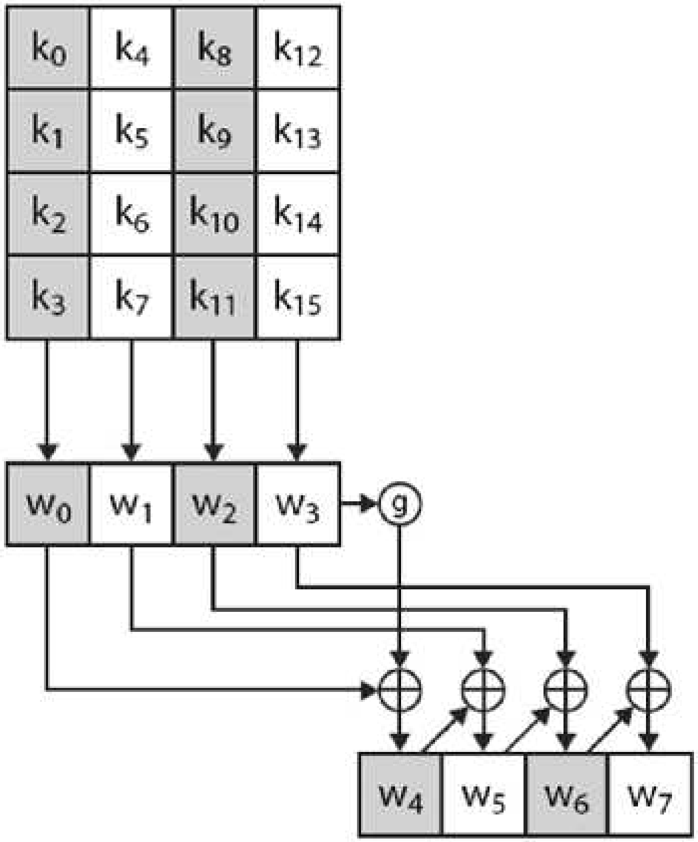
AES Round


State
SubBytes
S S S S S S S S S S S S S S S S
State
ShiftRows
State
MixColumns
M M M M
State
AddRoundKey
r_0 r_1 r_2 r_3 r_4 r_5 r_0 r_7 r_8 r_9 r_{10} r_{11} r_{12} r_{13} r_{14} r_{15}
State

AES Key Expansion

- takes 128-bit (16-byte) key and expands into array of 44/52/60 32-bit words
- start by copying key into first 4 words
- then loop creating words that depend on values in previous & 4 places back
  - in 3 of 4 cases just XOR these together
  - 1st word in 4 has rotate + S-box + XOR round constant on previous, before XOR 4th back

AES Key Expansion


k_0 k_4 k_8 k_{12}
k_1 k_5 k_9 k_{13}
k_2 k_6 k_{10} k_{14}
k_3 k_7 k_{11} k_{15}
w_0 w_1 w_2 w_3
g
w_4 w_5 w_6 w_7

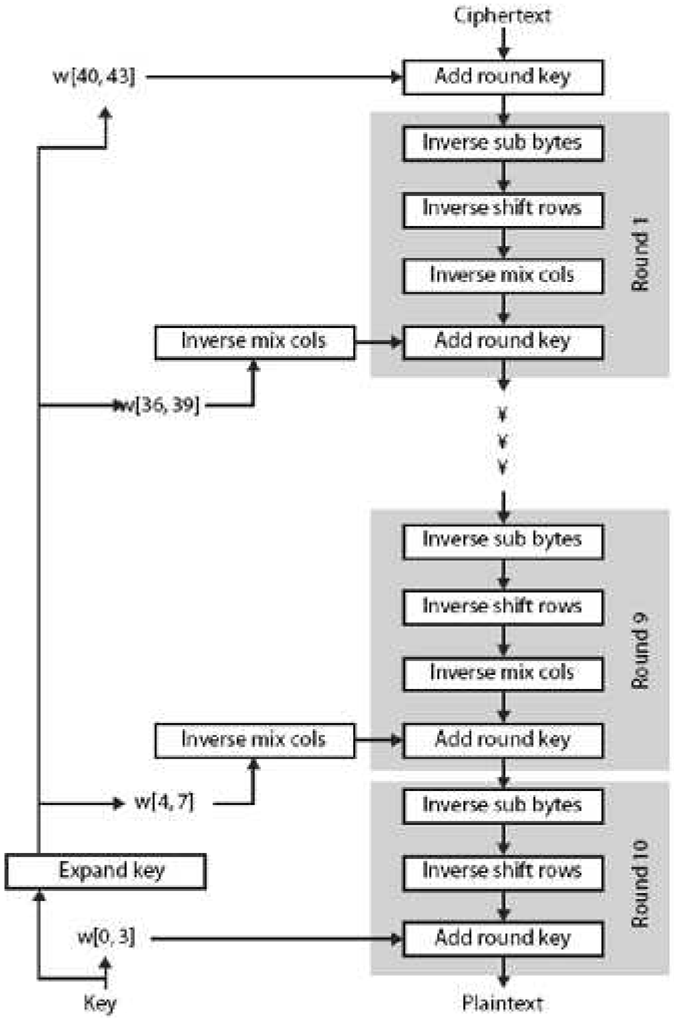
Key Expansion Rationale

- designed to resist known attacks
- design criteria included
  - knowing part key insufficient to find many more
  - invertible transformation
  - fast on wide range of CPU’s
  - use round constants to break symmetry
  - diffuse key bits into round keys
  - enough non-linearity to hinder analysis
  - simplicity of description

AES Decryption

- AES decryption is not identical to encryption since steps done in reverse
- but can define an equivalent inverse cipher with steps as for encryption
  - but using inverses of each step
  - with a different key schedule
- works since result is unchanged when
  - swap byte substitution & shift rows
  - swap mix columns & add (tweaked) round key

AES Decryption



Implementation Aspects

- can efficiently implement on 8-bit CPU
  - byte substitution works on bytes using a table of 256 entries
  - shift rows is simple byte shift
  - add round key works on byte XOR’s
  - mix columns requires matrix multiply in GF(2^8) which works on byte values, can be simplified to use table lookups & byte XOR’s

Implementation Aspects

- can efficiently implement on 32-bit CPU
  - redefine steps to use 32-bit words
  - can precompute 4 tables of 256-words
  - then each column in each round can be computed using 4 table lookups + 4 XORs
  - at a cost of 4Kb to store tables
- designers believe this very efficient implementation was a key factor in its selection as the AES cipher

Multiple Encryption & DES

- clear a replacement for DES was needed
  - theoretical attacks that can break it
  - demonstrated exhaustive key search attacks
- AES is a new cipher alternative
- prior to this alternative was to use multiple encryption with DES implementations
- Triple-DES is the chosen form

Double-DES?

- could use 2 DES encrypts on each block
  - $C = E_{K2}(E_{K1}(P))$
- issue of reduction to single stage
- and have “meet-in-the-middle” attack
  - works whenever use a cipher twice
  - since $X = E_{K1}(P) = D_{K2}(C)$
  - attack by encrypting P with all keys and store
  - then decrypt C with keys and match X value
  - can show takes $O(2^{56})$ steps

Triple-DES with Two-Keys

- hence must use 3 encryptions
  - would seem to need 3 distinct keys
- but can use 2 keys with E-D-E sequence
  - $C = E_{K1}(D_{K2}(E_{K1}(P)))$
  - nb encrypt & decrypt equivalent in security
  - if K1=K2 then can work with single DES
- standardized in ANSI X9.17 & ISO8732
- no current known practical attacks

Triple-DES with Three-Keys

- although are no practical attacks on two-key Triple-DES have some indications
- can use Triple-DES with Three-Keys to avoid even these
  - $C = E_{K3}(D_{K2}(E_{K1}(P)))$
- has been adopted by some Internet applications, eg PGP, S/MIME

Modes of Operation

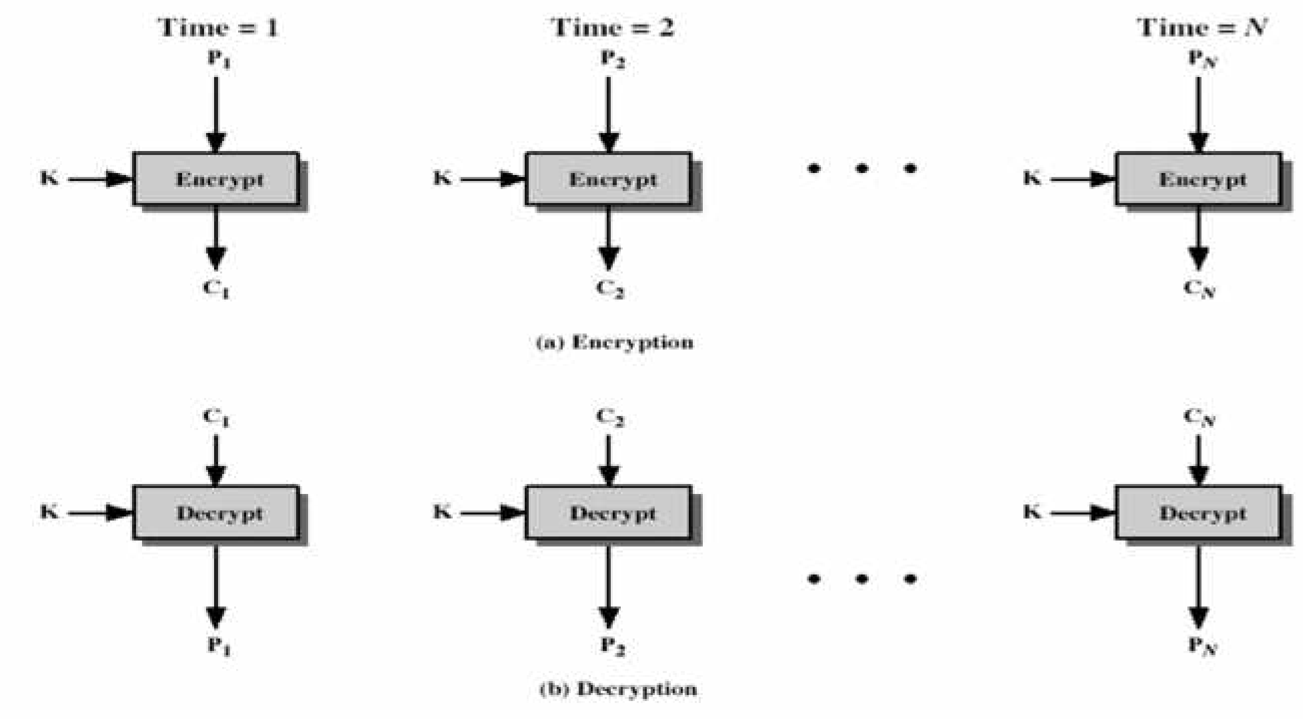
- block ciphers encrypt fixed size blocks
  - eg. DES encrypts 64-bit blocks with 56-bit key
- need some way to en/decrypt arbitrary amounts of data in practise
- **ANSI X3.106-1983 Modes of Use** (now FIPS 81) defines 4 possible modes
- subsequently 5 defined for AES & DES
- have **block** and **stream** modes

Electronic Codebook Book (ECB)

- message is broken into independent blocks which are encrypted
- each block is a value which is substituted, like a codebook, hence name
- each block is encoded independently of the other blocks

  $C_i = DES_{K1}(P_i)$
- uses: secure transmission of single values

Electronic Codebook Book (ECB)



(a) Encryption

(b) Decryption

Advantages and Limitations of ECB

- message repetitions may show in ciphertext
  - if aligned with message block
  - particularly with data such graphics
  - or with messages that change very little, which become a code-book analysis problem
- weakness is due to the encrypted message blocks being independent
- main use is sending a few blocks of data

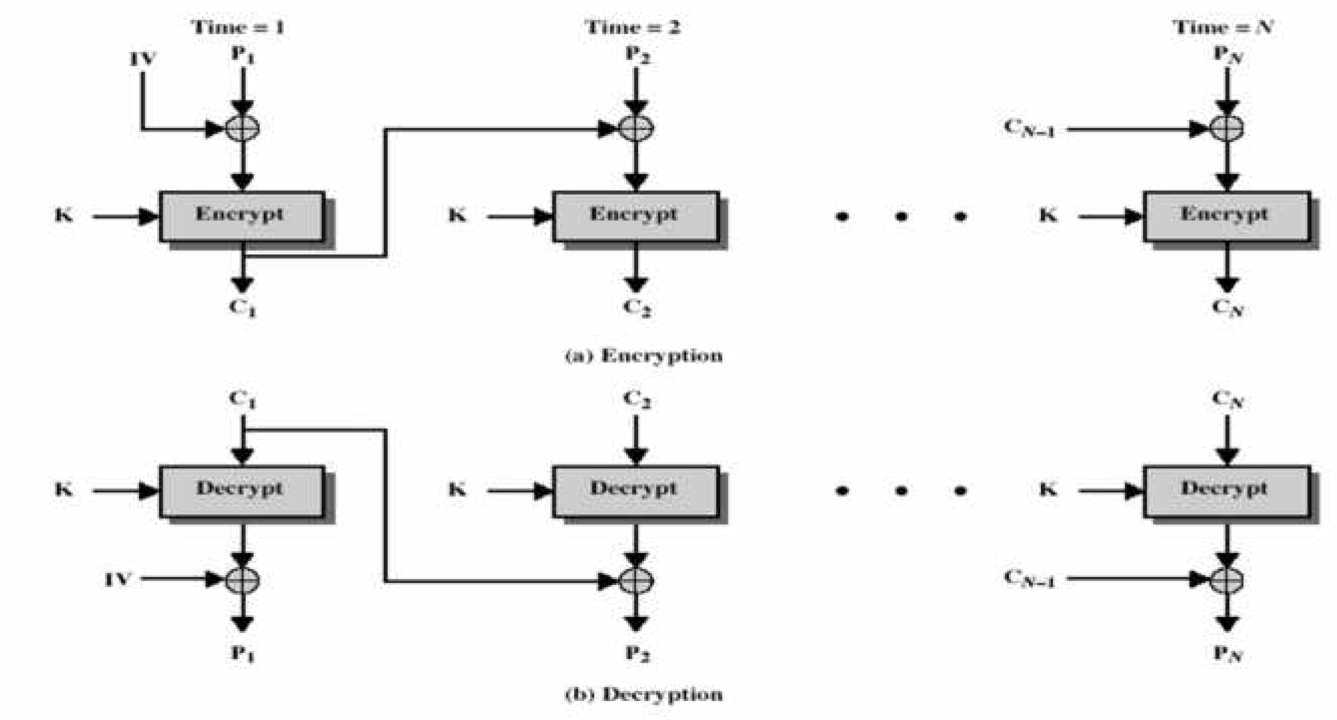
Cipher Block Chaining (CBC)

- message is broken into blocks
- linked together in encryption operation
- each previous cipher blocks is chained with current plaintext block, hence name
- use Initial Vector (IV) to start process

  $C_i = DES_{K1}(P_i \text{ XOR } C_{i-1})$

  $C_{-1} = IV$
- uses: bulk data encryption, authentication

Cipher Block Chaining (CBC)



Message Padding

- at end of message must handle a possible last short block
  - which is not as large as blocksize of cipher
  - pad either with known non-data value (eg nulls)
  - or pad last block along with count of pad size
    - eg. [b1 b2 b3 0 0 0 0 5]
    - means have 3 data bytes, then 5 bytes pad+count
  - this may require an extra entire block over those in message
- there are other, more esoteric modes, which avoid the need for an extra block

Advantages and Limitations of CBC

- a ciphertext block depends on **all** blocks before it
- any change to a block affects all following ciphertext blocks
- need **Initialization Vector** (IV)
  - which must be known to sender & receiver
  - if sent in clear, attacker can change bits of first block, and change IV to compensate
  - hence IV must either be a fixed value (as in EFTPOS)
  - or must be sent encrypted in ECB mode before rest of message

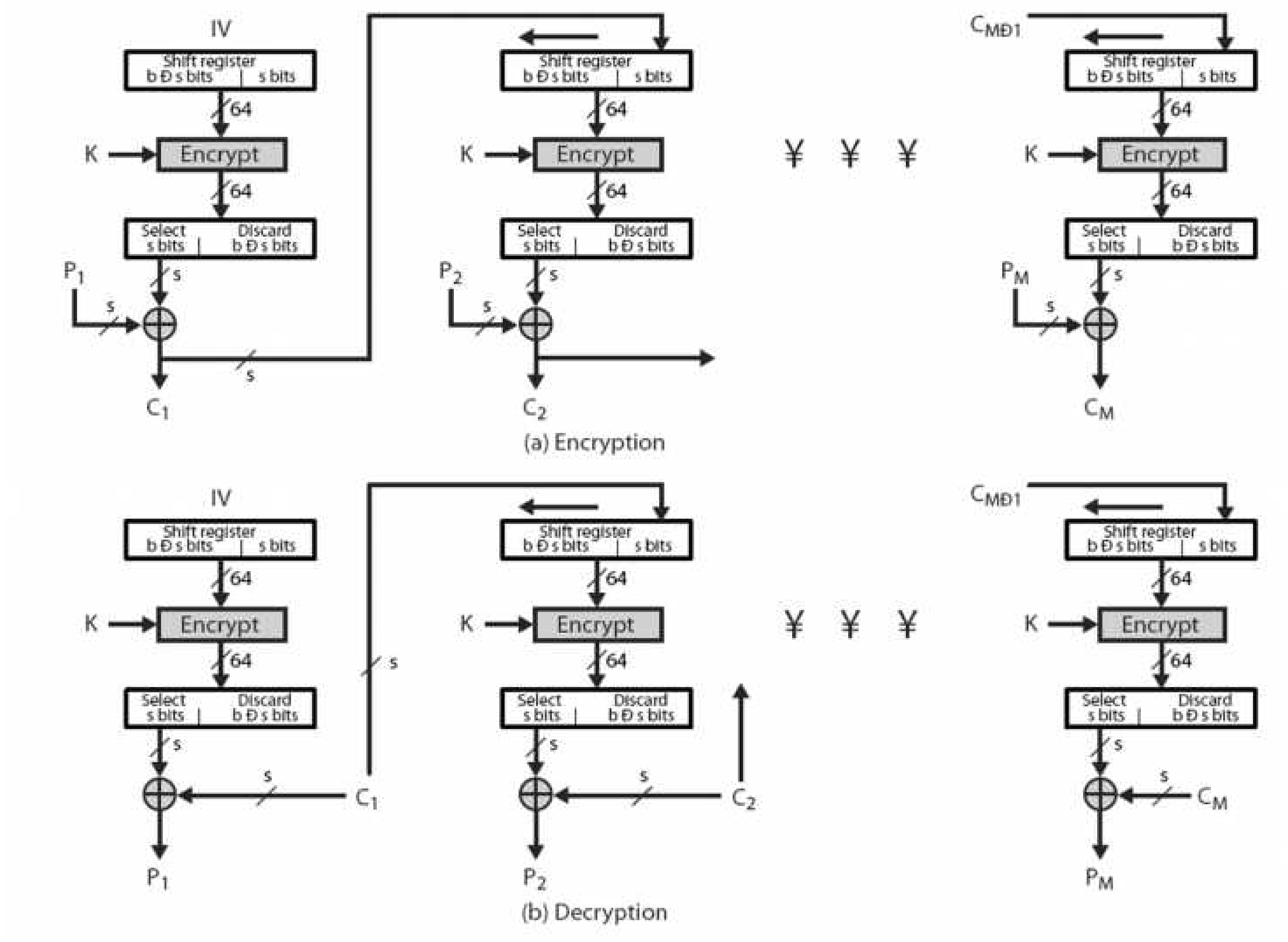
Cipher FeedBack (CFB)

- message is treated as a stream of bits
- added to the output of the block cipher
- result is feed back for next stage (hence name)
- standard allows any number of bit (1,8, 64 or 128 etc) to be feed back
  - denoted CFB-1, CFB-8, CFB-64, CFB-128 etc
- most efficient to use all bits in block (64 or 128)

  $C_i = P_i \text{ XOR } DES_{K1}(C_{i-1})$

  $C_{-1} = IV$
- uses: stream data encryption, authentication

Cipher FeedBack (CFB)



Advantages and Limitations of CFB

- appropriate when data arrives in bits/bytes
- most common stream mode
- limitation is need to stall while do block encryption after every n-bits
- note that the block cipher is used in **encryption** mode at **both** ends
- errors propogate for several blocks after the error

Output FeedBack (OFB)

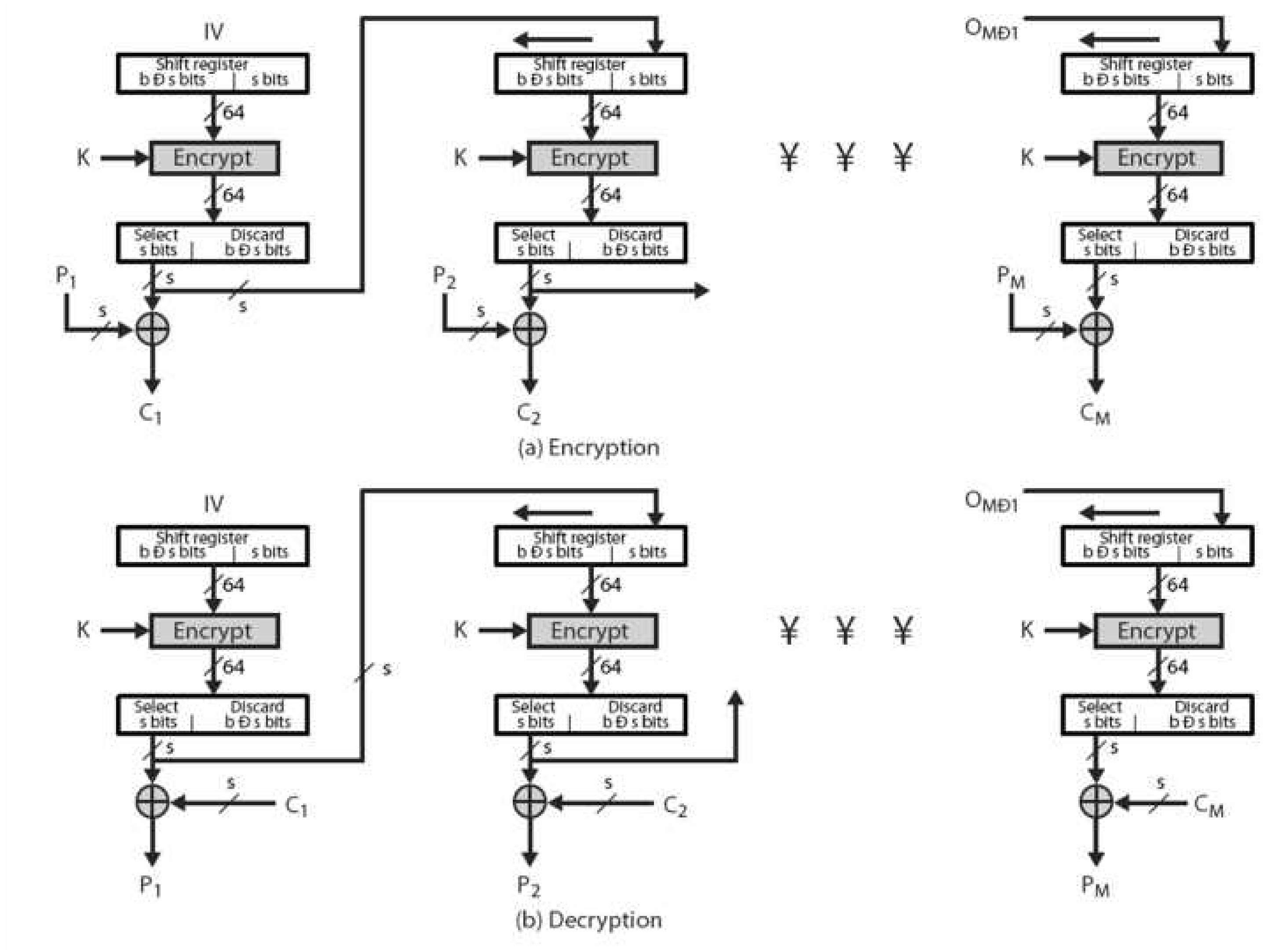
- message is treated as a stream of bits
- output of cipher is added to message
- output is then feed back (hence name)
- feedback is independent of message
- can be computed in advance

  $C_i = P_i \text{ XOR } O_i$

  $O_i = DES_{K1}(O_{i-1})$

  $O_{-1} = IV$
- uses: stream encryption on noisy channels

Output FeedBack (OFB)



Advantages and Limitations of OFB

- bit errors do not propagate
- more vulnerable to message stream modification
- a variation of a Vernam cipher
  - hence must **never** reuse the same sequence (key+IV)
- sender & receiver must remain in sync
- originally specified with m-bit feedback
- subsequent research has shown that only **full block feedback** (ie CFB-64 or CFB-128) should ever be used

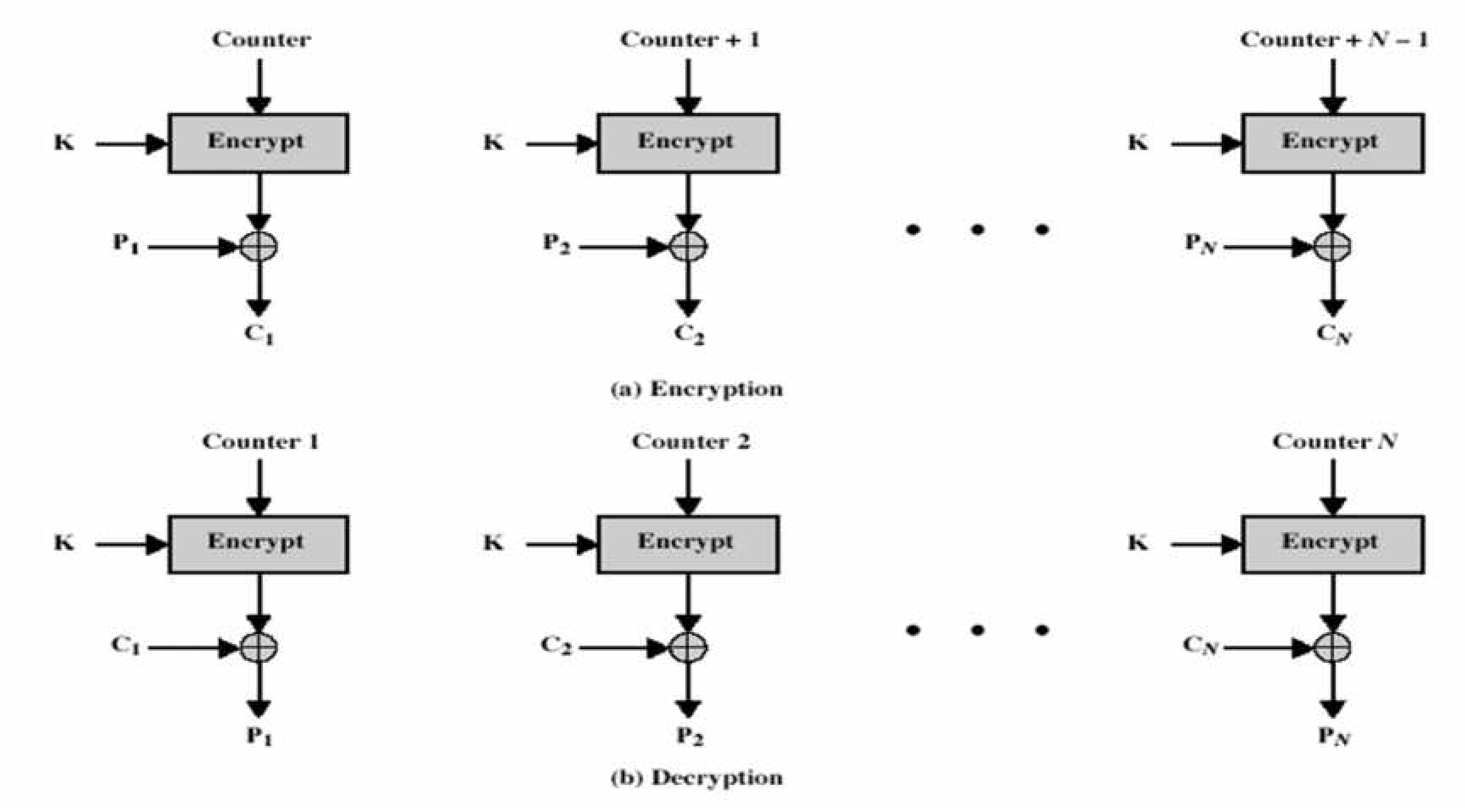
Counter (CTR)

- a “new” mode, though proposed early on
- similar to OFB but encrypts counter value rather than any feedback value
- must have a different key & counter value for every plaintext block (never reused)

  $C_i = P_i \text{ XOR } O_i$

  $O_i = DES_{K1}(i)$
- uses: high-speed network encryptions

Counter (CTR)



(a) Encryption

(b) Decryption

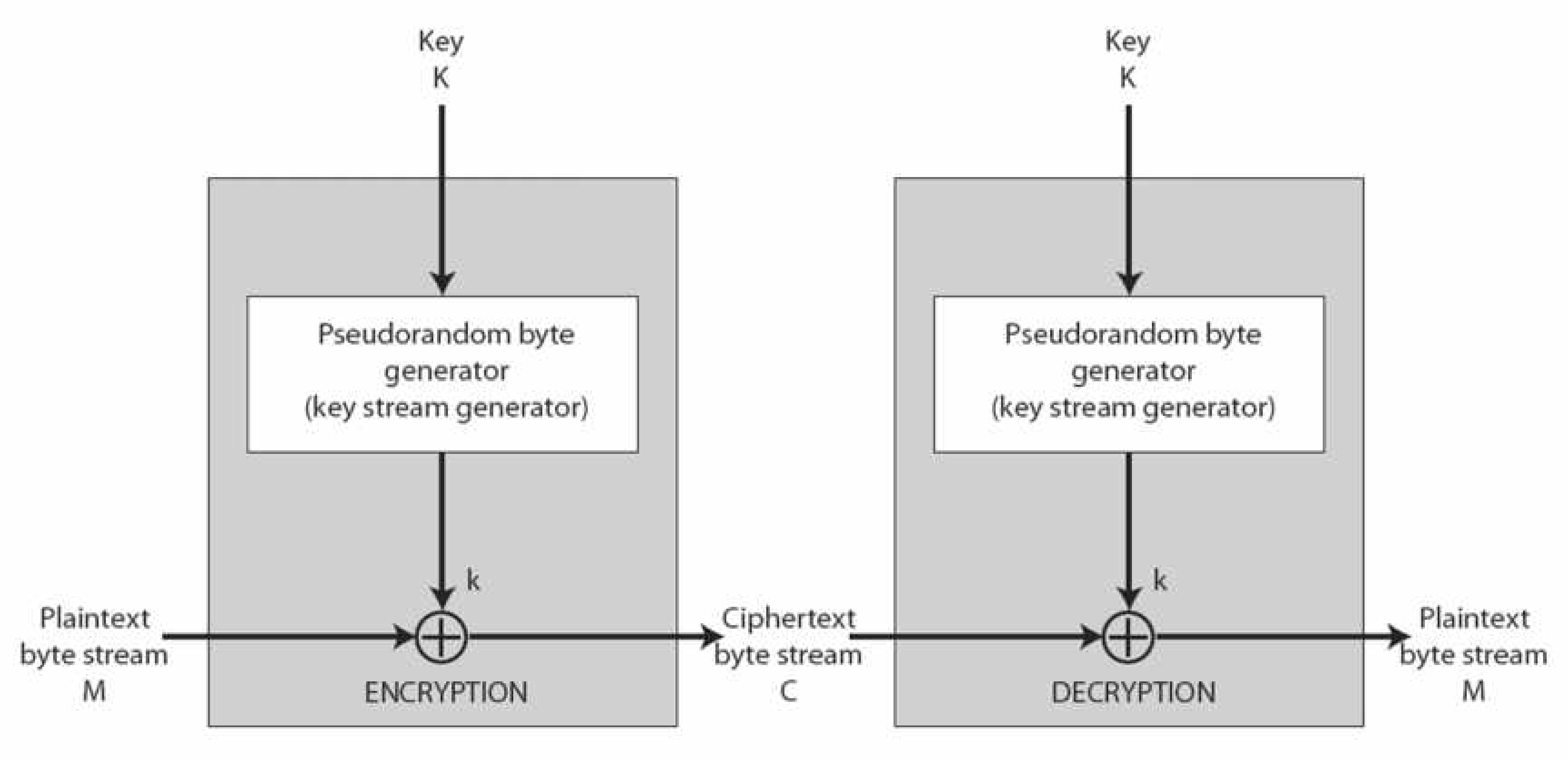
Advantages and Limitations of CTR

- efficiency
  - can do parallel encryptions in h/w or s/w
  - can preprocess in advance of need
  - good for bursty high speed links
- random access to encrypted data blocks
- provable security (good as other modes)
- but must ensure never reuse key/counter values, otherwise could break (cf OFB)

Stream Ciphers

- process message bit by bit (as a stream)
- have a pseudo random **keystream**
- combined (XOR) with plaintext bit by bit
- randomness of **stream key** completely destroys statistically properties in message
  - C_i = M_i XOR $StreamKey_i$
- but must never reuse stream key
  - otherwise can recover messages (cf book cipher)

Stream Cipher Structure


Key
K
Pseudorandom byte generator (key stream generator)
k
Plaintext byte stream M
ENCRYPTION
Ciphertext byte stream C
Key
K
Pseudorandom byte generator (key stream generator)
k
DECRYPTION
Plaintext byte stream M

Stream Cipher Properties

- some design considerations are:
  - long period with no repetitions
  - statistically random
  - depends on large enough key
  - large linear complexity
- properly designed, can be as secure as a block cipher with same size key
- but usually simpler & faster

RC4

- a proprietary cipher owned by RSA DSI
- another Ron Rivest design, simple but effective
- variable key size, byte-oriented stream cipher
- widely used (web SSL/TLS, wireless WEP)
- key forms random permutation of all 8-bit values
- uses that permutation to scramble input info processed a byte at a time

RC4 Key Schedule

- starts with an array S of numbers: 0..255
- use key to well and truly shuffle
- S forms **internal state** of the cipher

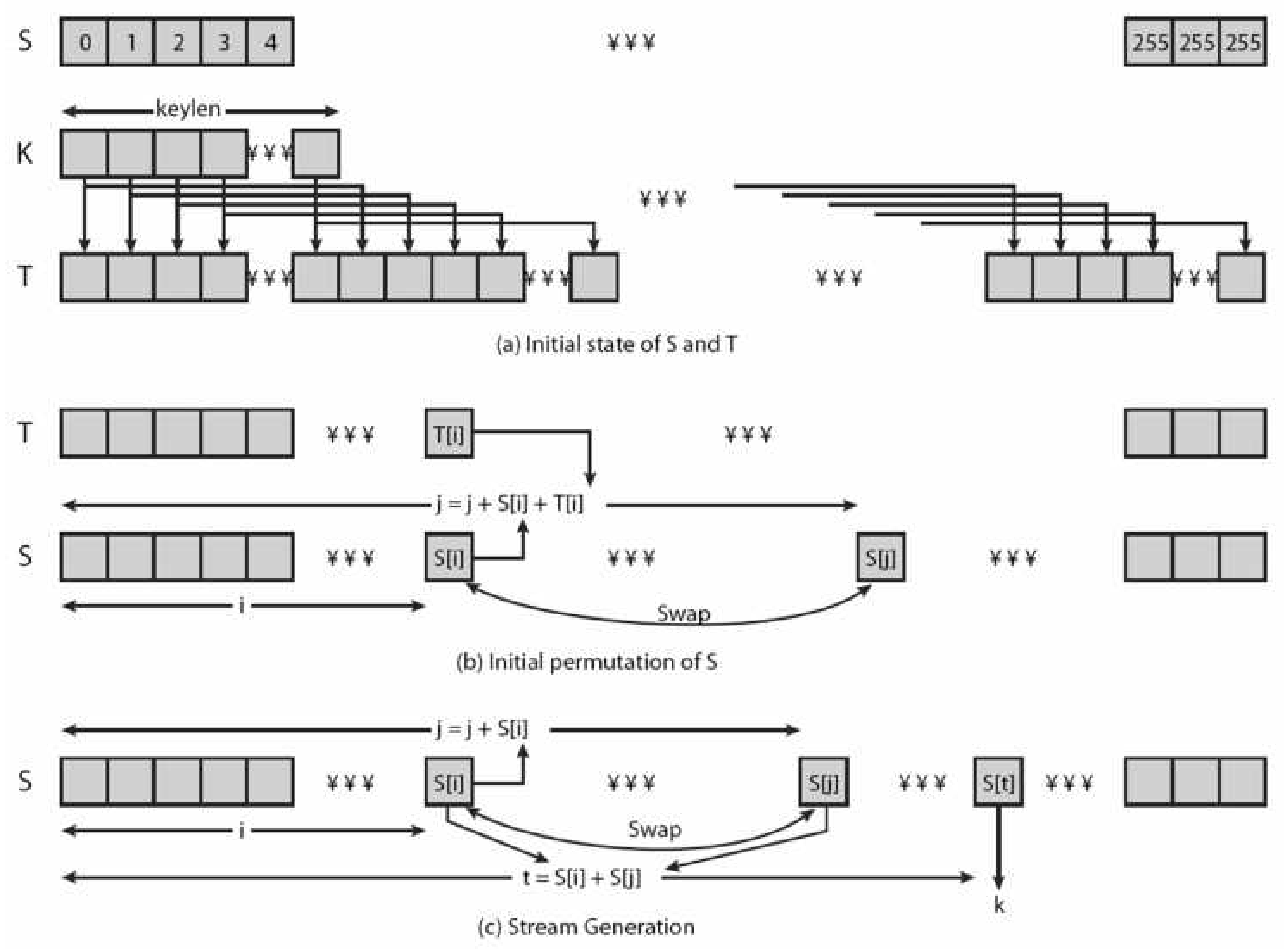
```
for i = 0 to 255 do
    S[i] = i
    T[i] = K[i mod keylen])
j = 0
for i = 0 to 255 do
    j = (j + S[i] + T[i]) (mod 256)
    swap (S[i], S[j])
```

RC4 Encryption

- encryption continues shuffling array values
- sum of shuffled pair selects "stream key" value from permutation
- XOR S[t] with next byte of message to en/decrypt

```
i = j = 0
for each message byte $M_i$
   i = (i + 1) (mod 256)
   j = (j + S[i]) (mod 256)
   swap(S[i], S[j])
   t = (S[i] + S[j]) (mod 256)
   $C_i$ = $M_i$ XOR S[t]
```

RC4 Overview



(a) Initial state of S and T

(b) Initial permutation of S

(c) Stream Generation

RC4 Security

- claimed secure against known attacks
  - have some analyses, none practical
- result is very non-linear
- since RC4 is a stream cipher, must **never reuse a key**
- have a concern with WEP, but due to key handling rather than RC4 itself

The basics – and a few minor details

- Modulo arithmetic
  - Addition and additive inverse are easy
  - Multiplicative inverse doesn’t always exist
- Properties of primes
  - A prime is divisible only by itself and one
  - Determining primality is not all that easy
- Multiword arithmetic
  - Additional method – Chinese remainder theorem
- Finding inverses in finite fields
  - Modified Euclid’s algorithm applies here also

Useful results of number theory

- Private key crypto
  - RSA algorithm
  - Elliptic curve cryptography
- Diffie-Hellman algorithm
  - Generates a shared secret key
- Chinese remainder theorem
  - Sometimes results in easier multiword arithmetic algorithms
- Generation and testing of large primes
  - Useful in all the above

The prime factorization theorem

- A prime is a number divisible only by itself and one
- Any number can be factored uniquely into a product of primes to some power
  - Example $1100 = 2^2 5^2 11^1$
- Relatively prime means (a,b)=1
  - (a,b) means gcd(a,b)
  - (a,b) is found using Euclid’s algorithm

Useful theorems involving a^x mod n

- Fermat's
  - a^{p-1} = 1 mod p, p doesn't divide a
- Euler's phi function
  - ♐(n) = number of numbers <n and relatively prime to n
  - Easily found if factorization is known
- Euler's theorem
  - $a^{♐(n)}$ = 1 mod n – reduces to Fermat's for n prime
- Miller-Rabin test
  - Based on inverse of Fermat's theorem
    n is not prime if an-1 ĸ1 mod n
- Fast exponentiation
  - Convert x to binary – for example x^8 is x squared three times

Prime Numbers

- prime numbers only have divisors of 1 and self
  - they cannot be written as a product of other numbers
  - note: 1 is prime, but is generally not of interest
- eg. 2,3,5,7 are prime, 4,6,8,9,10 are not
- prime numbers are central to number theory
- list of prime number less than 200 is:

```
2 3 5 7 11 13 17 19 23 29 31 37 41 43 47 53 59
61 67 71 73 79 83 89 97 101 103 107 109 113 127
131 137 139 149 151 157 163 167 173 179 181 191
193 197 199
```

Prime Factorisation

- to **factor** a number n is to write it as a product of other numbers: n=a x b x c
- note that factoring a number is relatively hard compared to multiplying the factors together to generate the number
- the **prime factorisation** of a number n is when its written as a product of primes
  - eg. 91=7x13 ; $3600=2^4$x3^2x5^2

$$a = \prod_{p \in \mathrm{P}} p^{a_p}$$

Relatively Prime Numbers & GCD

- two numbers `a, b` are **relatively prime** if have **no common divisors** apart from 1
  - eg. 8 & 15 are relatively prime since factors of 8 are 1,2,4,8 and of 15 are 1,3,5,15 and 1 is the only common factor
- conversely can determine the greatest common divisor by comparing their prime factorizations and using least powers
  - eg. $300=2^1 x 3^1 x 5^2$ $18=2^1 x 3^2$ hence $GCD(18,300)=2^1 x 3^1 x 5^0=6$

Prime Distribution

- prime number theorem states that primes occur roughly every (`ln n`) integers
- but can immediately ignore evens
- so in practice need only test `0.5 ln(n)` numbers of size `n` to locate a prime
  - note this is only the “average”
  - sometimes primes are close together
  - other times are quite far apart

Primitive Roots

- from Euler's theorem have $a^{ø(n)} \bmod n=1$
- consider $a^m=1 \pmod n$, $GCD(a,n)=1$
  - must exist for $m = ø(n)$ but may be smaller
  - once powers reach m, cycle will repeat
- if smallest is $m = ø(n)$ then a is called a **primitive root**
- if p is prime, then successive powers of a "generate" the group $\bmod p$
- these are useful but relatively hard to find

Private-Key Cryptography

- traditional **private/secret/single key** cryptography uses **one** key
- shared by both sender and receiver
- if this key is disclosed communications are compromised
- also is **symmetric**, parties are equal
- hence does not protect sender from receiver forging a message & claiming is sent by sender

Public-Key Cryptography

- probably most significant advance in the 3000 year history of cryptography
- uses **two** keys – a public & a private key
- **asymmetric** since parties are **not** equal
- uses clever application of number theoretic concepts to function
- complements **rather than** replaces private key crypto

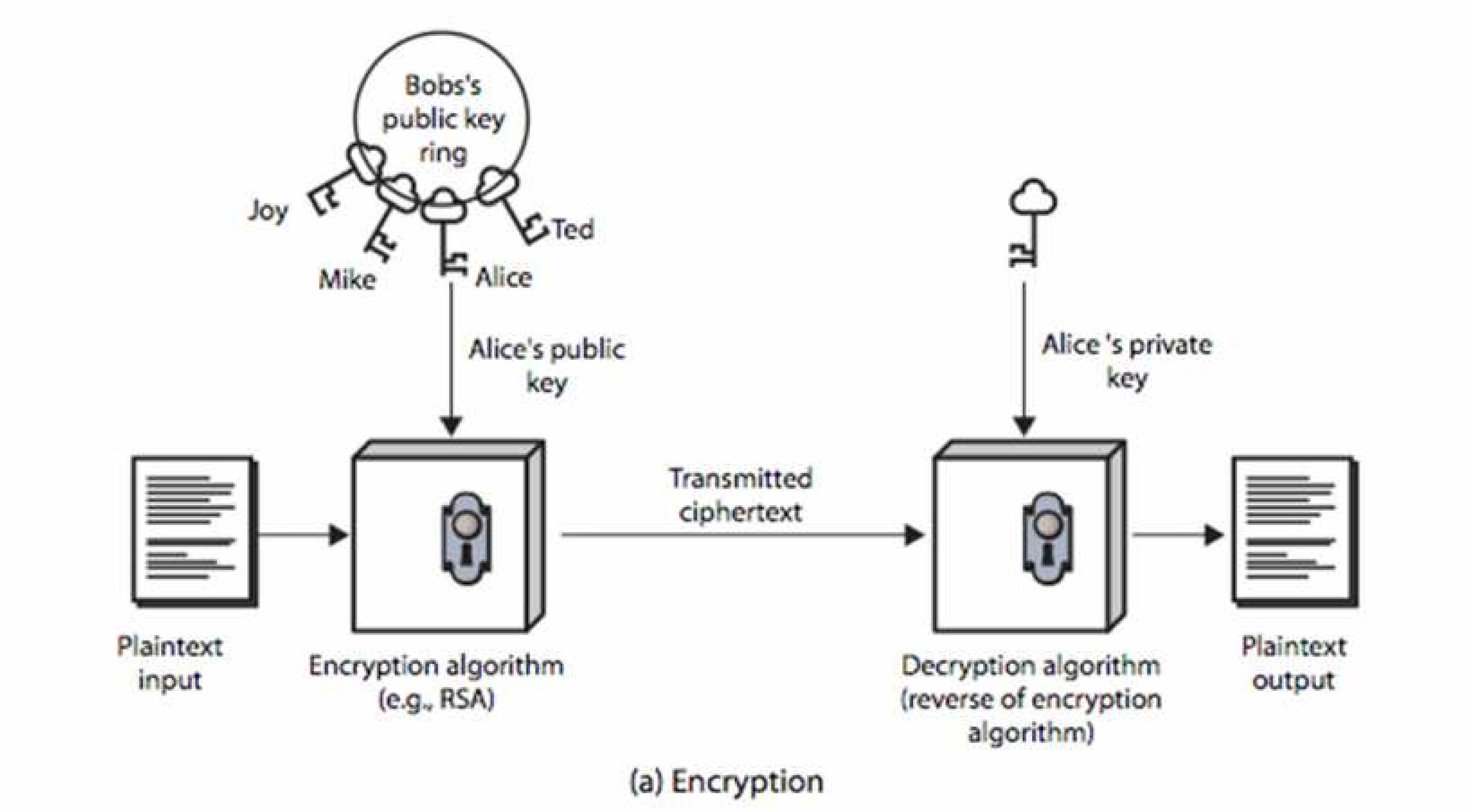
Why Public-Key Cryptography?

- developed to address two key issues:
  - **key distribution** – how to have secure communications in general without having to trust a KDC with your key
  - **digital signatures** – how to verify a message comes intact from the claimed sender
- public invention due to Whitfield Diffie & Martin Hellman at Stanford Uni in 1976
  - known earlier in classified community

Public-Key Cryptography

- **public-key/two-key/asymmetric** cryptography involves the use of **two** keys:
  - a **public-key**, which may be known by anybody, and can be used to **encrypt messages**, and **verify signatures**
  - a **private-key**, known only to the recipient, used to **decrypt messages**, and **sign** (create) **signatures**
- is **asymmetric** because
  - those who encrypt messages or verify signatures **cannot** decrypt messages or create signatures

Public-Key Cryptography

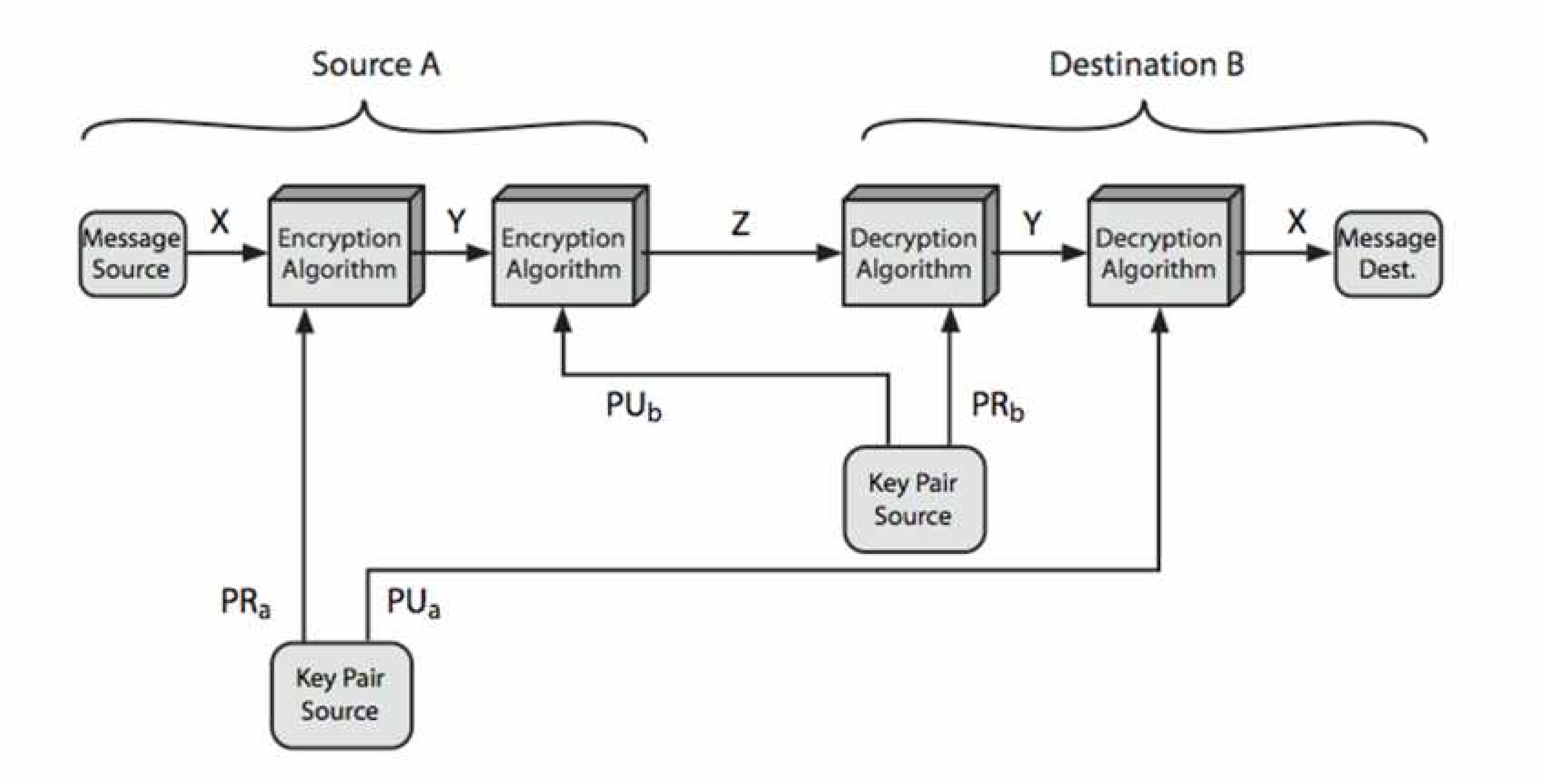


(a) Encryption

Public-Key Characteristics

- Public-Key algorithms rely on two keys where:
  - it is computationally infeasible to find decryption key knowing only algorithm & encryption key
  - it is computationally easy to en/decrypt messages when the relevant (en/decrypt) key is known
  - either of the two related keys can be used for encryption, with the other used for decryption (for some algorithms)

Public-Key Cryptosystems



Public-Key Applications

- can classify uses into 3 categories:
  - **encryption/decryption** (provide secrecy)
  - **digital signatures** (provide authentication)
  - **key exchange** (of session keys)
- some algorithms are suitable for all uses, others are specific to one

Security of Public Key Schemes

- like private key schemes brute force **exhaustive search** attack is always theoretically possible
- but keys used are too large (>512bits)
- security relies on a **large enough** difference in difficulty between **easy** (en/decrypt) and **hard** (cryptanalyse) problems
- more generally the **hard** problem is known, but is made hard enough to be impractical to break
- requires the use of **very large numbers**
- hence is **slow** compared to private key schemes

RSA

- by Rivest, Shamir & Adleman of MIT in 1977
- best known & widely used public-key scheme
- based on exponentiation in a finite (Galois) field over integers modulo a prime
  - nb. exponentiation takes $O((\log n)^3)$ operations (easy)
- uses large integers (eg. 1024 bits)
- security due to cost of factoring large numbers
  - nb. factorization takes $O(e^{\log n \log \log n})$ operations (hard)

RSA Key Setup

- each user generates a public/private key pair by:
- selecting two large primes at random - p, q
- computing their system modulus $n=p.q$
  - note $ø(n)=(p-1)(q-1)$
- selecting at random the encryption key e
  - where $1<e<ø(n), \gcd(e,ø(n))=1$
- solve following equation to find decryption key d
  - $e.d=1 \bmod ø(n)$ and $0 \le d \le n$
- publish their public encryption key: PU={e,n}
- keep secret private decryption key: PR={d,n}

RSA Use

- to encrypt a message M the sender:
  - obtains **public key** of recipient $PU=\{e,n\}$
  - computes: $C = M^e \bmod n$, where $0 \le M < n$
- to decrypt the ciphertext C the owner:
  - uses their private key $PR=\{d,n\}$
  - computes: $M = C^d \bmod n$
- note that the message M must be smaller than the modulus n (block if needed)

Why RSA Works

- because of Euler's Theorem:
  - $a^{ø(n)} \bmod n = 1$ where $\gcd(a,n)=1$
- in RSA have:
  - $n=p.q$
  - $ø(n)=(p-1)(q-1)$
  - carefully chose e & d to be inverses mod ø(n)
  - hence $e.d=1+k.ø(n)$ for some k
- hence :

$$C^{d} = M^{e.d} = M^{1+k.ø(n)} = M^{1}.(M^{ø(n)})^{k}$$

$$= M^{1}.(1)^{k} = M^{1} = M \bmod n$$

RSA Example - Key Setup

1. Select primes: p=17 & q=11
2. Compute $n = pq$ =17 x 11=187
3. Compute ø(n)=(p–1)(q-1)=16 x 10=160
4. Select e: gcd(e,160)=1; choose e=7
5. Determine d: de=1 mod 160 and d < 160
   Value is d=23 since 23x7=161= 10x160+1
6. Publish public key PU={7,187}
7. Keep secret private key PR={23,187}

RSA Example - En/Decryption

- sample RSA encryption/decryption is:
- given message `M = 88` (nb. `88<187`)
- encryption:

  `C = 88`^{7}` mod 187 = 11`
- decryption:

  `M = 11`^{23}` mod 187 = 88`

Exponentiation

- can use the Square and Multiply Algorithm
- a fast, efficient algorithm for exponentiation
- concept is based on repeatedly squaring base
- and multiplying in the ones that are needed to compute the result
- look at binary representation of exponent
- only takes O($\log_2$ n) multiples for number n
  - eg. $7^5 = 7^4.7^1 = 3.7 = 10 \bmod 11$
  - eg. $3^{129} = 3^{128}.3^1 = 5.3 = 4 \bmod 11$

Exponentiation

```
c = 0; f = 1
for i = k downto 0
    do c = 2 x c
       f = (f x f) mod n
    if $b_i$ == 1 then
       c = c + 1
               f = (f x a) mod n
 return f
```

Efficient Encryption

- encryption uses exponentiation to power e
- hence if e small, this will be faster
  - often choose e=65537 (2^{16}-1)
  - also see choices of e=3 or e=17
- but if e too small (eg e=3) can attack
  - using Chinese remainder theorem & 3 messages with different modulii
- if e fixed must ensure `gcd(e,ø(n))=1`
  - ie reject any p or q not relatively prime to e

Efficient Decryption

- decryption uses exponentiation to power d
  - this is likely large, insecure if not
- can use the Chinese Remainder Theorem (CRT) to compute mod p & q separately. then combine to get desired answer
  - approx 4 times faster than doing directly
- only owner of private key who knows values of p & q can use this technique

RSA Key Generation

- users of RSA must:
  - determine two primes at random - p, q
  - select either e or d and compute the other
- primes p, q must not be easily derived from modulus n=p.q
  - means must be sufficiently large
  - typically guess and use probabilistic test
- exponents e, d are inverses, so use Inverse algorithm to compute the other

RSA Security

- possible approaches to attacking RSA are:
  - brute force key search (infeasible given size of numbers)
  - mathematical attacks (based on difficulty of computing ø(n), by factoring modulus n)
  - timing attacks (on running of decryption)
  - chosen ciphertext attacks (given properties of RSA)

Factoring Problem

- mathematical approach takes 3 forms:
  - factor `n=p.q`, hence compute `ø(n)` and then d
  - determine `ø(n)` directly and compute d
  - find d directly
- currently believe all equivalent to factoring
  - have seen slow improvements over the years
    - as of May-05 best is 200 decimal digits (663) bit with LS
  - biggest improvement comes from improved algorithm
    - cf QS to GHFS to LS
  - currently assume 1024-2048 bit RSA is secure
    - ensure p, q of similar size and matching other constraints

Timing Attacks

- developed by Paul Kocher in mid-1990’s
- exploit timing variations in operations
  - eg. multiplying by small vs large number
  - or IF's varying which instructions executed
- infer operand size based on time taken
- RSA exploits time taken in exponentiation
- countermeasures
  - use constant exponentiation time
  - add random delays
  - blind values used in calculations

Chosen Ciphertext Attacks

RSA is vulnerable to a Chosen Ciphertext Attack (CCA)
attackers chooses ciphertexts & gets decrypted plaintext back
choose ciphertext to exploit properties of RSA to provide info to help cryptanalysis
can counter with random pad of plaintext
or use Optimal Asymmetric Encryption Padding (OASP)