14. Simultaneous-equation Models

14.1. SIMULTANEOUS DEPENDENCE OF ECONOMIC VARIABLES

The application of least squares to a single equation assumes, among others, that the explanatory variables are truly exogenous, that there is one-way causation between the dependent variable Y and the explanatory X's. If this is not true, that is if the X's are at the same time determined by Y, Assumption 6 of OLS is violated ($E(X_i u) \neq 0$), and the application of this method yields biased and inconsistent estimates. (For a proof see below, page 334.)

If we have a two-way causation in a function this implies that the function cannot be treated in isolation as a single equation model, but belongs to a wider system of equations which describes the relationships among all the relevant variables. If $Y = f(X)$, but also $X = f(Y)$ we are not allowed to use a single-equation model for the description of the relationship between Y and X. We must use a multi-equation model, which would include separate equations in which Y and X would appear as endogenous variables, although they might appear as explanatory in other equations of the model. A system describing the joint dependence of variables is called a *system of simultaneous equations*.

Given the nature of economic phenomena it is almost certain that any equation will belong to a wider system of simultaneous equations. Some examples will illustrate the meaning of simultaneous relationships and the violation of Assumption 6 of ordinary least squares, which creates what is known as *simultaneous equations bias*.

Example 1. Suppose we want to estimate the demand for food. We know from economic theory that the demand for any particular commodity depends on its price, P, on other prices, P_o, and on income, Y. Thus we may write the demand function for food as

$$Q = b_0 + b_1 P + b_2 P_o + b_3 Y + u$$

where Q = quantity demanded
P = price of food
P_o = price of other commodities
Y = income
u = random variable.

If we apply least squares to this equation we will obtain biased estimates of b_0 and b_1, because P and u are not independent. The demand for any commodity is a function of its price (*inter alia*), but at the same time the price in the market is influenced by the quantity demanded of that commodity. Consequently the above single equation cannot be treated as a complete (single-equation) model. There should be at least one more equation in the model giving the relationship between P and Q, for example

$$P = c_0 + c_1 Q + c_2 W + v$$

where W = index of weather conditions.

Substituting Q in this equation with its equal, we obtain

$$P = c_0 + c_1(b_0 + b_1 P + b_2 P_0 + b_3 Y + u) + c_2 W + v$$

Obviously P is dependent on u and hence we have violation of Assumption 6 of the method of least squares. P is not an exogenous variable in the demand function.

Example 2. Suppose we want to estimate the supply of money. This of course is regulated by the government in an attempt to avoid inflation. Thus we may say that the main determinant of the decision of the government about the supply of money is the real level of income. Hence we may write the supply function of money as

$$M = b_0 + b_1 Y + u$$

where M = money supply
Y = level of real income.

However, the level of real income is in turn influenced by the supply of money as well as by other real forces, like the investment decisions of businessmen, the welfare policies of the government, and so on. Consequently the supply of money cannot be treated as a single-equation model. Y is not truly exogenous. There is a joint dependence between M and Y and hence we must construct a model with simultaneous equations, one of which would be

$$Y = a_0 + a_1 M + a_2 I_t + \ldots + v$$

Substituting M, we obtain

$$Y = a_0 + a_1(b_0 + b_1 Y + u) + a_2 I_t + \ldots + v$$

Obviously $Y = f(u)$ and hence in the function of the supply of money the explanatory variable Y is not independent of the random variable u.

The bias arising from the application of classical least squares to an equation belonging to a system of simultaneous relations is called *simultaneous equations bias*. It originates from the violation of Assumption 6 of OLS, that is it arises from the dependence of the explanatory variables and u, $[E(uX) \neq 0]$

This creates several problems. Firstly, there arises the problem of identification of the parameters of individual relationships. Secondly, there arise problems of estimation. The application of OLS yields biased and inconsistent estimates. One should therefore choose other estimation methods.

14.2. CONSEQUENCES OF SIMULTANEOUS RELATIONS

We said that when there is a joint dependence between Y and X, their relationship cannot be described with a single equation, but with a system of simultaneous equations. In each relation there are explanatory variables which are endogenous to the system, that is, they appear as dependent in other equations of the system. Thus for any particular equation the random variable is not independent of the explanatory variable(s). Assumption 6 of OLS is not fulfilled $(E(uX) \neq 0)$ and as a consequence the estimates are both biased and inconsistent.

Assume we have the simple model

$$Y = b_0 + b_1 X + u \qquad E(u) = 0 \qquad E(v) = 0$$

$$X = a_0 + a_1 Y + a_2 Z + v \qquad E(u^2) = \sigma_u^2 \qquad E(v^2) = \sigma_v^2$$

$$E(u_i u_j) = 0 \qquad E(v_i v_j) = 0$$

$$E(uv) = 0$$

The model is mathematically complete: it contains two equations in two endogenous variables, X and Y. Z is assumed to be exogenously determined (for example by the government). Substituting X in the second equation we obtain

$$X = a_0 + a_1(b_0 + b_1 X + u) + a_2 Z + v$$

or

$$X = \frac{a_0 + b_0 a_1}{1 - b_1 a_1} + \frac{a_2}{1 - b_1 a_1} Z + \left(\frac{a_1 u + v}{1 - b_1 a_1}\right)$$

X and the disturbance term u are related. X is not a truly exogenous variable in the first equation.

It can be proved that the covariance of X and u is not zero.

$$\text{cov}(Xu) \neq 0$$

Proof. By definition the covariance of u and X is

$$\text{cov}(uX) = E\{[u - E(u)][X - E(X)]\}$$

But $E(u) = 0$. Therefore

$$\text{cov}(uX) = E\{u[X - E(X)]\}$$

Given that

$$X = \frac{a_0 + b_0 a_1}{1 - b_1 a_1} + \frac{a_2}{1 - b_1 a_1} Z + \left(\frac{a_1 u + v}{1 - b_1 a_1}\right)$$

and Z is exogenously determined we have

$$E(X) = \frac{a_0 + b_0 a_1}{1 - b_1 a_1} + \frac{a_2}{1 - b_1 a_1} Z$$

Therefore

$$\text{cov}(uX) = E\left[\frac{u}{1 - b_1 a_1}\{a_0 + a_1 b_0 + a_2 Z + a_1 u + v - (a_0 + a_1 b_0 + a_2 Z)\}\right]$$

$$= E\left[\frac{u}{1 - b_1 a_1}(a_1 u + v)\right]$$

$$= \frac{1}{1 - b_1 a_1} E(a_1 u^2 + uv) = \frac{a_1}{1 - b_1 a_1} E(u^2) \neq 0$$

As a consequence, if we apply the method of least squares to the first function the estimates of the coefficients will be biased and inconsistent.

Now we concluded in chapter ... that the first normal equation is obtained by multiplying the structural equation through by x and summing over all sample observations. In deviation form we have

$$y = b_1 x + u$$

$$\Sigma xy = b_1 \Sigma x^2 + \Sigma xu \qquad (14.1)$$

The last term which expresses the covariance of x and the random term u could be ignored by either of the following assumptions:

(a) If the X's are a set of fixed values in (hypothetical) repeated sampling, it is clear that the covariance of the X's and the u's is zero

$$E(\Sigma xu) = 0$$

(b) Even if the X's are stochastic (not fixed), the covariance of u's and the X's will still be zero so long as the X's are independent of the error term u.

With either of these conditions satisfied $E(\Sigma xu) = 0$ and we can obtain an unbiased estimate of b_1 by dividing (14.1) through by Σx^2

$$\frac{\Sigma xy}{\Sigma x^2} = b_1 + \frac{\Sigma xu}{\Sigma x^2} \qquad (14.2)$$

Setting $\hat{b}_1 = \Sigma xy/\Sigma x^2$ and taking expected values we obtain

$$E(\hat{b}_1) = b_1$$

However, if the X's and the u's are not independent, their covariance is different from zero, so that

$$E\{\Sigma(xu)\} \neq 0$$

The bias in $\hat{b}_1$ can be established by taking expected values of (14.2)

$$E\left(\frac{\Sigma xy}{\Sigma x^2}\right) = E(b_1) + E\left(\frac{\Sigma xu}{\Sigma x^2}\right)$$

or

$$E(\hat{b}_1) = b_1 + E\left(\frac{\Sigma xu}{\Sigma x^2}\right)$$

The bias is measured by the second term on the right-hand side and depends on the model being studied and the particular form of the dependence between X and u

$$\text{bias} = \{E(\hat{b}_1) - b_1\} = E\left(\frac{\Sigma xu}{\Sigma x^2}\right) \neq 0 \qquad (14.3)$$

In our example of the consumption function it can be proved that

$$\hat{b}_1 = \frac{b_1 \Sigma z^2 + (1 + b_1)\Sigma zu + \Sigma u^2}{\Sigma z^2 + 2\Sigma zu + \Sigma u^2}$$

(See J. Johnston, *Econometric Methods*, 1972, p. 344.) Letting $n \to \infty$ and noting that investment (I) is exogenous, the middle terms in the numerator and denominator will tend to zero. Hence the limiting value of the estimate b_1 is

$$\text{plim } \hat{b}_1 = \frac{b_1 \sigma_z^2 + \sigma_u^2}{\sigma_z^2 + \sigma_u^2}$$

or

$$\text{plim } \hat{b}_1 = b_1 + \frac{(1 - b_1)\sigma_u^2}{\sigma_z^2 + \sigma_u^2}$$

If $0 < b_1 < 1$ then the second additive term will be positive and the OLS estimate $\hat{b}_1$ will overestimate the true b_1.

If the first equation is a consumption function, b_1 is the MPC which on a priori grounds is positive but less than unity $(0 < b_1 < 1)$, hence the estimate $\hat{b}_1$ will have an upward bias, while $\hat{b}_0$ will have a downward bias. For the particular example of the consumption function the consequences of the violation of the assumption of the independence of the explanatory variable Y and u may be shown diagrammatically as in figure 14.1.

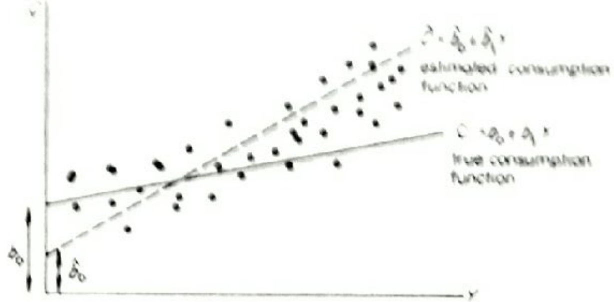


Figure 14.1

An intuitive explanation of the bias may be formulated on the following lines. In applying OLS for explaining the variation in Y we give as little emphasis as possible to the error term u and as much as possible to the explanatory variables. But u is unobservable and will not appear in the estimated equation. If u and the X's are correlated this specification bias (omission of u) will cause an error in the b's, because some of the effect of u will be wrongly absorbed by the coefficients of the X's.

It is clear from (14.3) that the bias is independent of the sample size; as n increases the terms that we sum increase in both the numerator and the denominator. Hence the bias cannot be eliminated by increasing the number of observations in the sample. Thus the first condition for consistency (the condition of asymptotic unbiasedness) does not hold and the estimates obtained from OLS will be inconsistent.

14.3. SOLUTION TO THE SIMULTANEOUS-EQUATION BIAS

Since the application of ordinary least squares to an equation belonging to a system of simultaneous equations yields biased and inconsistent estimates, the obvious solution is to apply other methods of estimation which give better estimates of the parameters. There are several methods for this purpose. The most common are:

(1) The reduced form method, or indirect least squares (ILS).
(2) The method of instrumental variables (IV).
(3) Two-stage least squares (2SLS).
(4) Limited information maximum likelihood (LIML).

(5) The mixed estimation method.
(6) Three-stage least squares (3SLS).
(7) Full information maximum likelihood (FIML).

The first five methods are called *single-equation methods*, because they are applied to one equation of the system at a time. The three-stage least squares and the full information maximum likelihood are called *systems methods*, because they are applied to all the equations of the system simultaneously. The above methods will be developed in chapters 16–19 of this book. The choice among the alternative techniques for the estimation of the parameters of a particular model is a difficult task and will be discussed in some detail in Chapter 21. Before proceeding with the discussion of these techniques, it is necessary to develop further some definitions and to discuss briefly the problem of deciding which variables are endogenous and which are truly exogenous or may be considered as exogenous in any particular econometric model of simultaneous equations.

14.4. SOME DEFINITIONS

1. Structural models

A structural model is a complete system of equations which describe the structure of the relationships of the economic variables. Structural equations express the endogenous variables as functions of other endogenous variables, predetermined variables and disturbances (random variables).

As an illustration we will use the following simple model for a closed economy.

$$C_t = a_0 + a_1 Y_t + u_1$$
$$I_t = b_0 + b_1 Y_t + b_2 Y_{t-1} + u_2$$
$$Y_t = C_t + I_t + G_t$$

The first equation is a consumption function, the second is an investment function, the third is a definitional equation. The system is complete in that it contains three equations in three endogenous variables, C_t, I_t, Y_t. The model contains two predetermined variables, government expenditure, G, and lagged income, Y_{t-1}.

In the remainder of this chapter, for the sake of simplicity, we will ignore the constant intercepts of the structural equations. (If one wants to retain the intercepts in the analysis, one should introduce a dummy variable, X_0, in the set of explanatory variables, which would always assume the value of 1.)

The structural parameters are, in general, propensities, elasticities or other parameters of economic theory. A structural parameter expresses the *direct effect* of each explanatory variable on the dependent variable. Indirect effects can be computed only by the solution of the structural system, but not by the individual structural parameters. Factors not appearing in any function explicitly may have an indirect influence on the dependent variable of that

function. For example a change in consumption will affect investment indirectly, through the increase that the consumption, C, will produce on income, Y, which is a determinant of investment. The effect of C on I cannot be measured directly by any of the structural parameters, but it will be taken into account by the simultaneous solution of the system.

Traditionally the structural parameters are represented by β's when they refer to endogenous variables, and by γ's when they are attached to a predetermined variable. Similarly endogenous variables are denoted by y's while exogenous variables are represented by x's. Using the conventional notation (and ignoring the constant intercepts) the structural system above becomes

$$y_1 = \beta_{13} y_3 + u_1$$
$$y_2 = \beta_{23} y_3 + \gamma_{21} x_1 + u_2$$
$$y_3 = y_1 + y_2 + x_2$$

where

$$y_1 = C \qquad y_2 = I \qquad y_3 = Y$$
$$x_1 = Y_{t-1} \qquad x_2 = G$$

Transferring all the observable variables to the left-hand side we may obtain the complete table of structural parameters as follows

$$y_1 + 0y_2 - \beta_{13} y_3 + 0x_1 + 0x_2 = u_1$$
$$0y_1 + y_2 - \beta_{23} y_3 - \gamma_{21} x_1 + 0x_2 = u_2$$
$$-y_1 - y_2 + y_3 + 0x_1 - x_2 = 0$$

Table of structural coefficients

1	0	$-a_1$	0	0
0	1	$-b_1$	$-b_2$	0
−1	−1	1	0	−1

Table of structural coefficients in standard notation

1	0	$-\beta_{13}$	0	0
0	1	$-\beta_{23}$	γ_{21}	0
−1	−1	1	0	−1

Values of the structural parameters may be obtained by using sample observations on the variables of the model and applying an appropriate econometric method. (See Chapters 16–21.)

2. Reduced form models

The reduced form of a structural model is the model in which the endogenous variables are expressed as a function of the predetermined variables only. The reduced form is obtained in two ways. The first is to express the endogenous variables directly as functions of the predetermined variables

$$y_i = \pi_{i1} x_1 + \pi_{i2} x_2 + \ldots + \pi_{ik} x_k + v_i \quad (i = 1, 2, \ldots, G)$$

and proceed with the estimation of the π's by applying some appropriate technique to this expression (*see below*). In our example of the simple three-equation model the *reduced form* would be

$$C_t = \pi_{11} Y_{t-1} + \pi_{12} G_t + v_1$$
$$I_t = \pi_{21} Y_{t-1} + \pi_{22} G_t + v_2$$
$$Y_t = \pi_{31} Y_{t-1} + \pi_{32} G_t + v_3$$

The second method for obtaining the reduced form of a model is to solve the structural system of endogenous variables in terms of the predetermined variables, the structural parameters and the disturbances. The structural system of our example gives the following reduced form model:

$$C_t = \frac{a_1 b_2}{1-a_1-b_1} Y_{t-1} + \frac{a_1}{1-a_1-b_1} G_t + \frac{u_1 + a_1 u_2 - b_1 u_1}{1-a_1-b_1}$$
$$I_t = \frac{b_2(1-a_1)}{1-a_1-b_1} Y_{t-1} + \frac{b_1}{1-a_1-b_1} G_t + \frac{u_2 + b_1 u_1 - a_1 u_2}{1-a_1-b_1}$$
$$Y_t = \frac{b_2}{1-a_1-b_1} Y_{t-1} + \frac{1}{1-a_1-b_1} G_t + \frac{u_1 + u_2}{1-a_1-b_1}$$

Clearly for the two reduced forms to be consistent the following relationships between the π's and the structural parameters must hold

$$\pi_{11} = \frac{a_1 b_2}{1-a_1-b_1} \qquad \pi_{12} = \frac{a_1}{1-a_1-b_1}$$
$$\pi_{21} = \frac{b_2(1-a_1)}{1-a_1-b_1} \qquad \pi_{22} = \frac{b_1}{1-a_1-b_1}$$
$$\pi_{31} = \frac{b_2}{1-a_1-b_1} \qquad \pi_{32} = \frac{1}{1-a_1-b_1}$$

It should be clear that there is a definite relationship between the reduced-form parameters and the structural parameters: the π's are functions of the structural parameters.

Derivation of the reduced form parameters.
(a) Substitute C_t and I_t in the third structural equation

$$Y_t = (a_1 Y_t + u_1) + (b_1 Y_t + b_2 Y_{t-1} + u_2) + G_t$$

By rearranging we obtain

$$Y_t = \frac{b_2}{1-a_1-b_1} Y_{t-1} + \frac{1}{1-a_1-b_1} G_t + \frac{u_1 + u_2}{1-a_1-b_1}$$

This is the reduced form of the third structural equation.
(b) Substitute Y_t into the consumption function

$$C_t = a_1 \left[\frac{b_2}{1-a_1-b_1} Y_{t-1} + \frac{1}{1-a_1-b_1} G_t + \frac{u_1 + u_2}{1-a_1-b_1}\right] + u_1$$

or

$$C_t = \frac{a_1 b_2}{1-a_1-b_1} Y_{t-1} + \frac{a_1}{1-a_1-b_1} G_t + \frac{u_1 + a_1 u_2 - b_1 u_1}{1-a_1-b_1}$$

This is the reduced form of the consumption function.

(c) Substitute Y_t into the investment function

$$I_t = b_1 \left[\frac{b_2}{1-a_1-b_1} Y_{t-1} + \frac{1}{1-a_1-b_1} G_t + \frac{u_1 + u_2}{1-a_1-b_1}\right] + b_2 Y_{t-1} + u_2$$

or

$$I_t = \frac{b_2(1-a_1)}{1-a_1-b_1} Y_{t-1} + \frac{b_1}{1-a_1-b_1} G_t + \left(\frac{u_2 + b_1 u_1 - a_1 u_2}{1-a_1-b_1}\right)$$

This is the reduced form of the investment function.

The reduced-form parameters measure the *total effect*, direct and indirect, of a change in the predetermined variable on the endogenous variables, after taking account of the interdependences among the jointly dependent endogenous variables, while a structural parameter indicates only the direct effect within a single sector of the economy.[1] For example π_{21} measures the effect of a unit increase in Y_{t-1} on the value of investment. This effect consists of two parts: firstly, there is the direct effect on I through the coefficient b_2 as set out in the structural equation of investment; secondly there is the additional effect due to the fact that an increase in Y_{t-1} affects $I_t \rightarrow$ and I_t influences $Y_t \rightarrow$ which in turn affects I_t; finally Y_t affects $C_t \rightarrow$ which in turn affects Y_t and hence I_t. Thus the total effect (measured by π_{21}) of Y_{t-1} on I_t may be split into the following components

$$\pi_{21} = \frac{b_2(1-a_1)}{1-a_1-b_1} = b_2 \left(1 + \frac{b_1}{1-a_1-b_1}\right)$$

or

$$\pi_{21} = b_2 + \frac{b_2 b_1}{1-a_1-b_1}$$
$$[\text{total effect}] = [\text{direct effect}] + [\text{indirect effect}]$$

The reduced form coefficients are used for forecasting and policy analysis, since it is the total effect of a change in the exogenous variables on the dependent variable(s) that is of interest to the policy maker.

The above two ways of defining the reduced-form model suggest that estimates of the reduced-form coefficients may be obtained in two ways.

Firstly. Direct estimation of the reduced-form coefficients. The reduced-form π's may be estimated by the method of least-squares-no-restrictions

[1] See A. Walters, *An Introduction to Econometrics*, Macmillan, London 1968, p. 181–4.

(LSNR). We express all the endogenous variables as functions of all the predetermined variables of the system and we apply ordinary least squares to these reduced-form functions. This method of obtaining the π's is called least-squares-no-restrictions (LSNR), because it does not take into account any information on the structural parameters, that is, it does not use any restrictions imposed by the form of the structural system. For example the structural equations define that some coefficients are zero if the respective variables are not included in a function; this information is not taken into account by the method of LSNR. This method does not require complete knowledge of the structural system. What is required is knowledge of the predetermined variables appearing in the whole system.

Secondly. Indirect estimation of the reduced-form coefficients. We saw that there is a definite relationship between the reduced-form coefficients and the structural parameters. It is thus possible first to obtain estimates of the structural parameters by any appropriate econometric technique and then substitute these estimates into the *system of parameters' relationships* to obtain (indirectly) values for the π's. This indirect method involves three steps:
(1) Solve the system of endogenous variables so that each equation contains only predetermined explanatory variables. This, as we saw, may be done by continuous substitutions of variables, until we arrive at the reduced-form of all the equations. In this way we obtain *the system of parameters' relations*, that is to say the system which defines the relations between the π's and the β's and γ's. (2) Obtain estimates of the structural parameters by any appropriate econometric method. (3) Substitute the estimates β's and γ's into the system of parameters' relations to find the estimates of the reduced-form coefficients.

This method is more complicated, but it has several advantages over the direct estimation of π's from LSNR. (a) The derivation of reduced-form π's from the structural β's and γ's is more efficient because in this way we take into account all the information (that is all the *a priori* restrictions imposed by the structure on the parameters) incorporated into the structural model.
(b) Structural changes occur continuously over time. If we know these changes in the β's and γ's we may easily recompute the π's. While if the π's are computed with the LSNR method it will not, in general, be possible to take this information into account, because each π is a function of several structural parameters, and if the exact relationship between π's, β's and γ's has not been established, we cannot incorporate into the former the changes that may have occurred to the latter. (c) Extraneous information on some structural parameters may become available from other studies; such information again will be useless if we have not estimated the π's from previous estimates of β's and γ's. (See Goldberger, *Econometric Theory*, pp. 379–80.)

3. Recursive models

A model is called recursive if its structural equations can be ordered in such a way that the first includes only predetermined variables in the right-hand side;

the second equation contains predetermined variables and the first endogenous variable (of the first equation) in the right-hand side; and so on. For example

$$y_1 = f(x_1, x_2 \ldots x_k; u_1)$$
$$y_2 = f(x_1, x_2 \ldots x_k; y_1; u_2)$$
$$y_3 = f(x_1, x_2 \ldots x_k; y_1, y_2; u_3)$$

and so on.

The random variables are assumed to be independent.

The special feature of a recursive model is that its equations may be estimated, one at a time, by OLS without simultaneous-equations bias. To understand this, let us write the above recursive model in its complete form. Assume that there are G endogenous variables k exogenous variables in the model

$$\begin{aligned}
y_1 &= \gamma_{11}x_1 + \gamma_{12}x_2 + \ldots + \gamma_{1k}x_k + u_1\\
y_2 &= \gamma_{21}x_1 + \gamma_{22}x_2 + \ldots + \gamma_{2k}x_k + \beta_{21}y_1 + u_2\\
y_3 &= \gamma_{31}x_1 + \gamma_{32}x_2 + \ldots + \gamma_{3k}x_k + \beta_{31}y_1 + \beta_{32}y_2 + u_3\\
&\vdots\\
y_G &= \gamma_{G1}x_1 + \gamma_{G2}x_2 + \ldots + \gamma_{Gk}x_k + \beta_{G1}y_1 + \beta_{G2}y_2 + \ldots + u_G
\end{aligned}$$

Given values of the exogenous variables (x_i) we may apply OLS to each equation individually, because by assumption the distribution variables u_i and u_j are independent, and hence the y's appearing in the right-hand side of each equation are independent of this equation's error term. For example, in the second equation y_1 is independent of u_2, given u_1 and u_2 are *ex hypothesi* independent.

Recursive systems are also called *triangular systems* because the coefficients of the endogenous variables (the β's) form a triangular array: the main diagonal of the array of β's contains units, and no coefficients appear above the main diagonal. For example assume that we have a model with four endogenous and five predetermined variables

$$\begin{aligned}
y_1 &= \gamma_{11}x_1 + \gamma_{12}x_2 + u_1\\
y_2 &= \beta_{21}y_1 + \gamma_{21}x_1 + \gamma_{22}x_2 + \gamma_{23}x_3 + u_2\\
y_3 &= \beta_{31}y_1 + \beta_{32}y_2 + \gamma_{31}x_1 + \gamma_{34}x_4 + u_3\\
y_4 &= \beta_{41}y_1 + \beta_{42}y_2 + \beta_{43}y_3 + \gamma_{44}x_4 + \gamma_{45}x_5 + u_4
\end{aligned}$$

To see whether this model is recursive it suffices to examine the form of the array of the β's. If it is triangular the system is recursive. The system may be

15. Identification

15.1. THE PROBLEM OF IDENTIFICATION STATED

Identification is a problem of model formulation, rather than of model estimation or appraisal. We say a model is identified if it is in a unique statistical form, enabling unique estimates of its parameters to be subsequently made from sample data. If a model is not identified then estimates of parameters of relationships between variables measured in samples may relate to the model in question, or to another model, or to a mixture of models.

An econometric model is frequently in the form of a system of simultaneous equations. The model may be said to be complete if it contains at least as many independent equations as endogenous variables. For identification of the entire model, it is necessary for the model to be complete and for each equation in it to be identified. Tests of identification are examined later in this chapter.

To illustrate the meaning of the identification problem let us take an example from the theory of market equilibrium. Assume that the market mechanism for a certain commodity is given by the following simple model

$$D = b_0 + b_1 P + u$$

$$S = a_0 + a_1 P + v$$

$$D = S$$

where D = quantity demanded, S = quantity supplied, P = price.

The first equation is the demand function, the second expresses the supply function and the third is the equilibrium condition of the market (or clearance equation). The model is complete in that there are three equations and three endogenous variables (S, D, P). But is each equation identified?

Assume we are interested in the measurement of the coefficients of the demand equation. To obtain estimates of b_0 and b_1 we normally use published time series reporting the quantity bought of the commodity. However, the quantity bought is identical with the quantity sold at any particular price. Market data register points of equilibrium of supply and demand at the price prevailing in the market at a certain point of time. A sample of time-series observations shows simultaneously the quantity demanded, D, *and* the quantity supplied, S, at the prevailing market price, P. If we use these data for estimation, we actually measure the coefficients of a function of the form $Q = f(P)$. This equation may

be either the demand function or the supply function (or even a 'bogus' equation, as we will presently see). How can we be sure which function we do really measure? If two econometricians use these data and one claims that he has estimated a demand function, while the other claims that he has estimated a supply function, how are we to decide who is right? Clearly, we need some criteria which will enable us to verify that the estimated coefficients belong to the one or the other relationship. Such criteria are known as 'identification conditions' of a function and will be developed in a subsequent section. For the time being let us return to our example. One might think that a scatter diagram of the sample observations might help. This is not always so. Suppose we plot the sample data on a diagram. The scatter of points may reveal one of the patterns shown in figures 15.1, 15.2, 15.3.

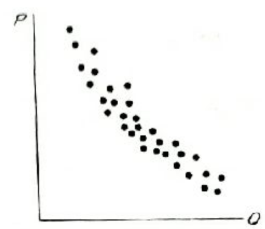


Figure 15.1

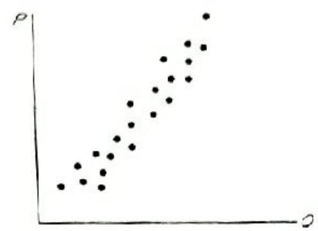


Figure 15.2

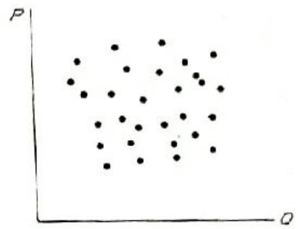


Figure 15.3

One might be tempted to conclude that the scatter of data in figure 15.1 identities the demand curve, the scatter of figure 15.2 identifies the supply curve, while the data of figure 15.3 identify neither relationship. This assertion is not necessarily true. In order to be able to say that the data identify the demand or the supply function we need to know the changes in the other factors which determine the supply and demand. Any model (like the one presented above) in which each equation contains the same explanatory variables, is statistically impossible to measure. Demand and supply are determined by many factors other than price. Changes in these factors cause shifts of the curves. We must have information on the shifts of the demand and the supply curves in order to be able to identify the coefficients of these relationships. It can be easily seen that the scatter of figure 15.1 will not identify the demand function, despite the apparent (spurious) high correlation between Q and P, if the observations have been generated by the intersection of shifting demand and supply curves. On figure 15.4 we plot imaginary sample points which represent the intersection of shifting demand and supply curves

Such sample data show a spurious high negative correlation, which is misleading: if we use the sample for estimating the relationship between Q and P that is $Q = f(P)$ we will obtain a high R^2 and a negative b_1, and we will be pretending that we measure the demand function, which obviously is not true. We are

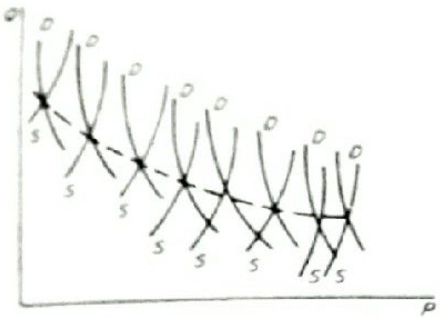


Figure 15.4

measuring a 'mongrel' function, that is, a function which is a mixture of supply and demand forces, and whose coefficients are really a mixture of the coefficients of both functions (see below). These spurious results are due to the omission of 'shift factors', in other words of variables which caused the shift of the demand and supply functions. If we have information on the shifts of the demand and supply curves (that is, on the changes of the other determining factors of demand and supply) we can say which function the data identify. For example in figure 15.5 we depict observations which show that the demand curve has remained fairly stable over the sample period, because the other factors which affect it – income, tastes, other prices – have remained almost constant, while the supply has been shifting widely due to changes in its other determinants (for example weather conditions). Such conditions, of fairly stable demand and widely shifting supply, give rise to observations which identify the demand function. This is the case with most agricultural products, whose supply is heavily influenced by weather conditions, while their demand does not shift much over time (they have low income elasticity of demand and the tastes of consumers for agricultural products do not change appreciably).

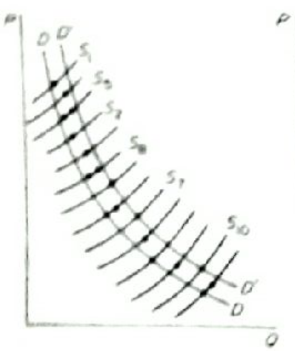


Figure 15.5

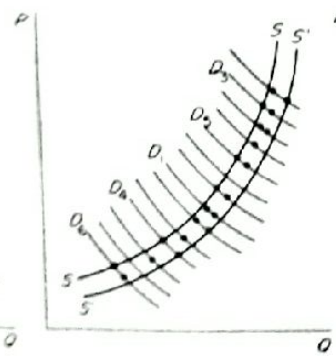


Figure 15.6

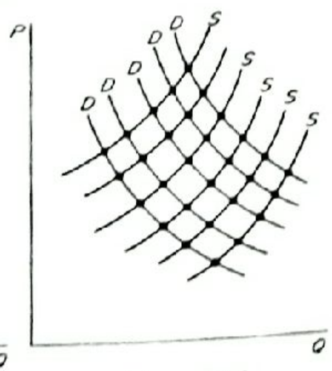


Figure 15.7

In figure 15.6 the supply is fairly stable while demand shifts within a wide range, because, say, of changes in tastes, incomes, war conditions. Under such

circumstances the observations generated by the interaction of demand and supply forces trace (identify) the supply function.

Finally, in figure 15.7, both supply and demand are shifting widely so that their interaction gives observations scattered all over the Q–P plane. However, if we know the factors that cause the shifts we may be able to identify both functions, or one of them. For example if D shifts due to income changes and S shifts due to weather conditions we will have the model $D = f(P, Y)$ and $S = f(P, W)$ and both functions can be identified despite the scatter of observations all over the Q–P plane (see below).

The above discussion may be summarised as follows. If we want to measure a given function belonging to a simultaneous-equations model, the function must be fairly stable over the sample period, that is, it must shift within a smaller range as compared with other relationships of the same model: we can measure the demand function if it is fairly stable while the supply function shows adequate variability. This condition is fulfilled if some factors *not included* in the demand function change considerably, causing a shift in the supply (or in other relevant equations). In other words, in order to identify the demand function, some factors *absent* from it but included in the supply function (or in other relations of the system) must be changing over the period of the sample.

Similarly, we can trace the supply function if it is fairly stable while demand shows enough variability. This implies that if the supply function is to be identified, some variables *absent* from it but affecting the demand function must be changing.

This may be called the *paradox of identification*: the identification of a function depends on variables *absent* from it, while at the same time being operative in the other function(s) of the model. We are able to identify a function by what variables *it does not include*.

The above was a diagrammatic presentation of the problem of identification of an econometric model. We may now examine the identification problem regarding a particular function in a more formal way.

A function belonging to a system of simultaneous equations is identified if it has a *unique statistical form*, that is if there is no other equation in the system, or formed by algebraic manipulation of the other equations of the system, which contains the same variables as the function in question. To illustrate this definition let us return to our earlier example of the model of the market mechanism of a certain product

$$D = b_0 + b_1 P + u \quad (15.1)$$

$$S = a_0 + a_1 P + v \quad (15.2)$$

$$D = S \quad (15.3)$$

We want to find out whether the estimates, which we may obtain by using sample data on demand and price can be identified as estimates of the true demand parameters b_0 and b_1. We may substitute S in equation (15.3) and obtain

$$D = a_0 + a_1 P + v \quad (15.4)$$

We thus have two equations (15.1) and (15.4) of the same statistical form, that is containing the same variables (D and P). However, the first contains the demand parameters (b_0, b_1), while the second contains the supply parameters (a_0, a_1). Regressing D on P with sample data we cannot be sure that the estimates which we will obtain are really the b's or the a's. The demand equation has not a unique statistical form, hence its parameters cannot be statistically identified.

Let us proceed further. By algebraic manipulations we may form an infinite number of equations which have the same statistical form with the demand function. For example multiplying equations (15.1) and (15.4) by k and c (arbitrary constants) respectively we obtain

$$kD = kb_0 + kb_1P + ku$$

$$cD = ca_0 + ca_1P + cv$$

Adding these expressions we obtain

$$(k + c)D = (kb_0 + ca_0) + (kb_1 + ca_1)P + (ku + cv)$$

or

$$D = \left(\frac{kb_0 + ca_0}{k + c}\right) + \left(\frac{kb_1 + ca_1}{k + c}\right) P + \left(\frac{ku + cv}{k + c}\right)$$

Setting

$$A_0 = \frac{kb_0 + ca_0}{k + c} \qquad A_1 = \frac{kb_1 + ca_1}{k + c} \qquad u^* = \frac{ku + cv}{k + c}$$

we may write the above expression in the form

$$D = A_0 + A_1P + u^*$$

This equation contains the same variables as the first equation of the structural model, but its parameters are a mixture (linear combination) of the parameters of the demand function, of the supply function and of the arbitrary constants (k, c). Thus by manipulating the relations of the structural model we obtained a 'bogus' equation, an equation which is neither the supply nor the demand function, but a mixture of both, which, however, has the same statistical form as the demand function. Consequently the demand function is not identified (or, more precisely, the parameters of the demand function are not identified). Under the above circumstances, if we use a sample of actual observations and perform the regression $D = f(P)$ we cannot be sure whether we obtain estimates of the b's, the a's, or of the mixed coefficients A_0 and A_1.

The conclusion from the above discussion is that (a) the identification of a system boils down to the identification of each one of its equations; (b) identification of the parameters of any equation is established if we can prove that its statistical form is unique. There are two formal rules with which we can establish the identification of a relationship. These rules set conditions for the identifiability of a relationship. They are (i) the *order condition* and (ii) the *rank condition* for identification. Before examining formally these conditions we must give some traditional definitions referring to identification.

In econometric theory two possible situations of identifiability are traditionally distinguished:

1. Equation underidentified.
2. Equation identified
   (a) Exactly identified.
   (b) Overidentified.

An equation is underidentified if its statistical form is not unique. *A system* is underidentified when one or more of its equations are underidentified.

If an *equation* has a unique statistical form we say that it is identified. It may be exactly identified or overidentified. But in both cases it is identified. A *system* is identified if *all* its equations are identified.

It should be noted that identification problems arise only for those equations which contain coefficients which must be estimated statistically (from sample data). Identification difficulties do not arise for definitional equations, identities, or statements of equilibrium conditions, because such relationships do not require measurement.

15.2. IMPLICATIONS OF THE IDENTIFICATION STATE OF A MODEL

Identification is closely related to the estimation of the model.

(1) If an equation (or a model) is underidentified it is impossible to estimate all its parameters with any econometric technique.

(2) If an equation is identified, its coefficient can, in general, be statistically estimated. In particular: (a) If the equation is exactly identified, the appropriate method to be used for its estimation is the method of indirect least squares (ILS, see Chapter 16). (b) If the equation is overidentified, indirect least squares cannot be applied, because it will not yield unique estimates of the structural parameters. There are various other methods which can be used in this case, for example two-stage least squares (2SLS), or maximum likelihood methods. These methods will be developed in subsequent chapters.

15.3. FORMAL RULES (CONDITIONS) FOR IDENTIFICATION

Identification may be established either by the examination of the specification of the structural model, or by the examination of the reduced form of the model (see below).

Traditionally identification has been approached via the reduced form. Actually the term 'identification' was originally used to denote the possibility (or impossibility) of deducing the values of the parameters of the structural relations from a knowledge of the reduced-form parameters. (See Johnston, *Econometric Methods*, 2nd ed., pp. 334–75.) In this section we will examine both approaches. However, we think that the reduced form approach is conceptually confusing and computationally more difficult than the structural model approach, because it requires the derivation of the reduced form first and then examination of the values of the determinant formed from some of the reduced form coefficients. The structural form approach is simpler and more useful.

In applying the identification rules we should either ignore the constant term, or, if we want to retain it, we must include in the set of variables a dummy variable (say X_0) which would always take on the value 1. Either convention leads to the same results as far as identification is concerned. In this chapter we will ignore the constant intercept.

15.3.1. ESTABLISHING IDENTIFICATION FROM THE STRUCTURAL FORM OF THE MODEL

We mentioned earlier that there are two conditions which must be fulfilled for an equation to be identified.

1. *The Order Condition for Identification*

This condition is based on a counting rule of the variables included and excluded from the particular equation. It is a necessary but not sufficient condition for the identification of an equation. The order condition may be stated as follows.

For an equation to be identified the total number of variables (endogenous and exogenous) excluded from it must be equal to or greater than the number of endogenous variables in the model less one. Given that in a complete model the number of endogenous variables is equal to the number of equations of the model, the order condition for identification is sometimes stated in the following equivalent form.

For an equation to be identified the total number of variables excluded from it but included in other equations must be at least as great as the number of equations of the system less one.

Let G = total number of equations (= total number of endogenous variables)
K = number of total variables in the model (endogenous and predetermined)
M = number of variables, endogenous and exogenous, included *in a particular equation.*

Then the order condition for identification may be symbolically expressed as

$$(K - M) \geqslant (G - 1)$$

$$\begin{bmatrix}\text{excluded}\\ \text{variables}\end{bmatrix} \geqslant [\text{total number of equations} - 1]$$

For example, if a system contains 10 equations with 15 variables, ten endogenous and five exogenous, an equation containing 11 variables is not identified, while another containing 5 variables is identified.

(a) For the first equation we have

$$G = 10 \qquad K = 15 \qquad M = 11$$

Order condition:

$$(K - M) \geqslant (G - 1)$$
$$(15 - 11) < (10 - 1)$$

that is, the order condition is not satisfied and the equation is underidentified.

(b) For the second equation we have

$$G = 10 \qquad K = 15 \qquad M = 5$$

Order condition:

$$(K - M) \geqslant (G - 1)$$
$$(15 - 5) > (10 - 1)$$

that is, the order condition is satisfied.

The order condition for identification is necessary for a relation to be identified, but it is not sufficient, that is, it may be fulfilled in any particular equation and yet the relation may not be identified.

2. *The Rank Condition for Identification*

The rank condition states that: *in a system of G equations any particular equation is identified if and only if it is possible to construct at least one non-zero determinant of order* (G – 1) *from the coefficients of the variables excluded from that particular equation but contained in the other equations of the model.*[1]

The practical steps for tracing the identifiability of an equation of a structural model may be outlined as follows.

Firstly. Write the parameters of all the equations of the model in a separate table, noting that the parameter of a variable excluded from an equation is equal to zero.

For example let a structural model be

$$y_1 = 3y_2 - 2x_1 + x_2 + u_1$$
$$y_2 = y_3 + x_3 + u_2$$
$$y_3 = y_1 - y_2 - 2x_3 + u_3$$

where the y's are the endogenous variables and the x's are the predetermined variables.

[1] This condition is called rank condition because it refers to the rank of the matrix of parameters of excluded variables. The rank of a matrix is the order of the largest non-zero determinant which can be formed from the matrix. In our case the relevant matrix is the submatrix of coefficients of the absent variables. Hence the rank condition may be also stated as follows: a sufficient condition for identification of a relationship is that the rank of the matrix of parameters of all the excluded variables (endogenous and predetermined) from that equation be equal to $(G - 1)$.

This model may be rewritten in the form

$$-y_1 + 3y_2 + 0y_3 - 2x_1 + x_2 + 0x_3 + u_1 = 0$$

$$0y_1 - y_2 + y_3 + 0x_1 + 0x_2 + x_3 + u_2 = 0$$

$$y_1 - y_2 - y_3 + 0x_1 + 0x_2 - 2x_3 + u_3 = 0$$

Ignoring the random disturbances the table of the parameters of the model is as follows.

Equations	*Variables*					
	y_1	y_2	y_3	x_1	x_2	x_3
1st equation	−1	3	0	−2	1	0
2nd equation	0	−1	1	0	0	1
3rd equation	1	−1	−1	0	0	−2

Secondly. Strike out the row of coefficients of the equation which is being examined for identification.

For example if we want to examine the identifiability of the second equation of the model we strike out the second row of the table of coefficients.

Thirdly. Strike out the columns in which a non-zero coefficient of the equation being examined appears. By deleting the relevant row and columns we are left with the coefficients of variables *not included* in the particular equation, but contained in the other equations of the model.

For example if we are examining for identification the second equation of the system, we will strike out the second, third and the sixth columns of the above table, thus obtaining the following tables.

Table of structural parameters

	y_1	y_2	y_3	x_1	x_2	x_3
1st	−1	3	0	−2	1	0
2nd	−0	−1	1	0	0	1
3rd	1	−1	−1	0	0	−2

Table of parameters of excluded variables

y_1	x_1	x_2
−1	−2	1
1	0	0

Fourthly. Form the determinant(s) of order $(G - 1)$ and examine their value. If at least one of these determinants is non-zero, the equation is identified. If all the determinants of order $(G - 1)$ are zero, the equation is underidentified.

In the above example of exploration of the identifiability of the second structural equation we have three determinants of order $(G - 1) = 3 - 1 = 2$. They are

$$\Delta_1 = \begin{vmatrix} -1 & -2 \\ 1 & 0 \end{vmatrix} \neq 0 \qquad \Delta_2 = \begin{vmatrix} -2 & 1 \\ 0 & 0 \end{vmatrix} = 0 \qquad \Delta_3 = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} \neq 0$$

(the symbol Δ stands for 'determinant'; see Appendix II) We see that we can form two non-zero determinants of order $G - 1 = 3 - 1 = 2$; hence the second equation of our system is identified.

Fifthly. To see whether the equation is exactly identified or overidentified we use the order condition $(K - M) \geqslant (G - 1)$. With this criterion, if the equality sign is satisfied, that is if $(K - M) = (G - 1)$, the equation is exactly identified. If the inequality sign holds, that is, if $(K - M) > (G - 1)$, the equation is overidentified.

In the case of the second equation we have

$$G = 3 \qquad K = 6 \qquad M = 3$$

and the counting rule $(K - M) \geqslant (G - 1)$ gives

$$(6 - 3) > (3 - 1)$$

Therefore the second equation of the model is overidentified.

The identification of a function is achieved by assuming that some variables of the model have a zero coefficient in this equation, that is, we assume that some variables do not directly affect the dependent variable in this equation. This, however, is an *assumption* which can be tested with the sample data. We will examine some *tests of identifying restrictions* in a subsequent section.

Some examples will illustrate the application of the two formal conditions for identification.

Example 1. Assume that we have a model describing the market of an agricultural product. From the theory of partial equilibrium we know that the price in a market is determined by the forces of demand and supply. The main determinants of the demand are the price of the commodity, the prices of other commodities, incomes and tastes of consumers. Similarly, the most important determinants of the supply are the price of the commodity, other prices, technology, the prices of factors of production, and weather conditions. The equilibrium condition is that demand be equal to supply. The above theoretical information may be expressed in the form of the following mathematical model

$$D = a_0 + a_1 P_1 + a_2 P_2 + a_3 Y + a_4 t + u$$

$$S = b_0 + b_1 P_1 + b_2 P_2 + b_3 C + b_4 t + w$$

$$D = S$$

where

D = quantity demanded
S = quantity supplied
P_1 = price of the given commodity
P_2 = prices of other commodities
Y = income
C = costs (index of prices of factors of production)
t = time trend. In the demand function it stands for 'tastes'; in the supply function it stands for 'technology'

The above model is mathematically complete in the sense that it contains three equations in three endogenous variables, D, S and P_1. The remaining variables, Y, P_2, C, t are exogenous. Suppose we want to identify the supply function. We apply the two criteria for identification:

10. Autocorrelation

10.1. THE MEANING OF THE ASSUMPTION OF SERIAL INDEPENDENCE

The fourth assumption of ordinary least squares is that the successive values of the random variable u are temporally independent, that is, that the value which u assumes in any one period is independent from the value which it assumed in any previous period. This assumption implies that the covariance of u_i and u_j is equal to zero

$$\text{cov}(u_i u_j) = E\{[u_i - E(u_i)]\,[u_j - E(u_j)]\}$$
$$= E(u_i u_j) = 0 \quad (\text{for } i \neq j)$$

given that by Assumption 2 $E(u_i) = E(u_j) = 0$.

If this assumption is not satisfied, that is, if the value of u in any particular period is correlated with its own preceding value (or values) we say that there is *autocorrelation* or *serial correlation* of the random variable.

It is convenient to change the subscripts of the u's and use $t, t-1, t-2$, etc. as subscripts, so as to show clearly the fact that we are at present concerned with the temporal dependence of the u's, their dependence through time. Thus we will write u_t for the value that u assumes in period t, u_{t-1} for the value of u in period $(t-1)$ and so on.

Autocorrelation is a special case of correlation. Autocorrelation refers to the relationship, not between two (or more) different variables, but between the successive values of the same variable. In this section we are particularly interested in the autocorrelation of the u's. However, autocorrelation may exist, and indeed it is a common phenomenon, in most economic variables. Thus we will treat autocorrelation of the u's in the same way as correlation in general.

Most of the standard econometric textbooks deal with the simple case of *linear* relationship between any two successive values of u

$$u_t = \rho u_{t-1} + v_t$$

This is known as a first-order autoregressive relationship (see below). We will begin our analysis with this form of simple relationship of the u's. In particular we will deal with the simple autocorrelation coefficient $\rho_{u_t u_{t-1}}$ (as a special form of the simple correlation coefficient ρ_{YX} developed in Chapter 3). Obviously $\rho_{u_t u_{t-1}}$ is subject to all the criticisms of the simple correlation coefficient cited in Chapter 3, for example $\rho_{u_t u_{t-1}}$ is not appropriate for non-linear relationships between u_t and u_{t-1}. Furthermore the simple $\rho_{u_t u_{t-1}}$ is not appropriate if the u's are related with more complex forms, with higher order auto-

regressive schemes. In a subsequent section we shall examine some solutions for such complex autoregressive structures.

We may obtain a rough idea of the existence or absence of autocorrelation in the u's by plotting the values of the regression residuals, e's, on a two-dimensiona diagram, as we did in simple correlation theory for the variables X and Y. The e's are estimates of the true values of u; thus if the e's are correlated this suggests autocorrelation of the true u's. Drawing the scatter diagram of the e's we should bear in mind the following:

(1) The 'variables' whose correlation we attempt to detect in this case are e_t and e_{t-1} (or some other lagged value of e, for example e_{t-2}).

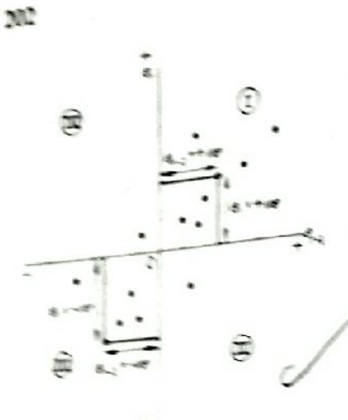
Variable I: e_t			Variable II: e_{t-1}		
e_{t+1}	or	(e_2)	e_t	or	(e_1)
e_{t+2}		(e_3)	e_{t+1}		(e_2)
e_{t+3}		(e_4)	e_{t+2}		(e_3)
.		.	.		.
.		.	.		.
.		.	.		.
$e_{t+(n-1)}$		(e_{n-1})	$e_{t+(n-2)}$		(e_{n-2})
e_{t+n}		(e_n)	$e_{t+(n-1)}$		(e_{n-1})

The observational points to be plotted are $e_t e_{t-1}$, or $(e_1, e_2), (e_2, e_3), (e_3, e_4) \ldots (e_n, e_{n-1})$.

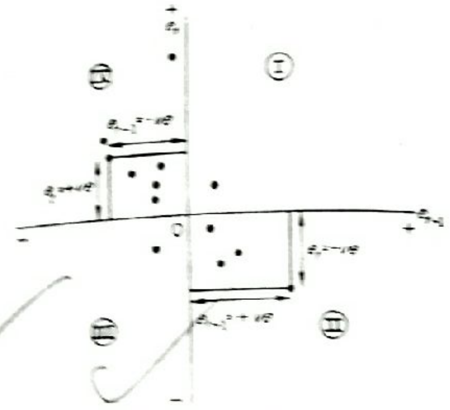
The mean of both 'variables' is zero ($\bar{e} = 0$) by definition. Hence the perpendiculars which pass through the 'means' are actually the two orthogonal axes. By analogy to what we said in Chapter 3, it is clear that if most of the points (e_t, e_{t-1}) fall in quadrants I and III (figure 10.1), the autocorrelation will be positive, since the products $(e_t)(e_{t-1})$ are positive. If most of the points (e_t, e_{t-1}) fall in quadrants II and IV (figure 10.2), the autocorrelation will be negative, because the products $(e_t)(e_{t-1})$ are negative.

It is obvious that autocorrelation, as indeed the simple correlation between any two variables, may be positive or negative in theory. However, *in practice autocorrelation is in most cases positive*. The main reasons for this are economic growth and cyclical movements of the economy. Most economic variables tend to grow in periods of growth, or they tend to show cyclical patterns. (See Fox, *Economic Statistics*, p. 199.)

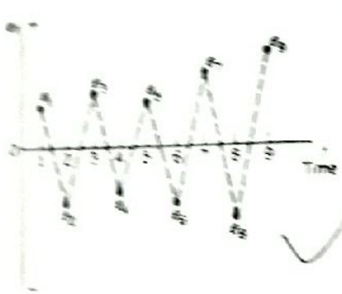
Another method commonly used in applied econometric research for the detection of autocorrelation is to plot the regression residuals, e's, against time. If the e's in successive periods show a regular time pattern (for example a sawtooth pattern, or a cyclical pattern) we conclude that there is autocorrelation in the function. In figures 10.3 and 10.4 we show hypothetical e's which are autocorrelated. In general if the successive (in subsequent time periods) values of the e's change sign frequently (figure 10.3) autocorrelation is negative. If the e's do not change sign frequently so that several positive e's are followed by several negative values of e (figure 10.4) autocorrelation is positive.



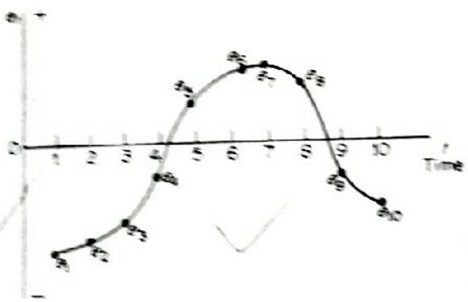
Positive autocorrelation
Figure 10.1



Negative autocorrelation
Figure 10.2



Negative autocorrelation
Figure 10.3



Positive autocorrelation
Figure 10.4

A measure of the first-order linear autocorrelation is provided by the autocorrelation coefficient

$$r_{e_t e_{t-1}} = \frac{\Sigma e_t e_{t-1}}{\sqrt{\Sigma e_t^2}\sqrt{\Sigma e_{t-1}^2}} = \hat{\rho}_{u_t u_{t-1}}$$

$r_{e_t e_{t-1}}$ is an estimate of the true autocorrelation coefficient $\rho_{u_t u_{t-1}}$ which measures the correlation of the true population of u's.

We will presently see that the test suggested by Durbin and Watson for autocorrelation reduces to the test of the statistical significance of the estimated autocorrelation coefficient ρ, that is we test whether $\rho = 0$ (see below).

Some definitions are appropriate at this stage. If the value of u in any particular period depends on its own value in the preceding period alone, we say

that the u's follow a *first-order autoregressive scheme* (or *first-order Markov process*). The relationship between the u's is then of the form

$$u_t = f(u_{t-1})$$

If u depends on the values of the two previous periods, that is $u_t = f(u_{t-1}, u_{t-2})$, the form of autocorrelation is called a *second-order autoregressive scheme*, and so on. In most applied research it is assumed that, when autocorrelation is present, it is of the simple first-order form $u_t = f(u_{t-1})$ and more particularly

$$u_t = a_1 u_{t-1} + v_t$$

where a_1 = the coefficient of the autocorrelation relationship
v = a random variable satisfying all the usual assumptions

$$E(v) = 0 \qquad E(v^2) = \sigma_v^2 \qquad E(v_i v_j) = 0$$

Clearly this is the simplest possible form of autocorrelation: a linear relationship between u_t and u_{t-1} (with suppressed constant intercept). If we apply ordinary least squares to this relationship we obtain

$$\hat{a}_1 = \frac{\sum_{t=2}^{n} u_t u_{t-1}}{\sum_{t=2}^{n} u_{t-1}^2}$$

On the other hand the autocorrelation coefficient $\rho_{u_t u_{t-1}}$ is given by the formula

$$\rho_{u_t u_{t-1}} = \frac{\Sigma u_t u_{t-1}}{\sqrt{\Sigma u_t^2}\sqrt{\Sigma u_{t-1}^2}}$$

Given that for large samples $\Sigma u_t^2 \approx \Sigma u_{t-1}^2$, we may write

$$\rho \approx \frac{\Sigma u_t u_{t-1}}{\sqrt{(\Sigma u_{t-1}^2)^2}} = \frac{\Sigma u_t u_{t-1}}{\Sigma u_{t-1}^2}$$

Clearly $\rho \approx \hat{a}_1$ for large samples. (See Kane, *Economic Statistics and Econometrics*, p. 366.) This is the reason why in most textbooks the simple first-order autoregressive model is given in the form

$$u_t = \rho u_{t-1} + v_t$$

where ρ = the first-order autocorrelation coefficient. Clearly if $\rho = 0$, $u_t = v_t$, that is u_t is not autocorrelated (given that by assumption v_t is not autocorrelated).

10.2. SOURCES OF AUTOCORRELATION

Autocorrelated values of the disturbance term u may be observed for many reasons.

1. *Omitted explanatory variables.* It is known that most economic variables tend to be autocorrelated. If an autocorrelated variable has been excluded from the set of explanatory variables, obviously its influence will be reflected in the random variable u, whose values will be autocorrelated. This case may be called 'quasi-autocorrelation' since it is due to the autocorrelated pattern of omitted explanatory variables (X's) and not to the behavioural pattern of the values of the true u. Of course, if several autocorrelated X's are omitted, u may not be autocorrelated, since the autocorrelation patterns of the omitted regressors may be such as to offset each other.

2. *Mis-specification of the mathematical form of the model.* If we have adopted a mathematical form which differs from the true form of the relationship, the u's may show serial correlation. For example if we have chosen a linear function while the true relationship between Y and the X's is of a cyclical form, the values of u will be temporally dependent.

3. *Interpolations in the statistical observations.* Most of the published time series data involve some interpolation and 'smoothing' processes which do average the true disturbances over successive time periods. As a consequence the successive values of the u are interrelated and exhibit autocorrelation patterns.

4. *Mis-specification of the true random term u.* It may well be expected in many cases for the successive values of the true u to be correlated. Thus even the purely random factors (wars, droughts, storms, strikes, etc.) exert influences that are spread over more than one period of time. For example a strike will have disruptive effects on the production process which will persist through several future periods. An exceptionally low cropping period in the agricultural sector, caused by abnormal weather conditions, will influence the performance of almost all other economic variables in several time periods; and so on. Such causes result in serially (temporally) dependent values of the disturbance term u, so that if we assume $E(u_i u_j) = 0$ we really mis-specify the true pattern of values of u. This case of autocorrelation may be called 'true autocorrelation' because its root lies in the u term itself.

It should be noted that the source of autocorrelation has a strong bearing on the solution which must be adopted for the 'correction' of the incidence of serial correlation. In other words the type of corrective action in each particular econometric application depends on the cause or source of autocorrelation. We will discuss this topic in a subsequent section.

10.3. PLAUSIBILITY OF THE ASSUMPTION OF NON-AUTOCORRELATED u's

From the discussion of the preceding paragraph it should be obvious that the assumption of temporal independence of the values of u can be easily violated in practice. Thus

(a) Taking into account that in most applied econometric research only the most important (three or four) explanatory variables are included explicitly in the function, it is natural to expect that omitted variables are a frequent cause of 'quasi-autocorrelation'. In particular, if we use time series it is almost certain that some at least of these omitted variables will be serially correlated, since in

economic life it is usual for the value of any variable in one particular period to be partly determined by its own value in the preceding period (or periods). For example output in period t depends on output in period $t - 1$; current income depends on past levels of income; investment decisions depend on past levels of investment; and so on. One can hardly think of any significant economic magnitude which is not somehow determined by the values which the same magnitude assumed in the past. Furthermore, in actual life, as we said, autocorrelation tends to be positive. If a disturbance (or an omitted variable) causes a positive u in period t, it is most probable that the u_{t+1} will also be positive. Similarly, if u assumes a negative value in t, the chances are that its value will be also negative in period $(t + 1)$.

(b) Interpolations and, in general, the customary data-collecting and processing techniques impart serial correlation in many aggregative time series.

(c) Random factors tend to persist in several time periods.

10.4. THE FIRST-ORDER AUTOREGRESSIVE SCHEME

In this section we will limit our analysis of the autocorrelation problem to the simple first-order autoregressive scheme, since most classical textbooks refer to this model as the most frequently assumed in applied econometric research. (In section 10.6 we will suggest a simple method for dealing with higher order autocorrelation structures.) We will first establish the mean, variance and covariance of u when its values are correlated with the simple Markov process. In this case the autoregressive structure is

$$u_t = \rho u_{t-1} + v_t \qquad \text{with } |\rho| < 1$$

where ρ = the coefficient of the autocorrelation relationship.[1]

v_t = a random term which fulfills all the usual assumptions of a random variable, that is,

$$E(v) = 0$$
$$E(v^2) = \sigma_v^2$$
$$E(v_i v_j) = 0 \qquad (\text{for } i \neq j)$$

The complete form of the first-order Markov process (the pattern of autocorrelation for all the values of u), is

$$\begin{aligned} u_t &= f(u_{t-1}) = \rho u_{t-1} + v_t \\ u_{t-1} &= f(u_{t-2}) = \rho u_{t-2} + v_{t-1} \\ u_{t-2} &= f(u_{t-3}) = \rho u_{t-3} + v_{t-2} \\ &\vdots \\ u_{t-r} &= f(u_{t-(r+1)}) = \rho u_{t-(r+1)} + v_{t-r} \end{aligned}$$

[1] ρ is also approximately equal to the first-order autocorrelation coefficient $\rho_{u_t u_{t-1}}$. See Page 203.

...ion, using the first differences of the original variables to re-estimate the ...
... the Durbin–Watson d statistic.
... value of d justify the 'first-differences' solution, adopted by the ...r?

... exercises are included in Appendix III.

11. Multicollinearity

11.1. THE MEANING OF MULTICOLLINEARITY

A crucial condition for the application of least squares is that the explanatory variables are not perfectly linearly correlated ($r_{x_ix_j} \neq 1$). The term multicollinearity is used to denote the presence of linear relationships (or near linear relationships) among explanatory variables. If the explanatory variables are perfectly linearly correlated, that is, if the correlation coefficient for these variables is equal to unity, the parameters become indeterminate: it is impossible to obtain numerical values for each parameter separately and the method of least squares breaks down. At the other extreme if the explanatory variables are not intercorrelated at all (that is if the correlation coefficient for these variables is equal to zero), the variables are called orthogonal[1] and there are no problems concerning the estimates of the coefficients, at least so far as multicollinearity is concerned. Actually, in the case of orthogonal X's, there is no need to perform a multiple regression analysis: each parameter, b_i, can be estimated by a *simple* regression of Y on the corresponding regressor: $Y = f(X_i)$. (See A. Goldberger, *Econometric Theory*, p. 201.)

In practice neither of the above extreme cases (of orthogonal X's or perfect collinear X's) is often met. In most cases there is some degree of intercorrelation among the explanatory variables, due to the interdependence of many economic magnitudes over time. In this event the simple correlation coefficient for each pair of explanatory variables will have a value between zero and unity, and the multicollinearity problems *may* impair the accuracy and stability of the parameter estimates, but the exact effects of collinearity have not as yet been theoretically established.

Multicollinearity is not a condition that either exists or does not exist in economic functions, but rather a phenomenon inherent in most relationships due to the nature of economic magnitudes. There is no conclusive evidence concerning the degree of collinearity which, if present, will affect seriously the parameter estimates. Intuitively, when any two explanatory variables are changing in nearly the same way, it becomes extremely difficult to establish the influence of each one regressor on Y separately. For example assume that the consumption expenditure of an individual depends on his income and liquid assets. If over a period of time income and the liquid assets change by the same proportion, the influence on consumption of one of these variables may be erroneously attributed to the other. The effects of these variables on consumption cannot be sensibly investigated, due to their high intercorrelation.

[1] Orthogonal variables are the variables whose covariance is zero: $\sum x_i x_j / n = 0$.

PLAUSIBILITY OF THE ASSUMPTION

Strictly speaking the assumption concerning multicollinearity, that is that the variables be not perfectly linearly correlated, is easily met in practice, because it is very rare for any two variables to be exactly intercorrelated in a linear form. However, the estimates of least squares may be seriously affected with a less than perfect intercorrelation between the explanatory variables (see below).

Multicollinearity may arise for various reasons. *Firstly*, there is a tendency of economic variables to move together over time. Economic magnitudes are influenced by the same factors and in consequence once these determining factors become operative the economic variables show the same broad pattern of behaviour over time. For example in periods of booms or rapid economic growth the basic economic magnitudes grow, although some tend to lag behind others. Thus income, consumption, savings, investment, prices, employment, tend to rise in periods of economic expansion and decrease in periods of recession. Growth and trend factors in time series are the most serious cause of multicollinearity. *Secondly*, the use of lagged values of some explanatory variables as separate independent factors in the relationship. Models with distributed lags have given satisfactory results in many fields of applied econometrics, and their use is expanding fast. For example in consumption functions it has become customary to include among the explanatory variables past as well as the present levels of income. Similarly, in investment functions distributed lags concerning past levels of economic activity are introduced as separate explanatory variables. Naturally the successive values of a certain variable are intercorrelated, for example income in the current period is partly determined by its own value in the previous period, and so on. Thus multicollinearity is almost certain to exist in distributed lag models. (Distributed lag models are discussed in Chapter 13.)

Taking the above considerations into account it is clear that some degree of collinearity is expected to appear in most economic relationships.

It should be noted that although multicollinearity is usually connected with time series, it is quite frequent in cross-section data as well. For example in a cross-section sample of manufacturing firms labour and capital inputs are almost always highly intercorrelated, because large firms tend to have large quantities of both factors while small firms usually have smaller quantities of both labour and capital. However, multicollinearity tends to be more common and more serious a problem in time series.

11.3. CONSEQUENCES OF MULTICOLLINEARITY

If the intercorrelation between the explanatory variables is perfect ($r_{x_ix_j} = 1$), then (a) the estimates of the coefficients are indeterminate, and (b) the standard errors of these estimates become infinitely large.

[...]at the relation to be estimated is

$$Y = b_0 + b_1X_1 + b_2X_2 + u$$

and that X_1 and X_2 are related with the exact relation $X_2 = kX_1$, where k is any arbitrary constant number.

The formulae for the estimation of the coefficients $\hat{b}_1$ and $\hat{b}_2$ are

$$\hat{b}_1 = \frac{(\Sigma x_1 y)(\Sigma x_2^2) - (\Sigma x_2 y)(\Sigma x_1 x_2)}{(\Sigma x_1^2)(\Sigma x_2^2) - (\Sigma x_1 x_2)^2}$$

$$\hat{b}_2 = \frac{(\Sigma x_2 y)(\Sigma x_1^2) - (\Sigma x_1 y)(\Sigma x_1 x_2)}{(\Sigma x_1^2)(\Sigma x_2^2) - (\Sigma x_1 x_2)^2}$$

Substituting kX_1 for X_2 we obtain

$$\hat{b}_1 = \frac{k^2(\Sigma x_1 y)(\Sigma x_1^2) - k^2(\Sigma x_1 y)(\Sigma x_1^2)}{k^2(\Sigma x_1^2)^2 - k^2(\Sigma x_1^2)^2} = \frac{0}{0}$$

$$\hat{b}_2 = \frac{k(\Sigma x_1 y)(\Sigma x_1^2) - k(\Sigma x_1 y)(\Sigma x_1^2)}{k^2(\Sigma x_1^2)^2 - k^2(\Sigma x_1^2)^2} = \frac{0}{0}$$

Therefore the parameters are indeterminate: there is no way of finding separate values of each coefficient.

Proof (b). If $r_{x_1x_2} = 1$ the standard errors of the estimates become infinitely large. In the two-variable model

$$Y = b_0 + b_1X_1 + b_2X_2 + u$$

if X_1 and X_2 are perfectly correlated ($X_2 = kX_1$) the variances of $\hat{b}_1$ and $\hat{b}_2$ will be

$$\text{var}(\hat{b}_1) = \sigma_u^2 \frac{\Sigma x_2^2}{\Sigma x_1^2 \Sigma x_2^2 - (\Sigma x_1 x_2)^2}$$

and

$$\text{var}(\hat{b}_2) = \sigma_u^2 \frac{\Sigma x_1^2}{\Sigma x_1^2 \Sigma x_2^2 - (\Sigma x_1 x_2)^2}$$

Substituting kX_1 for X_2 we obtain

$$\text{var}(\hat{b}_1) = \sigma_u^2 \frac{k^2 \Sigma x_1^2}{k^2 \Sigma x_1^2 \Sigma x_1^2 - k^2(\Sigma x_1^2)^2} = \frac{\sigma_u^2 \Sigma x_1^2}{0} = \infty$$

Thus the variances of the estimates become infinite unless $\sigma_u^2 = 0$. However, there is no *a priori* reason why σ_u^2 should tend to zero when intercorrelation of the explanatory variables increases. Haavelmo has suggested that 'the estimate of σ_u^2 is not impaired by the fact that the independent variables are highly intercorrelated'. (See 'Remarks on Frisch's Confluence Analysis and its Use in Econometrics', chapter 5 in T. Koopmans (ed.), *Statistical Inference in Dynamic Economic Models*, Wiley 1950, p. 260.)

To illustrate the problem let us take the following example of a relationship including three explanatory variables. Suppose that the true consumption function for a certain country is

$$Y = b_0 + b_1X_1 + b_2X_2 + b_3X_3 + u$$

where Y = total consumption

X_1 = income of rural areas

X_2 = income of urban areas

X_3 = tax on income

On *a priori* grounds one should expect $b_1 < b_2$, since the marginal propensity [illegible] and X_2 change together. We may [illegible] impossible to obtain a [illegible]

$$Y = b_0 + b_1 X_1 + b_2 X_2 + u$$

$$Y = b_0 + (b_1 + b_2) X_1 + b_3 X_3 + u$$

Thus by dropping one of the two (equal) variables we can obtain an estimate of the sum of their coefficient, but not individual values for b_1 and b_2. That is, the sum $(b_1 + b_2)$ will be identified, but neither b_1 nor b_2 is identified separately. (There is a close relation between multicollinearity and identification which will be discussed in Chapter 13.)

If the X's are not perfectly collinear, but are to a certain degree correlated $(0 < r_{x_i x_j} < 1)$, the effects of collinearity are uncertain. The evidence from theoretical econometric studies (with controlled data) as well as from applied research is controversial and by no means conclusive. In some studies the values of the coefficients become unstable as additional collinear variables are introduced into the function, or as the size of the sample increases; in other studies the values of the estimates are not significantly affected. The same holds for the standard errors of the estimates: in some studies the standard errors of the estimates are considerably increased when collinear variables are present in the function, while in other instances the standard errors have not been affected by the incidence of multicollinearity.

Two points should be stressed. Firstly, the estimates of the coefficients are statistically unbiased $(E(\hat{b}_i) = b_i)$ even when multicollinearity is strong. The statistical property of unbiasedness of the OLS estimates does not require that the X's be uncorrelated. On the other hand, samples with multicollinear X's may render the values of the estimates seriously imprecise and unstable. Unfortunately no firm rules have been established for assessing the seriousness of such errors. Yet the instability of the estimates may be so serious as to even cause a change in the sign of the parameter estimates as the degree of collinearity increases. There is some evidence[1] that increasing multicollinearity produces various changes in the values of the parameters, depending on the importance of each explanatory variable.

[1] See K. A. Fox, *Intermediate Economic Statistics*, pp. 259–265. See also Farrar and Glauber: 'However real the dependency relationship between y and each member of a relatively large explanatory variable set X may be, the growth of interdependence within X as its size increases rapidly decreases the stability of each independent variable's contribution to explained variance.' D. E. Farrar and R. R. Glauber, 'Multicollinearity in Regression Analysis: The Problem Re-visited', *Review of Economics and Statistics*, vol. 49, 1967.

[illegible] that the effect of multicollinearity on the parameter estimates depends on [illegible] of interdependence as well as on the importance of the variables happen to be collinear. In any particular study certain explanatory variables are more important than others (for example in a demand function the price of the commodity, consumers' income and the price of close substitutes are probably more important than prices of distant substitutes of the commodity being studied). If these strategically crucial explanatory variables happen to be strongly intercorrelated the seriousness of the problem is greater than in the case of secondary factors being multicollinear, because the latter may be dropped from the analysis without seriously impairing the results. Secondly, many writers seem to accept that when multicollinearity is present in a function the standard errors of the estimates will in general be large. This is not always so, because both the numerator and the denominator of the formulae of the variances will usually be affected by terms involving sums of cross products of the X's so that the final size of the variance of the b's may not be large.

L. R. Klein seems to accept that multicollinearity is not necessarily a problem unless it is high relative to the overall degree of multiple correlation among all variables simultaneously. That is Klein argues that collinearity is harmful if

$$r^2_{x_i x_j} \geq R^2_{y \cdot x_1, x_2, \ldots, x_k}$$

where $r^2_{x_i x_j}$ is the simple correlation between any two explanatory variables (X_i and X_j) and R^2 is the overall (multiple) correlation of the relationship. (L. R. Klein, *Introduction to Econometrics*, Prentice Hall International, 1962, pp. 64 and 101. Klein's approach has been attacked by Farrar and Glauber, 'Multicollinearity in Regression Analysis', *Rev. Econ. & Statist.*, 1967.)

On the other hand Theil argues that in a model with more than two X's small intercorrelations between variables may lead to non significance due to increase of the standard errors. (See H. Theil, 'Specification Errors and the Estimation of Economic Relationships', *Rev. Int. Statist. Inst.*, vol. 25, 1957.) Finally Frisch (in *Statistical Confluence Analysis by means of Complete Regression Systems*, University Economics Institute, Oslo 1934) showed that the standard errors are not always large when multicollinearity is present. Thus we may obtain very inaccurate estimates of the coefficients due to multicollinearity, and yet the standard errors of these 'wrong' estimates may not show it. (See C. Leser, *Econometric Techniques and Problems*, p. 27.)

Summing up the above arguments we may say that although there may be exceptions, in general, increasing standard errors appear when we include intercorrelated variables as explanatory in the function. Thus with multicollinearity in a function we run the danger of misspecification, because we may reject a variable whose standard error appears high, although this variable is an important determinant of the variations of the dependent variable. This danger is very serious due to the traditional procedure followed in applied relationship, especially in practice, in which case the whole set of explanatory variables is not known and it has become a common practice to approximate it by experimentation, that is, by gradually introducing additional variables in the relationship and judging their importance by, among other things, the size of their standard errors. For example suppose that the true relationship is

$$Y = b_0 + b_1 X_1 + b_2 X_2 + u$$

Since the true specification is not known, the researcher usually starts his study with the simpler tentative formulation

$$Y = a_0 + a_1 X_1 + v$$

This model will most probably yield significant results, although a_1 will have a specification error due to the omission of X_2. The addition of X_2 in the relationship should normally improve the fit, since the model is in this case correctly specified. However, if X_1 and X_2 are highly correlated and their standard errors are large, the researcher will usually reject X_2, being misguided by its large standard error. In this case multicollinearity results in the wrong decision and hence in the wrong specification of the model, since X_2 is by assumption (that is in the postulated model) an important explanatory variable in the relationship. (See Theil, *Economic Forecasts and Policy*, p. 217. See also Farrar and Glauber, *op. cit.*, p. 94.)

However, large standard errors do not always appear even in functions in which the regressors are strongly multicollinear. For example production functions with overall correlations much in excess of 0.950 have been well estimated with intercorrelations between labour and capital as high as 0.800 to 0.900. If these functions were not well estimated, we would tend to find high sampling errors of the estimated coefficients. By conventional criteria the estimated parameters of most Cobb–Douglas production functions are large relative to their standard error. The coefficients are generally high multiples of their standard errors. (See L. R. Klein, *Introduction to Econometrics*, p. 101.) Recently D. D. Silvey has published a study on the problem of multicollinearity. (See D. Silvey, 'Multicollinearity and Imprecise Estimation', *Royal Statistical Society*, 1969.) We will not examine Silvey's approach here, since it is not substantially superior to Farrar's and Glauber's test, which will be developed in the next section.

11.4 TESTS FOR DETECTING MULTICOLLINEARITY

11.4.1 A METHOD BASED ON FRISCH'S CONFLUENCE ANALYSIS

The seriousness of the effects of multicollinearity seems to depend on the degree of intercorrelation ($r_{x_i x_j}$) as well as on the overall correlation coefficient ($R_{y \cdot x_1 x_2 \ldots x_k}$). Thus one might suggest that the standard errors, the partial correlation coefficients ($r_{x_i x_j}$'s) and the total R^2 may be used for testing for multicollinearity. Yet none of these criteria by itself is a satisfactory indicator of multicollinearity, because:

(a) Large standard errors do not always appear with multicollinearity (see the case of Cobb–Douglas production functions). Furthermore large standard errors may arise for various reasons and not only because of the presence of linear relationships among the explanatory variables.

(b) The intercorrelations of the explanatory variables need not be high for the values of the b's and their standard errors to be affected badly. That is $r_{x_i x_j}$ is not an adequate criterion by itself.

(c) The overall R^2 may be high relative to the $r_{x_i x_j}$'s and yet the results may be highly imprecise and insignificant (with wrong signs and/or large standard errors).

However, a combination of all these criteria may help the detection of multicollinearity. In order to gain as much knowledge as possible as to the seriousness of multicollinearity we suggest the adoption of the following method which is in its essence a revised version of Frisch's 'Confluence Analysis' (or 'Bunch Map Analysis').

The procedure is to regress the dependent variable on each one of the explanatory variables separately. Thus we obtain all the elementary regressions, and we examine their results on the basis of *a priori* and statistical criteria.

We choose the elementary regression which appears to give the most plausible results, on both *a priori* and statistical criteria. Then we gradually insert additional variables and we examine their effects on the individual coefficients, on their standard errors, and on the overall R^2. A new variable is classified as *useful, superfluous* or *detrimental*, as follows.

(1) If the new variable improves R^2 without rendering the individual coefficients unacceptable ('wrong') on *a priori* considerations, the variable is considered useful and is retained as an explanatory variable.

(2) If the new variable does not improve R^2 and does not affect to any considerable extent the values of the individual coefficients, it is considered as superfluous and is rejected (i.e. is not included among the explanatory variables).

(3) If the new variable affects considerably the signs or the values of the coefficients, it is considered as detrimental. If the individual coefficients are affected in such a way as to become unacceptable on theoretical, *a priori*, considerations, then we may say that this is a warning that multicollinearity is a serious problem. The new variable is important, but because of intercorrelations with the other explanatory variables its influence cannot be assessed statistically by ordinary least-squares. This does not mean that we must reject the detrimental variable. If we did so, we would ignore information valuable to our attempts of approaching as best we can the 'true' specification of the relationship. In order to avoid the complications of multicollinearity and take into account the influence of the detrimental variable we have to follow one of the solutions developed in the following section. If we omit the detrimental variable completely in an attempt to avoid its detrimental influence on the other coefficients, we must bear in mind that in so doing we simply leave its influence to be absorbed by the other coefficients (whose values thus become mixed) and by the random term which may become correlated with the variables left

in the function, with the consequence of violation of Assumption 6, since in this case $E(u_i X_j \neq 0)$.

The method described above for establishing multicollinearity differs from Frisch's Confluence Analysis in that the latter estimates all possible regressions between the variables which are present in a relationship, taking each variable successively as the dependent variable and considering all possible regressions of each variable on all others which are gradually introduced into the analysis. It is thus obvious that Confluence Analysis requires many more computations, so that comparisons of the results become more complicated as compared with the proposed 'experimental technique'.

Example. Table 11.1. includes time-series data for the period 1951–68 on clothing expenditure, disposable income, liquid assets, a price index for clothing items and a general price index for a certain country.

Table 11.1. Data for the estimation of the demand function for clothing

Year	*Expenditure on clothing (£ m)*	*Disposable income (£ m)*	*Liquid assets (£ m)*	*Price index for clothing 1963 = 100*	*General price index 1963 = 100*
1959	8·4	82·9	17·1	92	94
1960	9·6	88·0	21·3	93	96
1961	10·4	99·9	25·1	96	97
1962	11·4	105·3	29·0	94	97
1963	12·2	117·7	34·0	100	100
1964	14·2	131·0	40·0	101	101
1965	15·8	148·2	44·0	105	104
1966	17·9	161·8	49·0	112	109
1967	19·3	174·2	51·0	112	111
1968	20·8	184·7	53·0	112	111

On *a priori* grounds consumers' expenditure on clothing is influenced by all the factors included in the above table, so that the demand function for clothing should be

$$C = b_0 + b_1 Y + b_2 L + b_3 P_c + b_4 P_0 + u$$

where C = expenditure on clothing
Y = income
L = liquid assets
P_c = price of clothing
P_0 = price of other commodities.

Applying least squares to this function we obtain the following estimates:

$$C = -13{\cdot}53 + 0{\cdot}097\,Y + 0{\cdot}015\,L - 0{\cdot}199\,P_c + 0{\cdot}34\,P_0$$
$$S(\hat{b}_i) \quad (7{\cdot}5) \quad (0{\cdot}03) \quad (0{\cdot}05) \quad (0{\cdot}09) \quad (0{\cdot}15)$$

$$R^2 = 0{\cdot}998 \qquad \Sigma\hat{y}^2 = 28{\cdot}15 \qquad \Sigma e^2 = 0{\cdot}33 \qquad d = 3{\cdot}4$$

Applying analysis of variance to test the overall significance of the fit we find

$$F^* = \frac{\Sigma\hat{y}^2/(K-1)}{\Sigma e^2/(n-K)} = \frac{28{\cdot}15/4}{0{\cdot}33/5} = 15{\cdot}6$$

Multicollinearity

Since the theoretical $F_{0{\cdot}05}$ value with $v_1 = K - 1 = 4$ and $v_2 = n - K = 5$ degrees of freedom is 5·19, we reject the null hypothesis, accepting the alternative that there is a significant relationship between clothing expenditure and the explanatory variables.

However, all the explanatory variables are seriously multicollinear as can be seen by simple correlation coefficients

$$r_{YL} = 0{\cdot}993$$
$$r_{YP_c} = 0{\cdot}980$$
$$r_{YP_0} = 0{\cdot}987$$
$$r_{LP_c} = 0{\cdot}964$$
$$r_{LP_0} = 0{\cdot}973$$
$$r_{P_cP_0} = 0{\cdot}991$$

To explore the effects of multicollinearity we compute the elementary regressions

(1) $\hat{C} = \hat{a}_0 + \hat{a}_1 Y = -1{\cdot}24 + 0{\cdot}118\,Y$ $\quad$ (0·37) (0·002) $\qquad R^2 = 0{\cdot}995 \qquad d = 2{\cdot}6$

(2) $\hat{C} = \hat{b}_0 + \hat{b}_1 P_c = -38{\cdot}51 + 0{\cdot}516\,P_c$ $\quad$ (4·20) (0·04) $\qquad R^2 = 0{\cdot}951 \qquad d = 2{\cdot}4$

(3) $\hat{C} = \hat{c}_0 + \hat{c}_1 L = 2{\cdot}11 + 0{\cdot}327\,L$ $\quad$ (0·81) (0·02) $\qquad R^2 = 0{\cdot}967 \qquad d = 0{\cdot}4$

(4) $\hat{C} = \hat{d}_0 + \hat{d}_1 P_0 = -53{\cdot}65 + 0{\cdot}663\,P_0$ $\quad$ (3·63) (0·03) $\qquad R^2 = 0{\cdot}977 \qquad d = 2{\cdot}1$

We choose the first elementary regression ($C = f(Y)$) as the first step in our analysis, since income (Y) seems on *a priori* grounds to be the most important explanatory variable during the period under consideration. We then introduce the remaining explanatory variables gradually into the function. The results are shown in table 11.2.

Table 11.2

	b_0 *Constant*	$\hat{b}_1$ (Y)	$\hat{b}_2$ (P_c)	$\hat{b}_3$ (L)	$\hat{b}_4$ (P_0)	R^2	d
$C = f(Y)$	−1·24 (0·37)	0·118 (0·002)	–	–	–	0·995	2·6
$C = f(Y, P_c)$	1·40 (4·92)	0·126 (0·01)	−0·036 (0·07)	–	–	0·996	2·5
$C = f(Y, P_c, L)$	0·94 (5·17)	0·138 (0·02)	−0·034 (0·06)	−0·037 (0·05)	–	0·996	3·1
$C = f(Y, P_c, P_0)$	−12·76 (6·52)	0·104 (0·01)	−0·188 (0·07)	–	0·319 (0·12)	0·997	3·5
$C = f(Y, P_c, L, P_0)$	−13·53 (7·5)	0·097 ((0·03)	−0·199 (0·09)	0·015 (0·05)	0·34 (0·15)	0·998	3.4

Note. The numbers in brackets are the standard errors of the estimates.

Changes in income seems to be important in explaining the variation in clothing expenditure.