computed by the usual formula of the covariance

$$\hat{\sigma}_{e_1 e_2} = \frac{\Sigma e_{1i} e_{2i}}{n}$$

$$\hat{\sigma}_{e_1 e_3} = \frac{\Sigma e_{1i} e_{3i}}{n} \quad \text{and so on.}$$

The complete set of the variances and the covariances of the error terms is as follows

$$\begin{vmatrix} \hat{\sigma}^2_{e_1} & \hat{\sigma}_{e_1 e_2} & \hat{\sigma}_{e_1 e_3} \cdots \hat{\sigma}_{e_1 e_G} \\ \hat{\sigma}_{e_2 e_1} & \hat{\sigma}^2_{e_2} & \hat{\sigma}_{e_2 e_3} \cdots \hat{\sigma}_{e_2 e_G} \\ \vdots & & \ddots \quad \vdots \\ \hat{\sigma}_{e_G e_1} & \hat{\sigma}_{e_G e_2} & \cdots\cdots \hat{\sigma}^2_{e_G} \end{vmatrix} = \begin{vmatrix} \frac{\Sigma e_1^2}{n} & \frac{\Sigma e_1 e_2}{n} \cdots & \frac{\Sigma e_1 e_G}{n} \\ \frac{\Sigma e_1 e_2}{n} & \frac{\Sigma e_2^2}{n} \cdots & \vdots \\ \vdots & \ddots & \vdots \\ \frac{\Sigma e_1 e_G}{n} & \cdots\cdots & \frac{\Sigma e_G^2}{n} \end{vmatrix}$$

Stage III. We use the above variances and covariances of the error terms in order to obtain the transformations of the original variables for the application of generalised least squares (GLS).

The presentation of the computations of this stage becomes extremely complicated with the use of simple algebra and summations and will not be presented here. The interested reader is referred to more advanced textbooks of econometrics.

19.2.1. PROPERTIES OF THE 3SLS ESTIMATES

1. The 3SLS estimates are biased but consistent.
2. They are more efficient than 2SLS, since in their estimation we use more information than in 2SLS.

The method is simpler than full information maximum likelihood. However, it requires complete knowledge of the specification of the entire model and a large amount of data. If we are interested in only one relationship of the entire model, 3SLS seems rather a tedious, time consuming approach. Furthermore, it is sensitive to specification error in the equations: a single specification error is transmitted to all the equations of the model.

We said that 3SLS reduces to 2SLS if the random variables are contemporaneously independent. Thus (a) if one is not very sure about the accuracy of the specification of all the equations, or (b) if it seems on *a priori* grounds that the u's are not seriously interdependent, it is preferable to apply 2SLS.

20. Testing the Forecasting Power of an Estimated Model

We said at the beginning of this book that one of the main goals of applied econometric research is to use the estimated model in order to forecast the value of the dependent variables given the values of the explanatory variables. We can predict the value of a variable in two alternative ways: either our forecast is a single value, or we can estimate an interval within which the value of the variable will most probably lie. The first method yields a *point prediction*, the second an *interval prediction*. For example we may predict that the gross national product of the United Kingdom in 1975 will be £45,000 million. This is a point prediction of the GNP. Alternatively we may predict that the gross national product of the United Kingdom in 1975 will be between £44,000 mn and £46,000 mn. This is an example of interval prediction.

Forecasting with an econometric model is a complicated process. In this chapter we will attempt to give a simple exposition of the forecasting procedure starting with a simple example of single-equation regression prediction. In a subsequent section we will illustrate the process of forecasting with an aggregate (multi-equation) econometric model.

20.1. FORECASTING WITH A SINGLE-EQUATION LINEAR REGRESSION MODEL

20.1.1. POINT PREDICTION

Suppose that the relationship between Y and X has been estimated by OLS

$$\hat{Y}_i = \hat{b}_0 + \hat{b}_1 X_i$$

Given the estimates $\hat{b}_0$ and $\hat{b}_1$, and given the value of the explanatory variable X in any period, we may obtain an estimate of the value of the dependent variable by substituting in the estimated regression equation

$$\hat{Y}_F = \hat{b}_0 + \hat{b}_1 \hat{X}_F$$

where Y_F = forecast value of the dependent variable
X_F = given value of X in the forecast period.

For example if the above function is the consumption function of the United Kingdom and its estimated form is

$$Y_t = 2{,}550 + 0{\cdot}68\ X_t$$

we can obtain the following point prediction of the level of consumption in 1975, provided the income (X) in that year will be £45,000 mn

$$Y_{1975} = 2{,}250 + 0{\cdot}68\ (45{,}000) = 33{,}150 \text{ million pounds.}$$

This is known as *conditional forecasting*, because it is based on the condition that the explanatory variable will assume in the forecast period the value X_F. Furthermore this procedure assumes that the structural relationship between Y and X will continue to be the same in the forecast period as it was during the sample period, that is, we assume that the parameters will not change between the two periods.

20.1.2. CONFIDENCE INTERVAL FOR A POINT PREDICTION (INTERVAL PREDICTION)

The need for constructing a confidence interval for a point prediction arises from the fact that when we use an econometric model for forecasting we are making a statistical judgement, which is subject to error. Statistical judgements cannot be precise statements due to their nature. Hence we must construct a confidence interval for such predictions.

The construction of confidence intervals in general has been examined in Chapter 5 (and in Appendix I). For the construction of a confidence interval for the forecast value $\hat{Y}_F$ in particular we must have information on the mean and the variance of the distribution of $\hat{Y}_F$.

(1) We first note that $\hat{Y}_F$ will have a normal distribution since it is determined by $\hat{b}_0$ and $\hat{b}_1$, which have a bivariate normal distribution.

(2) The mean of $\hat{Y}_F$ will be the true value of the forecast

$$Y_F = b_0 + b_1 X_F + u_F$$

(3) As regards the variance of the distribution of $\hat{Y}_F$ we note the following. In the case of a prediction from an econometric function there are two possible sources of variation (error): (a) The estimates of the parameters. In predicting, we do not know the true parameters of the structural relationship, and hence we use their estimates, $(\hat{b}_0, \hat{b}_1)$, which have been obtained from a sample. Hence they are subject to sampling error, which is transmitted to the forecast of the value of the dependent variable. In other words the standard errors of the estimates are a component of the variance of the forecast. (b) When we use the above method of forecasting we assume that in the period of the forecast the disturbance variable u will assume its mean value, (which is zero by definition). However, in any particular period u may assume a value different from zero, due to the occurrence of a random event. We cannot predict the exact value that u may assume in any particular period, but from the variance of the u term we have a measure of the range within which its values may lie. Hence the variance of the random variable is a second component of the variance of the forecast.

In summary, our point prediction will be associated with a certain variance, due to the sampling errors of the parameter estimates and to the variance of the random variable u.

The variance of the forecast in our single-equation model is given by the

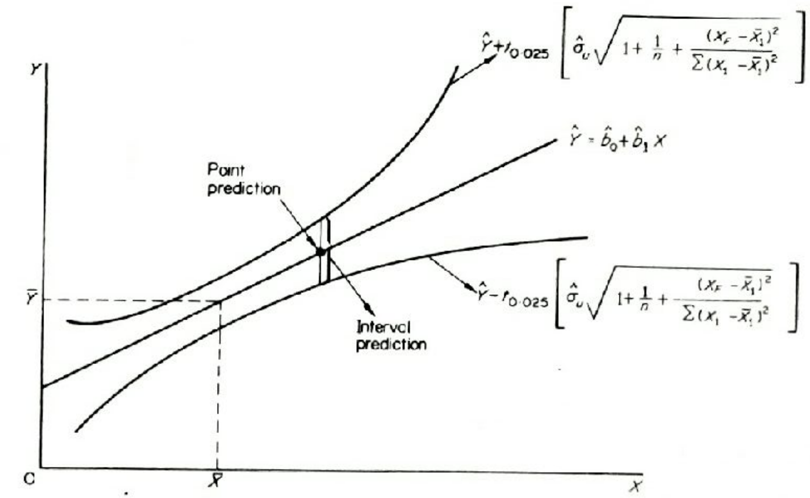

Figure 20.1

This expression is the same as the one derived for the two-variable function except for the third term which includes the covariances of the estimates of the coefficients, which allow for the joint dependence of the $\hat{b}$'s. See L. R. Klein, *Introduction to Econometrics*, pp. 244–51.

An arithmetic example will illustrate the construction of a confidence interval of a point prediction. Table 20.1 shows the consumption expenditure and income of the USA over the 1957–68 period.

Table 20.1. Consumption and income of the USA (in billions of $)

Year	*Consumption expenditure* C	*Income* X
1957	282·3	359·9
1958	291·1	370·1
1959	312·3	394·7
1960	326·5	414·5
1961	336·6	430·8
1962	356·6	458·7
1963	376·6	483·5
1964	402·9	515·5
1965	434·7	557·4
1966	468·3	613·1
1967	494·3	658·9
1968	538·9	721·0

Applying OLS to the data of table 20.1 we obtain the following estimate of the consumption function

$$\hat{Y}_t = 31{\cdot}76 + 0{\cdot}71\, X_t \qquad R^2 = 0{\cdot}998$$
$$(5{\cdot}39)\quad (0{\cdot}01)$$

The residual sum of squares, Σe^2, is found equal to 169·5. From this we get the following estimate of the variance of u

$$\hat{\sigma}_u = \frac{\Sigma e^2}{n-2} = \frac{169{\cdot}5}{12-2} = 16{\cdot}9$$

Assuming that the disposable income of the USA in 1975 will be $ 850 billion we can insert that value in our estimated function and obtain an estimate of the consumption expenditure in 1975

$$\hat{Y}_{1975} = 31{\cdot}76 + (0{\cdot}71)(850) \approx 635 \text{ billions of dollars}$$

To construct a confidence interval for the above point estimate we must compute the standard error of the forecast

$$s_{(\hat{Y}_{1975})} = \hat{\sigma}_u \sqrt{1 + \frac{1}{n} + \frac{(X_F - \bar{X})^2}{\Sigma(X - \bar{X})^2}}$$

The relevant data are

$\bar{X} = 498 \quad \hat{\sigma}_u = 16{\cdot}9 \quad n = 12 \quad (X_F - X)^2 = 123{,}904 \quad \Sigma(X - \bar{X})^2 = 151{,}482$

Hence

$$s_{(\hat{Y}_{1975})} = 16{\cdot}9\left(\sqrt{1 + \frac{1}{12} + \frac{123{,}904}{151{,}482}}\right) \approx 2{\cdot}32$$

Thus the confidence interval of the above point prediction is

$$\hat{Y}_F \pm t_{0{\cdot}025}(s_{(\hat{Y}_{1975})})$$

Given that $t_{0{\cdot}025} = 2{\cdot}23$ (with $n - 2 = 12 - 2 = 10$ degrees of freedom) we have

$$635 - (2{\cdot}23)(2{\cdot}32) < Y_{F(1975)} < 635 + (2{\cdot}23)(2{\cdot}32)$$
$$629{\cdot}83 < Y_{F(1975)} < 640{\cdot}17$$

that is the consumption expenditure of the USA in 1975 is expected (with 95 per cent probability) to lie between 630 and 640 millions of dollars (approximately).

20.2. FORECASTING WITH AN AGGREGATE MULTI-EQUATION ECONOMETRIC MODEL

Once the parameters of a structural model have been estimated with any appropriate technique, we can use the model for forecasting, provided we know the values of the predetermined variables in the period of the forecast.

We will illustrate the forecasting process by a simple Keynesian model. In this section we will use the structural parameters for forecasting. The forecasting procedure is greatly facilitated if we use the reduced-form coefficients of the model. However, we will not develop the reduced-form approach, since it is beyond the scope of this textbook. (For an introduction to the forecasting

procedure see D. B. Suits, 'Forecasting with an Aggregate Econometric Model', *American Economic Review*, vol. 52, 1962, pp. 104–32.)

Assume we have the following model of income determination.

$$C_t = a_0 + a_1(Y_t - T_t) + u_1$$
$$I_t = b_0 + b_1 Y_t + b_2 Y_{t-1} + u_2$$
$$T_t = cY_t + u_3$$
$$M_t = m_0 + m_1 Y_t + m_2 P_{t-1} + u_4$$
$$Y_t = C_t + I_t + G_t + E_t - M_t$$

The system contains five equations in five endogenous variables, C_t, I_t, T_t, M_t, Y_t. There are four predetermined variables in the model G_t, Y_{t-1}, P_{t-1}, E_t.

In total there are nine variables in the model:

C_t = consumption expenditure
I_t = investment
T_t = taxation
M_t = imports
Y_t = income
Y_{t-1} = lagged income
P_{t-1} = lagged price level
G_t = government expenditure
E_t = exports.

The first equation is the consumption function. Consumers' expenditure depends on disposable income.

The second equation is the investment function. Investment is determined by the current level of income as well as by the income in the previous period.

The third equation is a tax revenue function. Revenues from taxation are determined by current income.

The fourth equation is an import function. Imports depend on the level of national income and/on the price level of the country lagged one period.

The fifth equation is the usual accounting identity of national income.

Applying the counting rule for identification (see Chapter 15) we see that the four behavioural equations are overidentified. Choosing the method of 2SLS and using time series for the period 1948–1969, we obtain the following estimates of the structural parameters.

$$C_t = 20 + 0{\cdot}8(Y_t - T_t)$$
$$I_t = 2 + 0{\cdot}1 Y_t + 0{\cdot}3 Y_{t-1}$$
$$T_t = 0{\cdot}2 Y_t$$
$$M_t = 3 + 0{\cdot}1 Y_t + 0{\cdot}1 P_{t-1}$$
$$Y_t = C_t + I_t + G_t + E_t - M_t$$

Having estimated the structural parameters we may forecast the values of the endogenous variables for any period, given that we know the values which the predetermined variables will assume in that period. Suppose that for the period of the forecast the exogenous variables will assume the following values

$$G = 20 \qquad Y_{t-1} = 150$$
$$F = 10 \qquad P_{t-1} = 110$$

Inserting these values into the model and transferring all the endogenous variables in the left-hand side of the equations, we obtain

$$C_t - 0{\cdot}8\,Y_t - 0{\cdot}8\,T_t = 20$$
$$I_t - 0{\cdot}1\,Y_t = 2 + 0{\cdot}3(150) = 47$$
$$T_t - 0{\cdot}2\,Y_t = 0$$
$$M_t - 0{\cdot}1\,Y_t = 3 + 0{\cdot}1(110) = 14$$
$$Y_t - C_t - I_t + M_t = 20 + 10 = 30$$

This is a system of five equations in five unknowns which are the endogenous variables *C, I, T, M* and *Y*. The system may be solved with any method. Its solution yields the following forecast values

$$C = 1675 \qquad T = 46{\cdot}1 \qquad Y = 230{\cdot}5$$
$$I = 70 \qquad M = 37$$

On the above forecasting procedure we note the following.

Firstly. Any forecast with an econometric model is a *conditional* forecast. For example the above forecast should be read as follows:

If the exogenous variables take the values assumed in the forecast period, and
if the structural parameters remain constant, and
if the clause '*ceteris paribus*', which was made in estimating the model, remains valid for the period of the forecast,

then the endogenous variables will take the values given by the solution of the structural model.

In this sense it is apparent that one should not expect the forecast to be realised. Actually one of the reasons for making forecasts is precisely to avoid their realisation. For example if taxation and government expenditure are held constant, and our model forecasts an increase in unemployment due to a fall in the economic activity, this result is obviously undesirable: the forecast should not be realised. Thus taxation and/or government expenditure should change so as to avoid the occurrence of the forecast.

Secondly. The forecast computed from the above solution is a point forecast. It is based on the point estimates of the structural parameters, and the random terms have been set equal to zero (which is their mean value) in the period of the forecast. However, there are various reasons to expect the actual values of the endogenous variables to deviate from this 'point' value. Such reasons are: (a) The random variable *u*, which was set equal to its mean value of zero in the forecast period, may assume a value different than zero; (b) The coefficients used in the forecast are not the true parameters, but are estimates of the true parameters obtained from a sample, thus they are subject to sampling error. For these reasons we should construct confidence intervals for the values forecast by an econometric model. (See J. Johnston, *Econometric Methods*, 1972, pp. 152–5.)

Thirdly. If the values assumed by the exogenous variables are not realised in the period of the forecast, obviously the forecast will not be realised.

Fourthly. If the structural coefficients change in the period of the forecast, our model must be modified accordingly, otherwise it is unsuitable for forecasting.

Fifthly. The same holds for the '*ceteris paribus*' clause: if the factors assumed constant when estimating the model (for example tastes, population movements, social changes, etc.) do actually change in the period of the forecast, again our model must be modified before being used for forecasting.

It might be thought that since conditions underlying the structure of the model as well as the structure itself (coefficients) of the model do change, forecasting with econometric models is bound to be inaccurate. Fortunately there are various ways for modifying the model in such a way as to incorporate information which becomes available after the estimation of its coefficients. Some examples will illustrate this point.

Example 1. It is well known that investment functions are bad approximations to the actual behaviour of investors. Assume that we have information of the investment projects directly from a survey of businessmen's plans. Obviously this information is superior to any forecast obtained from an investment function. In this case we may ignore the investment equation and use the above information for forecasting purposes.

Example 2. A lot of information may be incorporated into the model, at least for short-term forecasting (and until a more effective way is found, for example re-computing the function or functions), by adjusting accordingly the constant intercept of the function, that is shifting the function, while keeping its slope(s) constant. For example assume that the government imposes a credit squeeze (or an easing of credit terms). This may well be expected to make the consumption function shift (downwards or upwards). From past experience of similar measures one usually is able to set a (lower or upper) limit to such a shift. This estimated amount of the probable shift is then subtracted (added) to the constant term of the consumption function. It should be noted that this shift is equivalent to ascribing to the *u* term a value different than its mean value (zero), a procedure perfectly justified from our knowledge that *u* will not be zero in the period of the forecast.

Example 3. Assume that the taxation laws are changed so as to increase total tax revenue without, however, affecting the marginal tax rate. This means that the slope of the tax equation (which is the marginal tax rate) will be constant, while the constant intercept will increase so as to account for the change in the tax structure. The amount by which the tax revenue will increase is usually estimated by some way or another. This amount is added to the constant term of the tax equation. Again this is equivalent to saying that $u \neq 0$ in the period of the forecast, and in fact that it will assume the value of the estimated change of the tax revenue.

It should be clear from the above examples that an econometric model is not a sterile tool but a highly flexible device, if appropriately used. It can then become a useful tool in policy formulation.

The above forecasting procedure should be repeated if the predetermined variables assume different values. We should include these new values into the structural model and solve all the components (Previous). This is a very tedious job when one wants to examine the values which the endogenous variables will assume under different sets of values of the exogenous variables. Fortunately the forecasting procedure may be greatly facilitated by using the reduced form of the model. For an introduction to this forecasting technique see D. B. Suits, *The Theory and Application of Econometric Models*, Athens 1963, Center of Planning and Economic Research.

20.5 TEST OF SIGNIFICANCE OF THE DIFFERENCE BETWEEN A SINGLE PREDICTION AND THE ACTUAL OBSERVATION

This test is frequently used as a basis for the evaluation of the forecasting power of an econometric model. It is a simple t test similar to the one developed in the preceding paragraph, based on the standard error of the forecast.

We said that the statistic

$$t^* = \frac{Y_F - \hat{Y}_F}{\hat{\sigma}_u \sqrt{1 + \frac{1}{n} + \frac{(X_F - \bar{X})^2}{\Sigma(X - \bar{X})^2}}}$$

has the Student's t distribution with $n - 2$ degrees of freedom.

Instead of Y_F, assume that we have an actual observation on the dependent variable, Y_A, which is *not included in the sample* used for the estimation of the function, and we want to test whether the difference between the observed value of Y and the predicted one from the econometric function ($\hat{Y}_F$) is statistically significant. In other words we want to answer the question: Is the difference compatible with the estimated relationship, or does this difference indicate a significant change in the estimated relationship between Y and X? Formally we want to test the null hypothesis

$$H_0: \hat{Y}_F = Y_A$$

against the alternative hypothesis $H_1: \hat{Y}_F \neq Y_A$.

For this purpose we use the sample data as well as the value actually assumed by the explanatory variable, and we compute the observed t^* statistic

$$t^* = \frac{Y_A - \hat{Y}_F}{\hat{\sigma}_u \sqrt{1 + \frac{1}{n} + \frac{(X_F - \bar{X})^2}{\Sigma(X - \bar{X})^2}}}$$

where t^* = observed value of the t statistic
$\hat{\sigma}_u$ = estimate of the variance of u
X_F = actual (observed) value of X in the period of forecast
Y_A = actual (observed) value of Y
$\hat{Y}_F$ = predicted (forecast) value of Y from the regression.

Subsequently we find the theoretical value of t from the t-Table with $n - 2$ degrees of freedom, that is the value that t would assume if the difference

between Y_A and $\hat{Y}_F$ were equal to zero, (with a given probability, for example with 95 per cent probability). We compare the observed t^* with its theoretical value, that is the value of t if the null hypothesis were true, (if there were no difference between Y_A and $\hat{Y}_F$), and decide whether the observed difference $Y_A - \hat{Y}_F$ is significant or not according to the following rule

If $t^* < t$ we conclude that there is no significant difference between actual and predicted value. The observation is compatible with the estimated relationship. In this case we accept that the predictive power of our model is good.

If $t^* > t$ we conclude that the difference between the actual and predicted value of Y is significant. The observation of Y is not compatible with the relationship which has been estimated.

The difference between the actual and the forecast value of Y may be due to one of the following reasons.

(a) *Abnormal conditions in the period of forecast*: u may assume an exceptionally high value out of the range suggested by its estimated variance. In this case our model is still valid and we do not need to modify or reject it.

(b) *Errors in the value of* X_F. Recall that $\hat{Y}_F$ is a conditional forecast. If X_F contains errors, the forecast value of Y will have errors. Again our model is valid in the sense that the 'poor' prediction is due to errors in X_F and not to wrong specification of the structural relationship.

(c) *Change in the structure of the relationship.* The poor predictive power in this case is due to the fact that the conditions have changed between the sample period and the period of the forecast, so that the estimated relationship between Y and X does not hold any more. In this case the model although valid for the sample period, cannot be used for forecasting, unless modified (restated) and recomputed.

As an illustration of the above test procedure consider the following consumption function of the United States which was estimated for the 1954–65 period. (See E. Kane, *Economic Statistics and Econometrics*, p. 226.)

$$\hat{Y}_t = -3.0 + 0.928 X_t$$
$$(3.5) \quad (0.01)$$

The disposable income in 1968 was 721 billions of dollars. Using this value we obtain the following estimate of the consumption expenditure for this year

$$\hat{Y}_{F,1968} = -3.0 + (0.928)(721) = 666$$

The actual consumption expenditure in 1968 was 539 billions of dollars. Is the difference $Y_A - \hat{Y}_F$ significant? Is the predictive power of the 1954–65 function good? Or does the difference suggest a change of the structural relationship between the period 1954–65 and the year 1968? We may answer these questions by performing the above test, using the following data

$$X_A = 721 \qquad Y_A = 539 \qquad \hat{Y}_F = 666$$

$$\hat{\sigma}_u = \sqrt{\frac{\Sigma e^2}{n-2}} = 4.48 \qquad \bar{X} = 350 \qquad \Sigma(X - \bar{X})^2 = 3875.8$$

$$\hat{\sigma}_u\left(\sqrt{1+\frac{1}{n}+\frac{(X_A-\bar{X})^2}{\Sigma(X-\bar{X})^2}}\right) = 4{\cdot}48\left(\sqrt{1+\frac{1}{12}+\frac{137{,}641}{3{,}876}}\right) \approx 27$$

Therefore

$$t^* = \frac{Y_A - \hat{Y}_F}{s(\hat{Y}_F)} = \frac{539-666}{27} = -4{\cdot}7$$

The theoretical value of t with $12-2=10$ degrees of freedom (at the 95 per cent level of significance) is $-2{\cdot}23$.

Since $t^* < t$ we conclude that the difference between $\hat{Y}_F$ and Y_t is significant. The forecasting power of the consumption function model is poor.

If the structural coefficients must be re-computed, we may either keep the same specification and increase only the sample to include more recent observations, or we may change the specification of the model, or both. Respecification of the function may take various forms, depending on the suspected or known cause of the change in structure. Some examples may illustrate the variety of possible restatements of the original relationship.

(a) One may introduce new observable explanatory variables directly in the function.

(b) One may use more equations and turn his simple model into a multi-equation model.

(c) Changes in the parameters may be measured indirectly by introducing appropriate dummy variables in the original function (see Chapter 12).

(d) If it is known that the coefficients change over time, one may include as a separate explanatory variable 'tX' (where t stands for 'time') in the function, apart from the simple variable X (see Chapter 12).

(e) If the distribution of income is changing, one may split the aggregate income variable into two or more parts, for example Y_1 for wage-income and Y_2 for non-wage income (see Chapter 17).

20.4. EVALUATION OF THE FORECASTING PERFORMANCE OF AN ESTIMATED MODEL

20.4.1. PREDICTION–REALISATION DIAGRAMS

Suppose we have several forecasts from an econometric model and the corresponding realised values of an endogenous variable over several periods. The forecasting performance of the econometric model is judged on the basis of the differences between predictions and realisations. The smaller the differences between predictions (P_i) and actual values (A_i) of the dependent variable, the better the forecasting performance of the model.

We may examine the forecasting performance of a model by using a *prediction–realisation diagram* (figure 20.2). This is a diagram on which we plot the points determined by the predictions and realisations on a certain variable. Along the vertical axis we measure the actual (observed) *changes* in

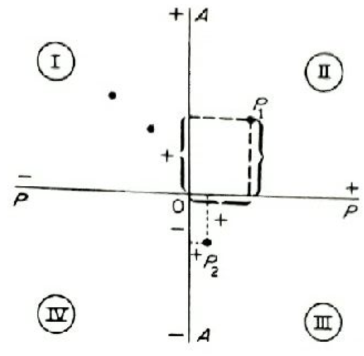

Figure 20.2

the values and along the horizontal axis we plot the predicted changes in values of the variable. Points lying in quadrants II and IV show that the model predicts the direction of the change in the dependent variable correctly. For example point P_1 implies that the model predicted a positive change in the dependent variable and the realised change was actually positive. On the other hand points falling in quadrants I and III show that the forecast was opposite to realisation; the model at point P_2 predicts a positive change (an increase in the value of the variable) while the realised change was negative (the value of the variable decreased). Such points are called *turning point errors.* Points falling in quadrants I and III show a very poor predictive performance of the model. Points falling in quadrants II and IV show that the model predicts correctly the direction of the change. We can say more about the accuracy of the predictions by introducing a 45° line with a positive slope through the origin (figure 20.3). This is called the *line of perfect forecast*: any point on this line shows equality (zero difference) between prediction and

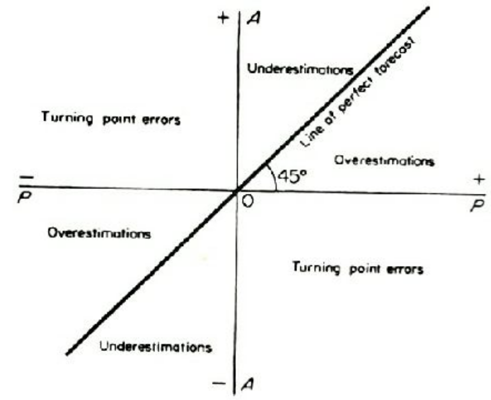

Figure 20.3

actual value. In quadrant II points above the 45° line show that the model underestimated the value of the dependent variable, while points below the 45° line show that the model overestimated the value of the endogenous variables. On the contrary in quadrant IV points above the 45° line show overestimation while points below the line show underestimation of the value of the dependent variable. The closer the points (defined by predicted and actual values) to the 45° line the better the forecasting performance of the model.

To illustrate the above procedure we plot in figure 20.4 the actual GNP

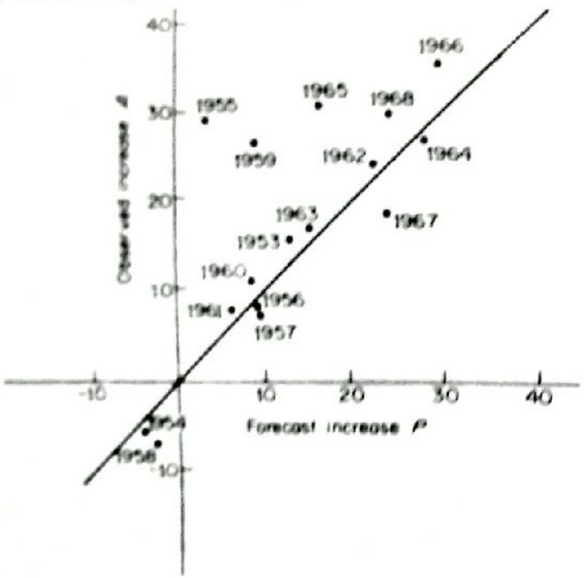

Figure 20.4

of the United States together with its forecast values by the econometric model of the Research Seminar in Quantitative Economics of the University of Michigan.[1] This model is known as the RSQE model of the US economy, from the initials of the Research Seminar. Inspection of the chart shows that the forecasts from the RSQE model are in most years satisfactorily accurate.

20.4.2 THEIL'S INEQUALITY COEFFICIENT

(H. Theil, *Applied Economic Forecasting*, pp. 26–36, North-Holland 1966. An earlier version of the inequality coefficient is to be found in H. Theil's, *Economic Forecasts and Policy*, pp. 31–48, North-Holland 1962.)

A systematic measure of the accuracy of the forecasts obtained from an econometric model has been suggested by H. Theil. This measure is called the *inequality coefficient* and is defined by the expression

$$U^2 = \frac{\Sigma(P_i - A_i)^2/n}{\Sigma A_i^2/n}$$

[1] Source: D. B. Suits, *Principles of Economics*, 1970, p. 206.

or

$$U = +\sqrt{\frac{\Sigma(P_i - A_i)^2/n}{\Sigma A_i^2/n}}$$

where P_i = predicted (forecast) *change* in the dependent variable
A_i = actual (realised) *change* in the dependent variable

The values that the inequality coefficient assumes lie between 0 and ∞

$$0 \leqslant U \leqslant \infty$$

The smaller the value of the inequality coefficient the better is the forecasting performance of the model.

If $P_i = A_i$, then $U = 0$, and we say that with our model we attain perfect forecasts.

If $P_i = 0$, then $U = 1$, and the model forecasts no better than a 'naïve' zero-change prediction.

If $U > 1$ the predictive power of the model is worse than the zero-change prediction. Thus if $U > 1$ it is preferable to accept the zero-change extrapolation that $Y_{t+1} = Y_t$, that is, to assume that there will be no change in the value of the dependent variable between the periods t and $t+1$.

The numerator of the inequality coefficient is the root mean square prediction error (RMS prediction error) and is the important term in this measure. The denominator is simply a way for achieving the independence of U from the units of measurement of the variables.

Further insight into the sources of the forecast error may be obtained by the following decomposition of the inequality coefficient.

The numerator can be decomposed into three terms each showing a different source of forecast error.[1]

$$\frac{1}{n}\Sigma(P_i - A_i)^2 = (\bar{P} - \bar{A})^2 + (S_P - S_A)^2 + 2(1 - r_{PA})S_P S_A$$

where $\bar{P}$ and $\bar{A}$ are the means of predictions and actual values

$$\bar{P} = \frac{1}{n}\Sigma P_i \qquad \bar{A} = \frac{1}{n}\Sigma A_i$$

S_P and S_A are the standard deviations of predictions and realisations

$$S_P^2 = \frac{1}{n}\Sigma(P_i - \bar{P})^2 \qquad S_A^2 = \frac{1}{n}\Sigma(A_i - \bar{A})^2$$

and r_{PA} is the correlation coefficient of predicted and realised changes

$$r_{PA} = \frac{\Sigma(P_i - \bar{P})(A_i - \bar{A})}{nS_P \; S_A}$$

[1] The rationale of this decomposition is the same as the decomposition of the total variation of any variable, which is the basis of the Analysis of Variance method, developed in Chapter 8.

The three components which form the sources of the forecast error are called *partial inequality coefficients*. The first component shows that the cause of the discrepancy between predictions and realisations is the difference between their means (or central tendencies); it is referred to as the *bias component* of the inequality coefficient. The second component shows that another cause of the discrepancy between P_i and A_i is the difference between their variance; it is referred to as the *variance component* of the inequality coefficient. The third component shows that still another cause of the discrepancy between P_i and A_i is their imperfect covariance; it is called the *covariance component* of the inequality coefficient.

The third source of forecast error is the most dangerous one, in the sense that not much can be done about it: we can never hope that forecasters will be able to produce predictions which would be perfectly correlated with the actual values of the variable. It is natural that $r_{PA} \neq 1$ and hence the 'covariance component' of the prediction error cannot be expected to be zero. The other two sources of error can be reduced in general in the course of time, by the incorporation of additional information in the forecasting process.

A useful way to present the various sources of the forecast error is to divide each component by the total forecast-variation $\Sigma(P_i - A_i)^2/n$. In this way we express each component as a proportion of the total prediction-error. This procedure leads to the following *inequality proportions*

$$\text{bias proportion} \quad U_M = \frac{(\bar{P} - \bar{A})^2}{\Sigma(P_i - A_i)^2/n}$$

$$\text{variance proportion} \quad U_S = \frac{(S_P - S_A)^2}{\Sigma(P_i - A_i)^2/n}$$

$$\text{covariance proportion} \quad U_C = \frac{2(1 - r_{PA})S_P S_A}{\Sigma(P_i - A_i)^2/n}$$

Clearly

$$U_M + U_S + U_C = 1$$

An example. The table below shows the forecast changes P_i from a simple linear regression equation of the imports of a country, and the realised changes of imports (A_i). Is the forecasting performance of our regression model 'good'? To answer this question we compute Theil's inequality coefficient. The relevant terms are included in table 20.2.

$$U^2 = \frac{\Sigma(P_i - A_i)^2}{\Sigma A_i^2} = \frac{101}{256} \approx 0.4$$

and

$$U = \sqrt{0.4} = 0.666$$

Given that the value of the inequality coefficient is low, ($U < 1$) we conclude that

the forecasting performance of the estimated import function (from which the predictions were derived) is fairly good. Decomposition of the forecast error yields the following inequality proportions (given $\bar{P} = 1{\cdot}3$ $\bar{A} = 1{\cdot}4$ $S_A^2 = 23{\cdot}64$ $S_P^2 = 10{\cdot}61$ $S_A = 4{\cdot}86$ $S_P = 3{\cdot}26$ $r_{PA} = 0{\cdot}813$)

Table 20.2. Realised and predicted levels of imports

Forecast period	*Predicted change in imports* P_i	*Actual changes in imports* A_i	A_i^2	$(P_i - A_i)^2$	$(A_i - \bar{A})^2$	$(P_i - \bar{P})^2$
1960	+5	+10	100	25	73·96	13·69
1961	+2	+2	4	0	0·36	0·49
1962	−4	−7	49	9	70·56	28·09
1963	0	+4	16	16	6·76	1·69
1964	+1	−3	9	16	19·36	0·09
1965	+4	+6	36	4	21·16	7·29
1966	+7	+4	16	9	6·76	32·49
1967	−2	−4	16	4	29·16	10·89
1968	−2	−1	1	1	5·76	10·89
1969	+2	+3	9	1	2·56	0·49
	$\Sigma P_i = 13$	$\Sigma A_i = 14$	$\Sigma A_i^2 = 256$	$\Sigma(P_i - A_i)^2 = 85$	$\Sigma(A_i - \bar{A})^2 = 236{\cdot}4$	$\Sigma(P_i - \bar{P})^2 = 106{\cdot}1$

$$U_M = \frac{(P - A)^2}{\Sigma(P_i - A_i)^2/n} = \frac{(1{\cdot}3 - 1{\cdot}4)^2}{8{\cdot}5} = 0{\cdot}001$$

$$U_S = \frac{(S_P - S_A)^2}{\Sigma(P_i - A_i)^2/n} = \frac{(3{\cdot}26 - 4{\cdot}86)^2}{8{\cdot}5} = 0{\cdot}301$$

$$U_C = \frac{2(1 - r_{PA})S_A S_P}{\Sigma(P_i - A_i)^2/n} = \frac{2(1 - 0{\cdot}813)(15{\cdot}84)}{8{\cdot}5} = 0{\cdot}708$$

Thus

$$\frac{1}{n}\Sigma(P - A)^2 = (\bar{P} - \bar{A})^2 + (S_P - S_A)^2 + 2(1 - r_{PA})S_P S_A$$

or

$$8{\cdot}5 = 0{\cdot}01 + 2{\cdot}56 + 5{\cdot}92$$

This shows that the error in the forecast is mainly due to the high correlation between P_i and A_i, that is to the source of error about which nothing can be done in order to improve the forecasting performance of the model. For a further discussion of the inequality coefficient the reader is referred to H. Theil's *Applied Economic Forecasting*, North Holland, 1966.

20.4.3 THE JANUS QUOTIENT

Another measure of the accuracy of the forecasts is the *Janus quotient*[1] defined by the expression

$$J^2 = \frac{\sum_{t=n+1}^{n+m} (P_t - A_t)^2/m}{\sum_{t=1}^{n} (A_t - P_t)^2/n}$$

The numerator is the sum of differences between predictions for *future periods outside the sample* which has been used for the estimation of the model, and realisations. The denominator contains the sum of differences of 'predictions' and realisations over the period of the sample.

Conceptually, the Janus quotient refers to the time period $t = n$, that lies at the end of the sample period and the beginning of the future prediction period. The Janus quotient takes into account both the prediction performance of the model in future periods as well as the 'prediction' performance in the past (sample) periods. Schematically we may present the periods involved in the estimation of the Janus Quotient as follows.

$t = 1, 2, \ldots, n$ $\quad$ $t = (n+1), (n+2), \ldots, (n+m)$

Sample period J $\quad$ Future prediction period

The definition of the Janus Quotient shows that J is non-negative. Its values may vary between zero and ∞

$$0 \leqslant J \leqslant \infty$$

Furthermore, it should be clear that if the structure of the model remains in the future the same as in the period of the sample, the Janus Quotient will be approximately equal to unity

$$J \approx 1$$

The higher the value of J the poorer the forecasting performance of the model. Furthermore, values of J higher than unity are suggestive, under certain conditions, of changes in the structure of the model.

EXERCISES

1 Consider the following consumption function estimated from 20 observations

$$\hat{C}_t = 81 + 0.75\,Y$$
$$(0.05)$$

$\bar{Y} = 750 \qquad \Sigma(Y - \bar{Y})^2 = 7{,}560 \qquad R^2 = 0.950 \qquad \hat{\sigma}_u^2 = 2.3$

[1] See H. Wold (editor), *Econometric Model Building*, North-Holland, 1964, article by A. Gadd and H. Wold, pp. 229–35.

where $\quad Y$ = disposable income

Find the value of C in 1980, given that Y_{1980} = 980 billions of dollars. Construct a 95 per cent confidence interval for your forecast.

2 The following equation is the cost function of a firm, estimated on the basis of yearly observations over the 1950–64 period

$$Y = 2434.6 + 85.70X - 0.03X^2 + 0.00004X^3$$

where $\quad Y$ = total costs (in £), X = output (in arbitrary units).

The output of the firm in 1970 was 1,950 and its cost of production amounted to £295,650.

Test the significance of the difference $Y_F - Y_A$ (at the 95 per cent level).

3 The following table shows the forecasts of the exports (in £ m) of a certain country obtained from two different models and the actual exports. Which model do you think yields better forecasts? (Note that the forecasts refer to the years included in the sample used for the estimation of the export models.)

Year	*Actual exports* Y_A	*Forecast exports from model I* Y_{F1}	*Forecast exports from model II* Y_{F2}
1950	20	30	28
1951	30	25	26
1952	32	28	36
1953	25	28	25
1954	29	25	21
1955	26	30	39
1956	32	30	34
1957	36	39	30
1958	32	34	40
1959	31	30	38

4. Assume that the estimated models in the previous example were used for forecasting the value of exports for the period 1965–70 (a period outside the sample). The forecasts and realised exports are as follows.

Year	Y_A	Y_{F1}	Y_{F2}
1965	38	38	37
1966	42	40	36
1967	46	52	48
1968	50	54	51
1969	52	48	50
1970	54	52	54

Given the above results would you revise your conclusions derived from Exercise 3 concerning the forecasting performance of the two export models?