Unit III

Techniques of derivatives – simple, partial and cross partial derivatives – maxima and minima of function of two variables-conditions

Derivative

❖ DEFINATION OF DERIVATION?

1. The derivatives is the exact rate at which one quantity changes with respect to another.
2. Geometrically, the derivatives is the slope of curve at point on curve.
3. The derivatives is often called the instantaneous rate of change.
4. The derivatives of a function represents an infinitely small change the fuction with respect to one of its variable.

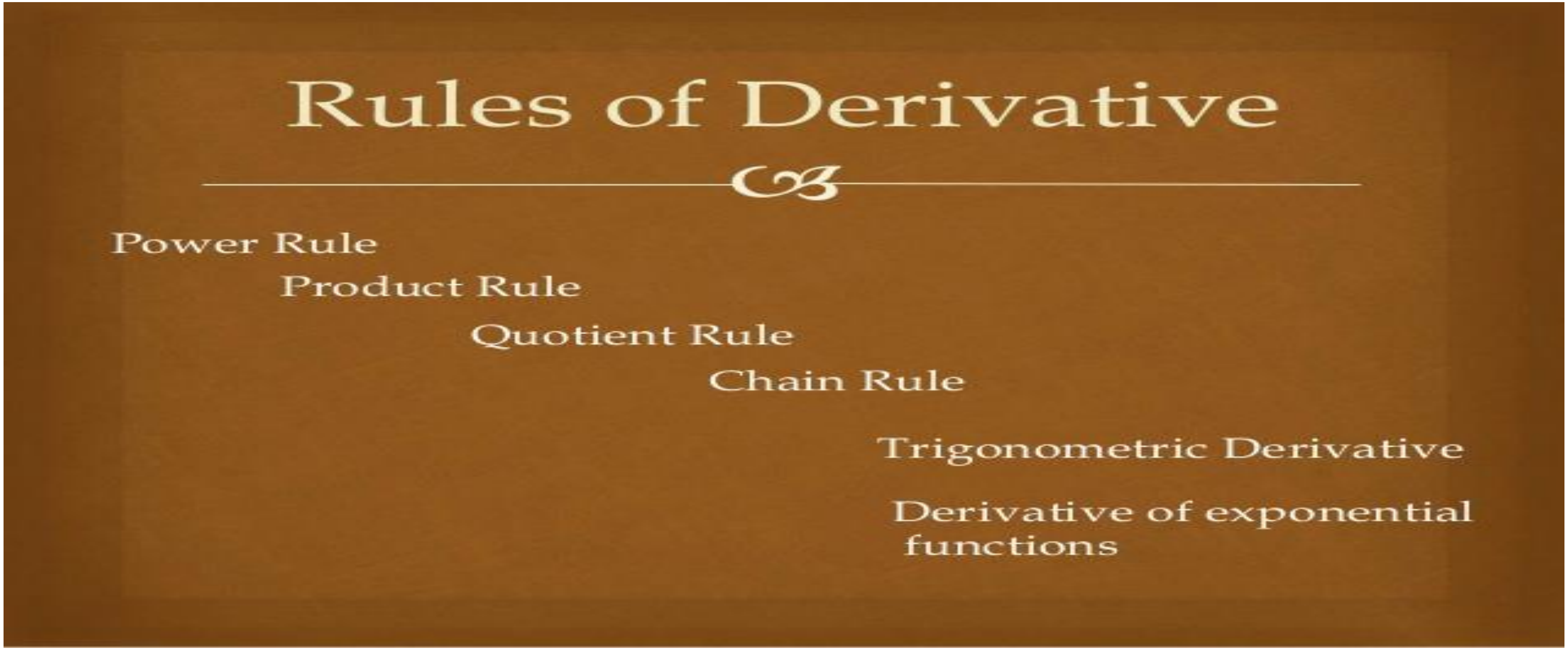
Its is written as $\frac{dy}{dx}$

Derivative

Notations of Derivative of $y=f(x)$

- $f'(x)$
- y'
- dy/dx
- $d/dx \, f(x)$

Rule …


Rules of Derivative
Power Rule
Product Rule
Quotient Rule
Chain Rule
Trigonometric Derivative
Derivative of exponential functions

Rule…

Power Rule:

Example: What is $\frac{d}{dx}x^3$?

The question is asking "what is the derivative of x^3 ?"

We can use the Power Rule, where n=3:

$\frac{d}{dx}x^n$ = nxn−1

$\frac{d}{dx}x^3 = 3x^{3-1} = 3x^2$

(In other words the derivative of $x^3$is$3x^2$)

Rule..

Sum Rule:

Example: What is the derivative of x^2+x^3 ?

The Sum Rule says:

the derivative of f + g = f' + g'

So we can work out each derivative separately and then add them.

Using the Power Rule:

$\frac{d}{dx} x^2 = 2x$

$\frac{d}{dx} x^3 = 3x^2$

And so:

the derivative of $x^2 + x^3 = 2x + 3x^2$

Rule…

Product Rule

- To find the derivative of a problem where we need to multiply (example: $(x^3)\ (x^2+2)$) then we use the product rule:

$y = u \cdot v$

$y' = u \cdot v' + v \cdot u'$

$y = (x^3)(x^2+2)$

$u = x^3$

$v = (x^2+2)$

$u' = 3x^2$

$v' = 2x$

$y' = (x^3)(2x) + (x^2+2)(3x^2)$

Rule..

Chain Rule

We use chain rule when there is a problem usually inside a parenthesis (then raised to a power), or under a radical or in the denominator. We substitute this with the letter *u*. so we solve the derivative for *u* and later replace it for the original function.

For example: y=(x3+1)9

y=(u)9

dy=9u8

du

u=x3+1

du=3x2

dx

Rule…

The Quotient Rule

If $k(x) = \dfrac{f(x)}{g(x)}$, then:

$$k`(x) = \frac{f`(x)g(x) - g`(x)f(x)}{(g(x))^2}$$

Using Leibniz notation,

If $y = \dfrac{f(x)}{g(x)}$, then:

$$\frac{dy}{dx} = \frac{\frac{df}{dx}.g(x) - \frac{dg}{dx}.f(x)}{(g(x))^2}$$

OR

$$\frac{dy}{dx} = \frac{f`g - g`f}{g^2}$$

Derivative

Higher derivatives. 1st ,2nd 3rd

- Any derivative beyond the first derivative can be referred to as a higher order derivative.
- The derivative of the function f(x) may be denoted by f'(x)
- Its double (or "second") derivative is denoted by f ''(x).
- This is read as "f double prime of x," or "The second derivative of f(x)."

Derivatives

Higher Order Derivative Ex.

Find the first four derivatives of $f(x) = 4x^3 + 5x^2 + 3$.

Solution

$$f'(x) = 12x^2 + 10x, \quad f''(x) = 24x + 10, \quad f'''(x) = 24, \quad f^{(4)}(x) = 0$$

Derivatives

Partial derivatives

Partial derivatives are defined as derivatives of a function of multiple variables when all but the variable of interest are held fixed during the differentiation

- **Let** $z = f(x, y)$.

  The partial derivative $f_x(x, y)$ **can also be written as**

  $$\frac{\partial f}{\partial x}(x, y) \text{ or } \frac{\partial z}{\partial x}.$$

  Similarly, $f_y(x, y)$ **can also be written as**

  $$\frac{\partial f}{\partial y}(x, y) \text{ or } \frac{\partial z}{\partial y}.$$

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Higher order…

Higher Orders Partial Derivatives

- For the function $z = f(x,y)$ the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are in turn functions of x and y, in general. So, we can think of second partial derivatives of z, but in this case there are three different second derivatives:

$$z_{xx} = f_{xx} = \frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$$

$$z_{yy} = f_{yy} = \frac{\partial\left(\frac{\partial z}{\partial y}\right)}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$$

Second-order direct partial derivatives

$$z_{xy} = f_{xy} = \frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y.\partial x}$$

Second-order cross partial derivative

Higher order…

The Equality of Mixed (Cross) Partial Derivatives

$$z_{yx} = f_{yx} = \frac{\partial\left(\frac{\partial z}{\partial y}\right)}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x.\partial y}$$

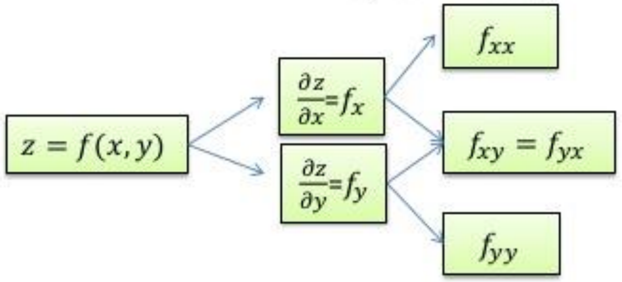
} Second-order cross partial derivative

- If the cross (mixed) partial derivatives f_{xy} and f_{yx} are continuous and finite in their domain then they are equal to one another; i.e.

$$f_{xy} = f_{yx}$$

Or

$$\frac{\partial^2 z}{\partial y.\partial x} = \frac{\partial^2 z}{\partial x.\partial y}$$



Deriatives …

Total Differential

- For a multi variables scalar function the same rule applies:

$$z = f(x_1, x_2, \dots, x_n)$$

$$dz = \frac{\partial z}{\partial x_1}.dx_1 + \frac{\partial z}{\partial x_2}.dx_2 + \cdots + \frac{\partial z}{\partial x_n}.dx_n$$

- in the case of two variables function $z = f(x, y)$ we assumed x and y are independent, but if they depend on other variables the differential of each one of them can be treated as the total differential of a dependent variable, that is;

$$z = f(x, y) \rightarrow dz = \frac{\partial z}{\partial x}.dx + \frac{\partial z}{\partial y}.dy \qquad A$$

$$x = h(r, s) \rightarrow dx = \frac{\partial x}{\partial r}.dr + \frac{\partial x}{\partial s}.ds \qquad B$$

$$y = k(r, s) \rightarrow dy = \frac{\partial y}{\partial r}.dr + \frac{\partial y}{\partial s}.ds \qquad C$$

Substituting B and C into A:

$$dz = \frac{\partial z}{\partial x}.\left(\frac{\partial x}{\partial r}.dr + \frac{\partial x}{\partial s}.ds\right) + \frac{\partial z}{\partial y}.\left(\frac{\partial y}{\partial r}.dr + \frac{\partial y}{\partial s}.ds\right)$$

Maxima and minima conditions..

- **Differentiate** the given function.
- let $f'(x) = 0$ and find critical numbers.
- Then find the second **derivative** $f''(x)$.
- Apply those critical numbers in the second **derivative**.
- The function $f(x)$ is **maxima** when $f''(x) < 0$.
- The function $f(x)$ is **minima** when $f''(x) > 0$.

Applications of Differentiation

- Single Variable Case In economics the differential calculus - It is convenient at this stage to list some of the functional relationships which recur most frequently in the work of the economists:
- A production function $Q = f(L)$ which records the maximum amount of output that can be produced with given amount of labour.
- A cost function $C = f(Q)$records the total expenses C associated with production level Q.
- A utility function $U(Q)$, which measures the pleasure that the individual derives from the ownership of some quantity of Q of some commodity.
- A revenue function $P.Q = Q.F(Q)$, which shows the total income of the firm when it sells Q units of a commodity at the price P per unit; Economists have then adopted the following terminology:
- Marginal product is the name given to dQ/dL
- Marginal cost refers to dC/dQ
- Marginal utility refers to dU/dQ Marginal revenue refers to $d(P.Q)\ dQ$

Application. ..

- Example 3. (a) For the total revenue function $TR = 500q - 2q^2$,

Find the value of MR when $q = 20$ (b) If $P = 80 - 4q$ is the linear demand function, write out the total revenue and hence the marginal revenue functions .

Solution: a) $MR = \frac{d(TR)}{dq} = 500 - 4q$, when $q = 20$

$MR = 500 - 80 = 420$

(b) We know by definition that $TR = pq$. $\Rightarrow TR = (80 - 4q)\,q = 80q - 4q^2$
$MR = \frac{d(TR)}{dq} = 80 - 8q$

Profit Maximisation of a Firm

- Profit Maximisation of a Firm -well acquainted with the marginal cost equals marginal revenue as a perquisite for profit maximisation.
- If R(Q) is the revenue function of a0 firm and C(Q) the cost function. From these it follows that a profit function **π** may be formulae as $\pi = R(Q) - C(Q)$ ---(1)
- (i) A firm sets its output where its marginal profit is zero. We obtain this result formally using first order condition for a profit maximisation.
- We set the first derivative of the profit function, equation (i) with respect to quantity equal to zero. $d\pi / dQ = 0$. (2)
- (ii) Equation (ii) is a necessary condition for profit to be maximised. Sufficiency requires, in addition, that the second order condition hold: $d^2\pi / dQ^2 < 0$.
- (iii) Because profit is a function of revenue and cost, we can state the above in one additional way. The first order condition can be stated by setting the first order derivative of $\pi = R(Q) - C(Q)$ equal to zero. $d\pi / dQ = dR / dQ - dC / dQ = MR - MC = 0 \Rightarrow MR = MC$.

Application…

- The first order condition can be stated by setting the first order derivative of $\pi = R(Q) - C(Q)$ equal to zero. $d\pi\ dQ = dR\ dQ - dC\ dQ = MR - MC = 0 \Rightarrow MR = MC$.
- . (iv) Equation (iv) is a first order condition which states that for profit to be maximised, the marginal cost must be equal to marginal revenue of the output. For profit to be maximised,
- the second order condition must hold. $d\ ^2\pi\ /dQ^2 = d\ ^2R\ /dQ^2 - d\ ^2C\ dQ^2 = d(MR)\ dQ - d(MC)\ dQ < 0$

That is, for profit to be maximised, the slope of the marginal revenue curve, d(MR)/dQ must be less than the slope of the marginal cost, (d(MC))/dQ curve.

References

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