Replacement Models

11.1 Introduction

The replacement problems are concerned with the situations that arise when some items such as men, machines, electric light bulbs etc need replacement due to their decreased efficiency, failure or breakdown. Such decreased efficiency or complete breakdown may either be gradual or all of a sudden.

The replacement problem arises because of the following factors :

i) The old item has become worse or requires expensive maintenance.

ii) The old item has failed due to accident.

iii) A more efficient design of equipment has become available in market.

Thus the problem of replacement is to decide the best policy to determine the age at which the replacement is most economical instead of continuing at increased cost due to factor like maintenance. The objective is to find the optimum period of replacement. We shall discuss the following main type of replacement situations:

i) Replacement of items that deteriorate with time.

ii) Replacement of items which do not deteriorate but fail after certain amount of use.

For items which do not deteriorate but fail all of a sudden, following are the two types of replacement policies :

i) ***Individual replacement policy*** : Under this policy, an item is replaced immediately after its failure.

ii) ***Group replacement policy*** : Under this policy, we take decisions as to when all the items must be replaced, irrespective of the fact that items have failed or have not failed, with the provision that, if any item fails before the replacement time it may be individually replaced.

Remarks : In numerical problems we consider the minimum value of the average annual cost (*i.e.*,) minimum of $\frac{P(n)}{n}$ to determine the optimum replacement period.

Example 1 : A machine owner finds from his past records that the costs per year of maintaining a machine whose purchase price is Rs. 6000 are as given below:

Year	**1**	**2**	**3**	**4**	**5**	**6**
Maintenance Cost (Rs)	**1000**	**1200**	**1400**	**1800**	**2300**	**2800**
Resale Value (Rs)	**3000**	**1500**	**750**	**375**	**200**	**200**

Determine at what age is replacement due ?

[MU. MBA Apr 97, MU. BE. 80]

Solution : See Table below. Here C = Rs. 6000, S = scrap value.

Year (n) (1)	Main. cost (R_n) (Rs) (2)	Total Main. cost ΣR_n (Rs) (3)	C – S (Rs) (4)	Total Cost = (3) + (4) = P(n) (5)	Ave. cost $= \frac{(5)}{(1)} = \frac{P(n)}{n}$ (6)
1	1000	1000	3000	4000	4000
2	1200	2200	4500	6700	3350
3	1400	3600	5250	8850	2950
4	1800	5400	5625	11,025	2756
5	2300	7700	5800	13,500	2700*
6	2800	10,500	5800	16,300	2717

Minimum cost is in 5th year $\Rightarrow$ optimum replacement plan : replace the machine at the end of 5th year.

Example 2 : The cost of a machine is Rs 6100 and its scrap value is Rs.100. The maintenance costs found from experience are as follows :

Year	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Main. Cost (Rs)	**100**	**250**	**400**	**600**	**900**	**1200**	**1600**	**2000**

When should the machine be replaced ? ***[MU. BE. Mech Oct 96]***

Solution : Since the scrap value of the machine is Rs.100 resale value of the machine after a year remains constant throughout. The costs required can now be calculated as follows.

Year (n) (1)	Main. cost (f_n) (2)	Cumulative Mai. cost Σf_n (3)	C – S (4)	Total Cost T=(3)+(4) (5) say	Ave. cost $T_A = (5)/n$ (6)
1	100	100	6000	6100	6100
2	250	350	6000	6350	3175
3	400	750	6000	6750	2250
4	600	1350	6000	7350	1837
5	900	2250	6000	8250	1650
6	1200	3450	6000	9450	1575*
7	1600	5050	6000	11,050	1579
8	2000	7050	6000	13,050	1631

Here T_A is minimum during 6th year.

Hence the machine should be replaced at the end of 6^{th} year.

Example 3 : **A taxi owner estimates from his past records that the costs per year for operating a taxi whose purchase price when new is Rs. 60,000 are as given below:**

Age :	**1**	**2**	**3**	**4**	**5**
Operting Cost (Rs)	**10,000**	**12,000**	**15,000**	**18,000**	**20,000**

After 5 years, the operating cost is Rs. 6000 k where k = 6, 7, 8, 9, 10 (k denoting age in years). If the resale value decreases by 10% of purchase price each year, what is the best replacement policy ? Cost of the money is zero. *[MU. BE. Apr 91]*

Solution : 10% of purchase price $= \text{Rs. } 60{,}000 \times \frac{10}{100}$

$= \text{Rs. } 6000$

Thus the resale value decreases by Rs. 6000 every year, which means (C – S) increases by Rs. 6000 every year.

Average annual cost of the taxi is computed as below.

(ii) In a similar fashion we prepare a table for B.

(1) year *(n)*	(2) Running cost R_n	(3) Commutative running cost	(4) C – S	(5) (5) = (3)+(4)	(6) $= \frac{(5)}{(1)}$
1	400	400	10,000	10,400	10,400
2	1200	1600	10,000	11,600	5800
3	2000	3600	10,000	13,600	4533
4	2800	6400	10,000	16,400	4100
5	3600	10,000	10,000	20,000	4000*
6	4400	14,400	10,000	24,400	4066

The above Table indicates that machine 'B' should be replaced at the end of 5^{th} year.

Since the lowest average cost of Rs.4000 for machine B is less than the lowest average cost of Rs.5200 for machine A, machine A can be replaced by machine B.

Now we have to determine as to when A should be replaced. **Machine A should be replaced when the cost for next year of running this machine becomes more than the average yearly cost for machine B**

Now total cost for machine A in the first year = Rs. 9200

Total cost for machine A in the

II year = Rs. 11,400 – Rs. 9200

= Rs.2200

III year = Rs. 4200 (15600 – 11400)

IV year = Rs. 6200 (21800 – 15600)

As the cost of running machine A in III year (Rs.4200) is more than the average yearly cost for machine B (Rs 4000) ; machine A should be replaced at the end of two years (i.e.,) one year after it is one year old.

Example 6 : **A machine shop has a press which is to be replaced after it wears out. A new press is to be installed now. Further an optimum replacement is to be found for next 7 years after which the press is no longer required.**

The following data is given :

Year	Installation cost at beginning of year (Rs)	Salvage Value at end of year (Rs)	Operating cost during the year (Rs)
1	200	100	60
2	210	50	80
3	220	30	100
4	240	20	120
5	260	15	150
6	290	10	180
7	320	0	230

Find the optimum replacement plan and the corresponding minimum cost.

Solution : Using the given information, the minimum average annual cost of the press is computed in the following table.

Year (n)	f_n	Σf_n	C – S	T	T_A
1	60	60	200 – 100 = 100	160	160
2	80	140	210 – 50 = 160	300	150
3	100	240	190	430	143*
4	120	360	220	580	145
5	150	510	245	755	151
6	180	690	280	970	162
7	230	920	320	1240	177

Since T_A is minimum for $n = 3$, the machine should be replaced every third year. [*Ans*]

11.3 Money Value, Present worth factor (pw$f$) and Discount Rate

Money Value : Since money has a value over time, we often speak: money is worth 10% per year. This can be explained in the following 2 ways. *(i)* In one way, spending Rs.100 today would be equivalent to spending Rs.110 in a year's time

(ii) consequently one rupee after a year from now is equivalent to $(1 \cdot 1)^{-1}$ rupee today.

Present worth factor (pwf) ***[MU. MBA Nov.96]***

As we have just seen above, one rupee a year from now is equivalent to $(1 \cdot 1)^{-1}$ rupee today at the interest rate 10% per year. one rupee spent two years from now is equivalent to $(1 \cdot 1)^{-2}$ today. Similarly we can say one rupee spent 'n' years from now is equivalent to $(1 \cdot 1)^{-n}$ today. The quantity $(1 \cdot 1)^{-n}$ is called present worth factor (pw$f$) or present value of one rupee spent n years from now.

Discount rate (Depreciation Value) ***[MU. MBA Nov.96]***

The present worth factor of unit amount to be spent after one year is given by $V = (1 + r)^{-1}$ were r is the interest rate. Then V is called discount rate (technically known as depreciation value)

Theorem : The maintenance cost incre

will give the desired result (minimum ...

Example 7 : Let the value of the money be 10% per year and suppose that machine A is replaced after every 3 years whereas machine B is replaced after every six years.

The yearly cost of both machines are given as under :

Age :	1	2	3	4	5	6
Machine A :	1000	200	400	1000	200	400
Machine B :	1700	100	200	300	400	500

Determine which machine should be purchased ?

[MU. BE. Nov 93]

Solution : Present worth factor $V = \dfrac{100}{100+10} = \dfrac{10}{11}$

$\therefore$ Total discount cost (present worth) of A for 3 years is

$$\text{Rs.} \left[1000 + 200 \times \frac{10}{11} + 400 \times \left(\frac{10}{11}\right)^2\right]$$

$$= \text{Rs. } 1512 \text{ [nearly]}$$

Again the total discount cost of B for six years is

$$\text{Rs.} \left[1700 + 100 \times \left(\frac{10}{11}\right) + 200 \times \left(\frac{10}{11}\right)^2 + 300 \times \left(\frac{10}{11}\right)^3 + 400 \times \left(\frac{10}{11}\right)^4 + 500 \times \left(\frac{10}{11}\right)^5\right]$$

$$= \text{Rs. } 2765$$

$$\text{Average yearly cost of A} = \frac{1512}{3} = \text{Rs. } 504$$

$$\text{and average yearly cost of B} = \frac{2765}{6} = \text{Rs. } 461$$

Although this shows that the apparent advantage with B, but the comparison is unfair because the periods of consideration are different. So if we consider 6 year period of machine A also, then the total discount of A will be $1000 + 200 \times \dfrac{10}{11} + \ldots + 400 \times \left(\dfrac{10}{11}\right)^5 = \text{Rs. } 2647$

which is 118 Rs less costlier than machine B over the same period.

$\Rightarrow$ Machine A should be purchased [*Ans*]

Example 8 : **A pipeline is due for repairs. It will cost Rs.10,000 and lasts for 3 years. Alternatively a new pipeline can be laid at a cost of Rs. 30,000 and lasts for 10 years. Assuming cost of capital to be 10% and ignoring salvage value, which alternative should be chosen.**

[MU. BE. May '96]

Solution : Consider two types of pipeline for infinite replacement cycles of 10 years for the new pipeline, and 3 years for the existing pipeline.

The present worth factor is $V = \dfrac{100}{100+10} = 0.9091.$

Let D_n denote the discounted value of all future costs associated with a policy of replacing they equipment after 'n' years. Then if we designate the initial outlay by C,

$$D_n = C + V^n C + V^{2n} C + \ldots\ldots$$

$$= C\,[1 + V^n + V^{2n} + \ldots\ \ldots]$$

$$= \frac{C}{1 - V^n}$$

Now substituting the values of C, V, n for two types of pipelines ; the discounted value for the existing pipeline is given by

$$D_3 = \frac{10{,}000}{1 - (0.9091)^3} = \text{Rs. } 40{,}258$$

and for the new pipeline

$$D_{10} = \frac{30{,}000}{1 - (0.9091)^{10}} = \text{Rs. } 48820$$

Since $D_3 < D_{10}$, the existing pipeline should be continued. [*Ans*]

Alternatively, the comparison may be made over $3 \times 10 = 30$ years.

Example 9 : **The cost patterns of 2 machines A and B, when money value is not considered is given below :**

Year	Machine A	Machine B
1	900	1400
2	600	100
3	700	700

Find the cost patterns for each machines when money is worth 10% per year, and hence find which machine is less costly.

[MU. BE. Nov 91]

Solution : The total outlay for three years for

Machine A = 900 + 600 + 700 = Rs. 2200 and also for

Machine B = 1400 + 100 + 700 = Rs. 2200

Here we observe that the total outlay for either machine is same for three years when money value is not taken into account. Hence both the machines appear equally good.

Now consider the money value at the rate of 10% per year and from the following table we get the discounted costs for A, B.

Year	1	2	3	Total cost
Machine A	900	$600 \times \frac{100}{110}$	$700 \times \left(\frac{100}{11}\right)^2$	Rs. 2023.93
Machine B	1400	$600 \times \frac{100}{110}$	$700 \times \left(\frac{100}{11}\right)^2$	Rs. 2069.43

Machine A is preferred [*Ans*]

Example 10 : **Assume that the present value of one rupee to be spent in a year's time is Re 0.9 and C = Rs. 3000, capital cost of equipment and the running costs are given in the table below. When should the machine be replaced?** ***[BRU. BE. Apr 97, MSU. BE. Apr 97]***

Year :	**1**	**2**	**3**	**4**	**5**	**6**	**7**
ung. cost (Rs) :	**500**	**600**	**800**	**1000**	**1300**	**1600**	**2000**

Solution : Consider the following table :

Year n (1)	R_n (2)	V^{n-1} (3)	R_nV^{n-1} (4)	ΣR_nV^{n-1} (5)	$C+\Sigma R_nV^{n-1}$ (6)	ΣV^{n-1} (7)	$W(n)=\frac{6}{7}$
1	500	1	500	500	3500	1	3500
2	600	0.90	540	1040	4040	1.90	2126.32
3	800	0.81	648	1688	4688	2.71	1729.88
4	1000	0.73	730	2418	5418	3.44	1575
5	1300	0.66	858	3276	6276	4.10	1530.73
6	1600	0.59	944.784	4220.78	7220.78	4.69	1539.61

Since W(n) is minimum at 6th year optimum replacement plan is end of sixth year.

Example 11 : **The cost of a new machine is Rs.5000. The maintenance cost of n^{th} year is given by $C_n = 500\ (n-1)$; $n = 1, 2$ ... suppose that money is worth 5% per year, after how many years will it be economical to replace the machine by a New one ?**

[MU. BE. Apr 96]

Solution : The present worth of the money to be spent a year from now is

$$V = (1 + 0.05)^{-1} = 0.9523$$

The optimum replacement time is determined in the following table.

Year n	R_n	V^{n-1}	R_nV^{n-1}	$C+\Sigma R_nV^{n-1}$	ΣV^{n-1}	$W(n)=\frac{5}{6}$
1	2	3	4	5	6	7
1	0	1.0000	0	5000	1.000	5000
2	500	0.9523	476	6476	1.9523	2805
3	1000	0.9073	907	6383	2.8593	2232
4	1500	0.8638	1296	7679	3.7231	2063
5	20000	0.8227	1645	9324	4.5458	2061
6	2500	0.7835	1959	11,283	5.3293	2117

Since $W(n)$ is minimum for $n = 5$ and $R_4 = 1500 < w(5)$ as well as $w(5) > R_6 = 2500$, it is economical to replace the machine by a new one at the end of 5 years. [*Ans*].

Example 12 : A production machine installed has initial investment of Rs. 30,000 and its salvage value at the end of *i* years of its use is estimated as Rs. $\frac{30,000}{i+1}$. The annual operating and maintenance cost in the first year is Rs. 15,000 and increases by Rs. 1000 in each subsequent years for first five years and increases by Rs. 5000 in each year thereafter. Replacement policy is to be planned over a period of seven years. During this period cost of capital may be taken as 10% per year. Solve the problem for optimal replacement.

Ans : Here $C = 30,000,\ V = \frac{1}{1.10}$

ΣV^{n-1}	Year (n)	R_n	V^{n-1}	R_nV^{n-1}	ΣR_nV^{n-1}	S_n	S_nV^n	$\frac{[C-S_nV^n+\Sigma R_nV^{n-1}]}{\Sigma V^{n-1}}$
1.000	1	15000	1.0000	15000	15000	15000	13635	31365
1.909	2	16000	0.909	14544	29544	10000	8260	26864
2.735	3	17000	0.826	14042	43586	7500	5633	24846
3.486	4	18000	0.751	13518	57104	6000	4098	23811
4.169	5	19000	0.683	12977	70081	5000	3105	23261*
4.790	6	24000	0.621	14904	84985	4286	2422	23500
5.355	7	29000	0.565	16385	101370	3750	1926	24173

Optimum replacement plan is after 5 years.

11.4 Group Replacement Policy

Example 15 : The following failure rates have been observed for certain items.

End of month	:	1	2	3	4	5
Probability of failure todate	:	0.10	0.30	0.55	0.85	1.00

The cost of replacing an individual item is Rs.1.25. The decision is made to replace all items simultaneously at fixed intervals and also replace individual items as they fail. If the cost of group replacement

is 50 paise, what is the best interval for group replacement. At what group replacement per item, would a policy of strictly individual replacement become preferable to the adopted policy.

[MU. BE. Nov 94]

Solution : Assume that items failing during a month are replaced at the end of the month.

Suppose that there are 1000 items in use. Let p_i be the probability that an item, which was new when placed in position for use, fails during i^{th} month of its life. Thus, we have

$$p_1 = 0.10$$

$$p_2 = 0.30 - 0.10 = 0.20$$

$$p_3 = 0.55 - 0.30 = 0.25$$

$$p_4 = 0.85 - 0.55 = 0.30$$

$$p_5 = 1.00 - 0.85 = 0.15$$

Since the sum of probabilities is one, all the probabilities beyond p_5 will be taken as zero.

Let N_i denote the number of replacements at the end of the month. Then we have

$$N_0 = \text{Number of items in the beginning} = 1000$$

$$N_1 = N_0 p_1 = 100$$

$$N_2 = N_0 p_2 + N_1 p_1 = 200 + 10 = 210$$

$$N_3 = N_0 p_3 + N_1 p_2 + N_2 p_1 = 250 + 20 + 21 = 291$$

$$N_4 = N_0 p_4 + N_1 p_3 + N_2 p_2 + N_3 p_1 = 396$$

$$N_5 = N_0 p_5 + N_1 p_4 + N_2 p_3 + N_3 p_2 + N_4 p_1 = 331$$

The expected life of each item $= \sum_{i=1}^{5} i p_i$

$$= 1 \times p_1 + 2 \times p_2 + 3 \times p_3 + 4 \times p_4 + 5 \times p_5$$

$$= 1 \times 0.10 + 2 \times 0.20 + 3 \times 0.25 + 4 \times 0.30 + 5 \times 0.15 = 3.20$$

Average number of failures per month

$$= \frac{1000}{3.2} = 313 \text{ (app)}$$

Since the replacement of all the 1000 items simultaneously costs Rs.0.50 per item and the replacement of individual item on failure costs Rs.1.25, the average cost for different group replacement policies is shown below :

End of month	Individual replacement	Total cost (Rs) individual + group	Average cost (Rs)
1	100	1000 × 0.50 + 1.25 × 100 = 625	625
2	310	1000 × 0.50 + 1.25 × 310 = 887.7	443.8
3	601	1000 × 0.50 + 1.25 × 601 = 1251.3	417.1*
4	997	1000 × 0.50 + 1.25 × 997 = 1746.3	436.6

Since the average cost is lowest in the 3rd month, it is optimal to have a group replacement after every 3rd month. Further, since the average cost is more than Rs. 391.25 for individual replacement, the policy of individual replacement is preferable.

Example 16 : Let $p(t)$ be the probability that a machine in a group 30 machines would break down in period t. The cost of repairing broken machine is Rs.200.00 Preventive maintenance is performed by servicing all the 30 machines at the end T units of time. Preventive maintenance cost is Rs.15 per machine. Find optimal T which will minimize the expected total cost per period of servicing, given that

$$p(t) = \begin{cases} 0.03 \text{ for } t = 1 \\ p(t-1) + 0.01 \text{ for } t = 2, 3, \ldots, 10 \\ 0.13 \text{ for } t = 11, 12, 13, \ldots \end{cases}$$

[MU. BE. Apr 98]

Solution : Here

t =	1	2	3	4	5	6	7	8	9	10	11	12
$p(t)$ =	.03	.04	.05	.06	.07	.08	.09	0.10	0.11	0.12	0.13	0

Since the sum of all probabilities can never be greater than one this means $p_{12} = 0, p_{13} = 0$ *etc.*

A machine which has lasted upto 11^{th} period is sure to fail in 12^{th} period.

Let N_i be the number of machines at the end of i^{th} period.

$$\Rightarrow N_0 = 30$$

$$N_1 = N_0 p_1 = 30 \times 0.03 = 0.9 \approx 1$$

$$N_2 = N_0 p_2 + N_1 p_1$$

$$= 30 \times 0.04 + 1 \times 0.03$$

$$= 1.23 \approx 1$$

$$N_3 = N_0 p_3 + N_1 p_2 + N_2 p_2$$

$$= 30 \times 0.05 + 1 \times 0.04 + 1 \times 0.03$$

$$= 1.57 \approx 2$$

$$N_4 = N_0 p_4 + N_1 p_3 + N_2 p_2 + N_3 p_1$$

$$= 1.95 \approx 2$$

Similarly $N_5 = 2,\ N_6 = 3,\ N_7 = 3, N_8 = 4,$

$N_9 = 4,\ N_{10} = 5,\ N_{11} = 6.$

Since the expected life of each machine, $\sum_{i=1}^{11} ip_i = 6.41$ time units we have average number of machines failed per period is $\frac{30}{6.41} = 5$ (app).

$\therefore$ Cost of individual replacement

$$= \text{Rs. } 5 \times 200 = \text{Rs. } 1000$$

Group maintenance cost is computed below :

End of Period	Cost of Maintenance in group	Average cost of Maintenance per period
1	Rs $(30 \times 15) + 1 \times 200 = 650$	Rs. 650
2	Rs $(30 \times 15) + 2 \times 200 = 850$	Rs. 425
3	Rs $(30 \times 15) + 4 \times 200 = 1250$	Rs. 417
4	Rs $(30 \times 15) + 6 \times 200 = 1650$	Rs. 412
5	Rs $(30 \times 15) + 8 \times 200 = 2050$	Rs. 410*
6	Rs $(30 \times 15) + 11 \times 200 = 2650$	Rs. 442

Since the minimum cost occurs in the 5^{th} period it is optimal to maintain all the machines upto 5^{th} period. [*Ans*]

Example 17 : **There is a large number of light bulbs, all of which must be kept in working order. If a bulb fails in service, it costs Re.1 to replace it, but if all the bulbs are replaced in the same operation, it costs only 35 paise a bulb. If the proportion of bulbs failing in successive time intervals is known, decide on the best replacement policy and give reason. The following mortality rates for light bulbs have been observed.**

Proportion failing during first week = 0.09
Proportion failing during second week = 0.16
Proportion failing during third week = 0.24
Proportion failing during fourth week = 0.36
Proportion failing during fifth week = 0.12
Proportion failing during sixth week = 0.03

Solution : Let number of bulbs initially be $N_0 = 10000$ (say)

If p_i denote the probability of failure during i^{th} week then $p_1 = 0.09$, $p_2 = 0.16, p_3 = 0.24, \; p_4 = 0.36, p_5 = 0.12, p_6 = 0.03,$

Now N_i denote the number of replacement at the end of i^{th} week.

$$\text{Then } N_0 = 10000$$

$$N_1 = N_0 p_1 = 1000 \times 0.09 = 900$$

$$N_2 = N_0 p_2 + N_1 p_1 = 1681$$

$$N_3 = N_0 p_3 + N_1 p_2 + N_2 p_1 = 2695$$

Similarly $N_4 = 4324, \; N_5 = 2747,$

$$N_6 = 2599$$

$$\text{The expected life of each bulbs } \sum_{i=1}^{i=6} i p_i = 3.35$$

$$\text{Average number of failures per week} = \frac{10,000}{3.35} = 2985 \text{ app.}$$

$$\text{The cost of individual replacement } 2985 \times 1 = \text{Rs. } 2985$$

Now the average cost of different group replacement is as follows:

End of week	Individual replacement	Total cost (Rs) individual + group	Average cost (Rs)
1	900	10,000 × 0.35 + 900 × 1 = 4400	4400
2	2581	10,000 × 0.35 + (2581 × 1) = 6081	3041
3	5276	10,000 × 0.35 + (5276 × 1) = 8776	2925*
4	9550	10,000 × 0.35 + (9550 × 1) = 13,050	3263
5	12,297	10,000 × 0.35 + (12,297 × 1)	3159

∴ It is optimal to have group replacement every 3rd week [4m]

Also average cost is less than Rs. 2985 for individual replacement, the policy of group replacement is preferred. [4m]