**DEPARTMENT OF STATISTICS,**

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## UNIT V

Sequential-sampling plans In item-by-item sequential sampling, the decision to: accept; reject; continue sampling; is made after each item is inspected. The acceptance and rejection numbers usually correspond to parallel sloping lines on the chart of (number nonconforming) versus (number inspected). Inspection could continue indefinitely but is usually terminated after a reasonable number of items have been inspected.

The parallel lines are found using Wald's sequential probability ratio test.

For given PRP = $(p_1, 1 - \alpha)$ and CRP = $(p_2, \beta)$, the lines are $X_A = -h1 + sn$ (acceptance line) and $X_R = h_2 + sn$ (rejection line) where $h_1$, $h_2$, and s are calculated from p1, p2, $\alpha$, and $\beta$.

The ultimate in multiple sampling is sequential sampling which provides for infinite number of stages for arriving at a decision. In sequential sampling, sample items are examined one at a time and after each item inspected one of three decisions, *viz*., to accept the lot, to reject the lot or to continue sampling is taken. This scheme provides for a minimum amount of inspection.

Sequential schemes are considered to require most care and supervision in operation. Where the inspection or testing costs per article are high and sampling destructive, utmost economy in the number of articles inspected is important and often outweighs administrative convenience.

***Sequential Probability Ratio Test*** **(S.P.R.T.).** A sampling plan satisfying the condition that the probability of rejecting the lot does not exceed $\alpha$ whenever $p \leq p_0$ and the probability of accepting lots does not exceed $\beta$ whenever $p \geq p_1$ is given by the ***sequential probability ratio test*** (SPRT), pioneered by Dr. Abraham Wald, for testing the hypothesis $H_0 : p = p_0$ against the hypothesis $H_1 : p = p_1$.

Here if we take AQL $= p_0$ ; LTPD $= 100p_1$ or lot tolerance fraction defective $p_1$ : $\alpha =$ Probability of Type I error and $\beta =$ Probability of Type II error then $\alpha$ and $\beta$ are the maximum producer's and consumer's risks respectively. SPRT is defined as follows :

Let the result of the inspection of the $i$th unit be denoted by a Bernoulli variate $X_i$, *i.e.*,

$$X_i = 1, \text{ if } i\text{th item inspected is found to be defective}$$
$$= 0, \text{ otherwise.}$$

For the incoming lot quality '$p$', if $f(x, p)$ represents the probability function of $X$ then

$$f(1, p) = p \quad \text{and} \quad f(0, p) = 1 - p$$

Let $p_{1m}$ and $p_{0m}$ be the probabilities of getting $d_m$ defectives in the sample $(X_1, X_2, \ldots, X_m)$ of size $m$ under $H_1$ and $H_0$ respectively. Then the Likelihood Ratio $\lambda_m$ is given by :

$$\lambda_m = \frac{p_{1m}}{p_{om}} = \frac{\prod_{i=1}^{m} f(x_i, p_1)}{\prod_{i=1}^{m} f(x_i, p_0)} = \prod_{i=1}^{m} \frac{f(x_i, p_1)}{f(x_i, p_0)} = \frac{p_1^{d_m} (1 - p_1)^{m - d_m}}{p_0^{d_m} (1 - p_0)^{m - d_m}} \qquad \ldots(1·23)$$

SPRT *is carried out as follows* : At each stage of the experiment, at the inspection of the $m$th for each possible integral value $m$, we compute $\lambda_m$ and

(*i*) If $\lambda_m \geq A$, we terminate the process with rejection of the lot.
(*ii*) If $\lambda_m \leq B$, we terminate the process with acceptance of the lot.
(*iii*) If $B < \lambda_m < A$, we continue the sampling by taking an additional observation, $\qquad \ldots(1·23a)$

where $A$ and $B$ are constants determined in terms of $\alpha$ and $\beta$ and are given by

$$A = (1 - \beta)/\alpha \quad \text{and} \quad B = \beta/(1 - \alpha) \qquad \ldots(1·23b)$$

For computational points of view, it would be much easier to deal with $\log \lambda_m$ rather than with $\lambda_m$. Thus SPRT can be restated as follows :

(*i*) If $\lambda_m \geq \log \log A$, reject the lot,
(*ii*) If $\log \lambda_m \leq \log B$, accept the lot, and
(*iii*) If $\log B < \log \lambda_m < \log A$, continue sampling by taking one more observation. $\qquad \ldots(1·23c)$

$$\log \lambda_m = d_m \log \left(\frac{p_1}{p_0}\right) + (m - d_m) \log \left(\frac{1-p_1}{1-p_0}\right)$$

Hence *accept the lot if*

$$d_m \log \left(\frac{p_1}{p_0}\right) + (m - d_m) \log \left(\frac{1-p_1}{1-p_0}\right) \le \log B \Rightarrow d_m \le \frac{\log B - m \log \left(\frac{1-p_1}{1-p_0}\right)}{\log \left(\frac{p_1}{p_0}\right) - \log \left(\frac{1-p_1}{1-p_0}\right)} = a_m \text{ (say)} \quad ...(1{\cdot}24a)$$

*Reject the lot if*

$$d_m \log \left(\frac{p_1}{p_0}\right) + (m - d_m) \log \left(\frac{1-p_1}{1-p_0}\right) \ge \log A \Rightarrow d_m \ge \frac{\log A - m \log \left(\frac{1-p_1}{1-p_0}\right)}{\log \left(\frac{p_1}{p_0}\right) - \log \left(\frac{1-p_1}{1-p_0}\right)} = r_m \text{ (say)} \quad ...(1{\cdot}24b)$$

*Continue sampling if* $a_m < d_m < r_m$ ...(1·24c)

For each $m$, $a_m$ and $r_m$ are known as acceptance number and rejection number respectively.

***Procedure.*** At each stage of the experiment, we compute $a_m$ and $r_m$ and we continue inspection as long as $a_m < d_m < r_m$. The first time when this inequality is violated, the inspection is stopped and then

(*i*) if $d_m \ge r_m$, lot is rejected, and

(*ii*) if $d_m \le a_m$, lot is accepted.

***Remark.*** If we write

$$g_1 = \log (p_1/p_0), g_2 = \log \left(\frac{1-p_0}{1-p_1}\right); \ \log A = a, \log B = -b \quad ...(1{\cdot}24d)$$

and $$s = \frac{\log (1-p_0/1-p_1)}{\log (p_1/p_0) - \log (1-p_1/1-p_0)} = \frac{g_2}{g_1+g_2}$$

then the acceptance and rejection lines $L_1$ and $L_2$ are given by the following equations :

$$\textit{Acceptance Line } L_1 : d_m = a_m = \frac{-b}{g_1+g_2} + \frac{mg_2}{g_1+g} \Rightarrow d_m = -h_1 + sm \quad ...(1{\cdot}24e)$$

where $$h_1 = \frac{b}{g_1+g_2} \quad ...(1{\cdot}24f)$$

and $-h_1$ gives the intercept of the line $L_1$ on the $d_m$ axis.

*Rejection Line* $L_2$ :

$$d_m = r_m = \frac{a}{g_1+g_2} + m\frac{g_2}{g_1+g_2} \Rightarrow d_m = h_2 + sm \quad ...(1{\cdot}24g)$$

where $h_2 = \dfrac{a}{g_1+g_2}$ ...(1·24h)

is the intercept of the line $L_2$ on the $d_m$ axis.

It is obvious from the equations (1·24*e*) and (1·24*g*) that the acceptance and the rejection lines are parallel to each other, their slope being *s*.

It may be pointed out that $d_m$ is the cumulative number of defectives, $m$ is the cumulative number of observations, at the stage considered.

First of all, we plot the two lines $L_1$ and $L_2$. If at any stage the point $(m, d_m)$ lies between the two lines, the sampling is to be continued by taking an additional observation. If the point $(m, d_m)$ lies above or one line $L_2$, the lot is rejected and if the point $(m, d_m)$ lies below or on the line $L_1$, lot is accepted.

***Remark.*** Dividing (1·24 *f* ) by (1·24*h*), we get :

$$\frac{h_1}{h_2} = \frac{b}{a} \quad \Rightarrow \quad ah_1 - bh_2 = 0 \qquad \text{...(1·24 } i\text{)}$$

**OC of Sequential Sampling Plan.** The *OC* function of a SPRT for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ in sampling from population with density function $f(x, \theta)$ is given by :

$$L(\theta) = P_a(\theta) = \frac{A^h(\theta) - 1}{A^h(\theta) - B^h(\theta)} \qquad \text{...(1·25)}$$

where, for each value of $\theta$, the value of $h(\theta)$ is to be determined so that $h(\theta) \neq 0$ and

$$E\left[\frac{f(x, \theta_1)}{f(x, \theta_0)}\right]^{h(\theta)} = 1, \qquad \text{...(1·25}a\text{)}$$

where $A$ and $B$ have been defined in (1·23*b*).

Thus the *O.C.* function of S.P.R.T. for testing $H_0 : p = p_0$ against $H_1 : p = p_1$ is given by :

$$L(p) = \frac{A^h - 1}{A^h - B^h} = L(p, h), \text{ (say)} \qquad \text{...(1·26)}$$

where $h = h(p)$ is obtained by the equation :

$$\sum_{x=0}^{1} \left[\frac{f(x, p_1)}{f(x, p_0)}\right]^h f(x, p) = 1$$

$$\Rightarrow \quad \left[\frac{f(1, p_1)}{f(1, p_0)}\right]^h f(1, p) + \left[\frac{f(0, p_1)}{f(0, p_0)}\right]^h f(0, p) = 1$$

$$\Rightarrow \quad p\left(\frac{p_1}{p_0}\right)^h + (1-p)\left(\frac{1-p_1}{1-p_0}\right)^h = 1 \qquad \text{...(1·27)}$$

The solution of (1·27) for $h = h(p)$ is very tedious. From practical. point of view, to draw the *OC* curve, it is necessary to solve (1·27) for *hg* instead we may regard $h$ as a parameter and solve (1·27) for $p$ thus giving

$$p = \frac{1 - \left(\dfrac{1-p_1}{1-p_0}\right)^h}{\left(\dfrac{p_1}{p_0}\right)^h - \left(\dfrac{1-p_1}{1-p_0}\right)^h} = p(h), \text{ (say)} \qquad \text{...(1·28)}$$

Now, various points on the *OC* curve are obtained by giving arbitrary values to $h$ and computing corresponding values of $p$ and $L(p)$ from (1·28) and (1·26) respectively.

**Remark.** If $h$ assumes negative values, *i.e.*, if instead of $h$ we take $-h$ where now $h > 0$, then

$$L(p, -h) = \frac{A^{-h} - 1}{A^{-h} - B^{-h}} = \left(\frac{1 - A^h}{B^h - A^h}\right) B^h = \left(\frac{A^h - 1}{A^h - B^h}\right) B^h$$

$$\therefore \quad L(p, -h) = B^h \, . \, L(p, h) \qquad \ldots(1\cdot29)$$

and

$$p(-h) = \frac{\left(\frac{1-p_1}{1-p_0}\right)^h - 1}{\left(\frac{1-p_1}{1-p_0}\right)^h - \left(\frac{p_1}{p_0}\right)^h} \cdot \left(\frac{p_1}{p_0}\right)^h = p(h) \, . \left(\frac{p_1}{p_0}\right)^h \qquad \ldots(1\cdot29a)$$

Thus for negative values of $h$, the points on the $OC$ curve can be obtained from equations (1·29) and (1·29$a$).

***Five Points on OC Curve.*** Often, a sufficient appraisal of the $OC$ can be obtained from the following five easily computed points on the curve.

Since a lot containing no defective ($p = 0$) will always be accepted and a lot with 100% defective ($p = 1$) is sure to be rejected, we have

$$L(0) = 1 \quad \text{and} \quad L(1) = 0$$

$$\begin{aligned} L(p_0) &= P\,(\text{Accepting a lot of quality } p_0) \\ &= 1 - P\,(\text{Rejecting a lot of quality } p_1) \\ &= 1 - \alpha \\ L(p_1) &= P\,(\text{Accepting a lot of quality } p_1) = \beta \end{aligned}$$

Let $p = p'$ when $h = 0$, *i.e.*, 

$$p' = \lim_{h_0 \to 0} p = \lim_{h_0 \to 0} \frac{1 - \left(\frac{1-p_1}{1-p_0}\right)^h}{\left(\frac{p_1}{p_0}\right)^h - \left(\frac{1-p_1}{1-p_0}\right)^h} \qquad \ldots(1\cdot30)$$

This is the indeterminate form $\frac{0}{0}$ and hence by L'Hospital's rule, we get

$$p' = \lim_{h \to 0} \frac{\left(\frac{1-p_1}{1-p_0}\right)^h \log\left(\frac{1-p_1}{1-p_0}\right)}{\left(\frac{p_1}{p_0}\right)^h \log \frac{p_1}{p_0} - \left(\frac{1-p_1}{1-p_0}\right)^h \log \frac{1-p_1}{1-p_0}} = \frac{-\log\left(\frac{1-p_1}{1-p_0}\right)}{\log\left(\frac{p_1}{p_0} - \log \frac{1-p_1}{1-p_0}\right)}$$

$$p' = \lim_{h \to 0} L(p) = \lim_{h \to 0} \frac{A^h - 1}{A^h - B^h} = \frac{\log A}{\log A - \log B} \quad \text{(L'Hospital's Rule)} \qquad \ldots(1\cdot30a)$$

***Remark.*** Using the notations of (1·24$d$), we get from (1·30) and (1·30$a$)

$$p' = \frac{g_2}{g_1 + g_2} = s \qquad \ldots(1\cdot30b)$$

$$L(p') = \frac{a}{a+b} = \frac{h_2}{h_1 + h_2} \qquad \ldots(1\cdot30c)$$

[On dividing numerator and denominator by $g_1 + g_2$ and using (1·24$f$) and (1·24$h$).]

The five points for the $OC$ curve are expressed in a tabular form in the adjoining table :

| $h$ | $p$ | $L(p)$ |
|---|---|---|
| $\infty$ | 0 | 1 |
| 1 | $p_0 = AQL$ | $1 - \alpha$ |
| 0 | $p'$ | $\frac{a}{a+b} = \frac{h_2}{h_1 + h_2}$ |
| $-1$ | $p_1 = LTFD$ | $\beta$ |
| $-\infty$ | 1 | 0 |

**ASN Function of Sequential Sampling Plan.** The sample size $n$ in sequential testing is a random variable which can be determined in terms of the density function $f(x, \theta)$. The ASN function of an SPRT for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ is given by :

$$E(n) = \frac{L(\theta) \log L(\theta) \log B + [1 - L(\theta)] \log A}{E(z)}, \text{ where } z = \log \frac{f(x, \theta_1)}{f(x, \theta_0)} \quad \ldots(1\cdot31)$$

Thus for sequential sampling plan with AQL $p_0$ and LTFD $p_1$ (*i.e.*, for testing $H_0 : p = p_0$ against $H_1 : p = p_1$), we have

$$E(n) = \frac{L(p) \log B + [1 - L(p)] \log A}{E(z)} \quad \ldots(1\cdot32)$$

where $$z = \log \frac{f(x, p_1)}{f(x, p_0)} \;;\; A = \frac{1-\beta}{\alpha} \;;\; B = \frac{\beta}{1-\alpha} \quad \ldots(1\cdot32a)$$

$$E(z) = E \log \frac{f(x, p_1)}{f(x, p_0)} = \sum_{x=0}^{1} f(x, p). \log \frac{f(x, p_1)}{f(x, p_0)}$$

$$= p \log \frac{p_1}{p_0} + (1-p) \log \frac{1-p_1}{1-p_0}$$

Hence, $$E(n) = \frac{L(p) \log B + [1 - L(p)] \log A}{p \log \left(\frac{p_1}{p_0}\right) + (1-p) \log \left(\frac{1-p_1}{1-p_0}\right)} \quad \ldots(1\cdot32b)$$

which is the required ASN function.

***Five Points on ASN Curve.*** A sufficiently good idea of ASN curve for the sequential sampling plan can be obtained from suitably chosen five points which are easy to obtain. The ASN curve so obtained is referred to as 5-point ASN curve.

The general 5-points on ASN curve corresponding to $p = 0, 1, p_1$ (LTFD), $p_0$ (AQL) and $s$ are obtained from (1·32$b$) as explained below.

When $p = 0, L(p) = 1$

$$\therefore \quad E(n) = \frac{\log B}{\log [(1-p_1)/(1-p_2)]} = \frac{-b}{-g_2} = \frac{b}{g_2} = \frac{b/(g_1+g_2)}{g_2/(g_1+g_2)} = \frac{h_1}{s} \quad \ldots(1\cdot33)$$

When $p_1 = p, L(p_1) = \beta$

$$\therefore \quad E(n) = \frac{\beta \log B + (1-\beta) \log A}{p_1 \log \frac{p_1}{p_2} + (1-p_1) \log \frac{1-p_1}{1-p_0}}$$

$$= \frac{a - (a+b)\beta}{p_1 g_1 - (1-p_1) g_2} = \frac{(1-\beta)a - b\beta}{(g_1+g_2) - g_2} \quad \ldots(1\cdot33a)$$

$$= \frac{(1-\beta) h_2 - \beta h_1}{p_1 - s} \quad \text{[Dividing numerator and denominator by } g_1 + g_2] \ldots(1\cdot33b)$$

When $p = p_0, L(p_0) = 1 - \alpha$

$$\therefore \quad E(n) = \frac{-(1-\alpha) b + \alpha a}{p_0 g_1 - (1-p_0) g_2} = \frac{\alpha (a+b) - b}{p_0 (g_1+g_2) - g_2} \quad \ldots(1\cdot33c)$$

$$= \frac{-(1-\alpha) h_1 + \alpha h_2}{p_0 - s}$$

$$= \frac{(1-\alpha) h_1 - \alpha h_2}{s - p_0} \quad \ldots(1\cdot33d)$$

When $p = 1, L(p) = 0$

$$\therefore \quad E(n) = \frac{a}{g_1} = \frac{h_2}{1-s} \qquad \ldots(1\cdot33e)$$

[c.f. (1·30b) and (1·30c)]

When $p = s, L(p) = \dfrac{h_2}{h_1 + h_2}$

$$\therefore \quad E(n) = \frac{-\left(\frac{h_2}{h_1+h_2}\right)b + \left(\frac{h_1}{h_1+h_2}\right)a}{sg_1 - (1-s)g_2}$$

$$= \frac{ah_1 - bh_2}{h_1 + h_2} \quad \frac{1}{s(g_1+g_2) - g_2} \left(\frac{0}{0}\text{ Form}\right)$$

[From (1·24i) and (1·30b)]

$$= \frac{h_1 h_2}{s(1-s)} \qquad \ldots(1\cdot33f)$$

These five points obtained in equations (1·33) to (1·33f) are expressed in the tabular form in the adjoining table :

FIVE POINTS ON ASN CURVE FOR SEQUENTIAL SAMPLING PLAN

| $P$ | ASN |
|---|---|
| 0 | $\dfrac{b}{g_2}$ or $\dfrac{h_1}{s}$ |
| $p_0$ (AQL) | $\dfrac{\alpha(a+b)-b}{p_0(g_1+g_2)-g_2}$ or $\dfrac{(1-\alpha)h_1 - \alpha h_2}{s-p_0}$ |
| $s$ | $\dfrac{h_1 h_2}{s(1-s)}$ |
| $p_1$ (LTFD) | $\dfrac{a-\beta(a+b)}{p_1(g_1+g_2)-g_2}$ or $\dfrac{(1-\beta)h_2 - \beta h_1}{p_1 - s}$ |
| 1 | $\dfrac{a}{g_1}$ or $\dfrac{h_2}{1-s}$ |

***Remarks*** **1.** Although sequential inspection plan provides for an infinite number of stages, it has been established mathematically that sequential process ultimately terminals with probability one. For a detailed discussion on SPRT the reader is referred to the book 'Sequential Analysis' by A. Wald, published by John Wiley & Sons, New York (1947).

**2.** The chief advantage of sequential plan is the reduction in the A.S.N. As compared with single sampling plan, sequential plan requires, on the average, 33% to 50% less inspection for the same degree of protection, *i.e.*, for same values of $\alpha$ and $\beta$.

**3.** *ASN* is maximum at $p = s$.

**4.** $p_0 < s < p_1$

**Proof.** In the usual notations : $s = \dfrac{g_2}{g_1+g_2} \quad \Rightarrow \quad 1-s = \dfrac{g_2}{g_1+g_2}$

Dividing, we get
$$\frac{s}{1-s} = \frac{g_2}{g_1} \qquad \ldots(1$$

We have
$$g_1 = \log(p_1/p_0) = \log p_1 - \log p_0$$

Using mean value theorem from differential calculus, *viz.*,
$$f(b) - f(a) = (b-a)f'(c); \quad a < c < b,$$
with
$$f(x) = \log x, \text{ we get}$$

$$g_1 = (p_1 - p_0)\left(\frac{1}{c_1}\right); \quad p_0 < c_1 < p_1$$

Similarly,
$$g_2 = \log\left(\frac{1-p_0}{1-p_1}\right) = \log(1-p_0) - \log(1-p_1)$$

$$= (p_1 - p_0)\frac{1}{c_2}, \; 1-p_1 < c_2 < 1-p_0$$

Substituting in (1), we get
$$\frac{s}{1-s} = \frac{c_1}{c_2}$$

$$\therefore \quad E(n) = \frac{a}{g_1} = \frac{h_2}{1-s} \quad \ldots(1\cdot33e)$$

When $p = s, L(p) = \dfrac{h_2}{h_1 + h_2}$ [c.f. (1·30b) and (1·30c)]

$$\therefore \quad E(n) = \frac{-\left(\frac{h_2}{h_1+h_2}\right)b + \left(\frac{h_1}{h_1+h_2}\right)a}{sg_1 - (1-s)g_2}$$

$$= \frac{ah_1 - bh_2}{h_1 + h_2} \cdot \frac{1}{s(g_1+g_2) - g_2} \left(\frac{0}{0} \text{ Form}\right)$$

[From (1·24i) and (1·30b)]

$$= \frac{h_1 h_2}{s(1-s)} \quad \ldots(1\cdot33f)$$

These five points obtained in equations (1·33) to (1·33f) are expressed in the tabular form in the adjoining table :

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| $s$ | $\dfrac{h_1 h_2}{s(1-s)}$ |
| $p_1$ (LTFD) | $\dfrac{a-\beta(a+b)}{p_1(g_1+g_2)-g_2}$ or $\dfrac{(1-\beta)h_2 - \beta h_1}{p_1 - s}$ |
| 1 | $\dfrac{a}{g_1}$ or $\dfrac{h_2}{1-s}$ |

***Remarks*** **1.** Although sequential inspection plan provides for an infinite number of stages, it has been established mathematically that sequential process ultimately terminals with probability one. For a detailed discussion on SPRT the reader is referred to the book 'Sequential Analysis' by A. Wald, published by John Wiley & Sons, New York (1947).

**2.** The chief advantage of sequential plan is the reduction in the A.S.N. As compared with single sampling plan, sequential plan requires, on the average, 33% to 50% less inspection for the same degree of protection, *i.e.*, for same values of $\alpha$ and $\beta$.

**3.** $ASN$ is maximum at $p = s$.

**4.** $p_0 < s < p_1$

**Proof.** In the usual notations : $s = \dfrac{g_2}{g_1 + g_2} \Rightarrow 1 - s = \dfrac{g_2}{g_1+g_2}$

Dividing, we get
$$\frac{s}{1-s} = \frac{g_2}{g_1} \quad \ldots(1)$$

We have $g_1 = \log(p_1/p_0) = \log p_1 - \log p_0$

Using mean value theorem from differential calculus, *viz.*,

$$f(b) - f(a) = (b-a)f'(c); \quad a < c < b,$$

with $f(x) = \log x$, we get

$$g_1 = (p_1 - p_0)\left(\frac{1}{c_1}\right); \quad p_0 < c_1 < p_1 \quad \ldots(2)$$

Similarly,
$$g_2 = \log\left(\frac{1-p_0}{1-p_1}\right) = \log(1-p_0) - \log(1-p_1)$$

$$= (p_1 - p_0)\frac{1}{c_2},\ 1 - p_1 < c_2 < 1 - p_0 \quad \ldots(3)$$

Substituting in (1), we get
$$\frac{s}{1-s} = \frac{c_1}{c_2} \quad \ldots(4)$$

∴ From (2) and (3), we get

$$c_1 < p_1,\ \frac{1}{c_2} < \frac{1}{1-p_1} \qquad\qquad c_1 > p_0,\ \frac{1}{c_2} > \frac{1}{1-p_0}$$

$$\Rightarrow \frac{c_1}{c_2} < \frac{p_1}{1-p_1} \qquad\qquad \Rightarrow \frac{c_1}{c_2} > \frac{p_0}{1-p_0}$$

Hence, $\dfrac{p_0}{1-p_0} < \dfrac{c_1}{c_2} < \dfrac{p_1}{1-p_1} \Rightarrow \dfrac{p_0}{1-p_0} < \dfrac{s}{1-s} < \dfrac{p_1}{1-p_2}$ (Using 4)

$$\Rightarrow 1+\frac{p_0}{1-p_0} < 1+\frac{s}{1-s} < 1+\frac{p_1}{1-p_1} \Rightarrow \frac{1}{1-p_0} < \frac{1}{1-s} < \frac{1}{1-p_1} \text{ or } 1-p_1 < 1-s < 1-p_0 \text{ or } p_0 < s < p_1.$$

***Example 1·18.*** *It is desired to run a risk of 1 in 100 in rejecting a lot which is as good as 15% defective and 2 in accepting a lot which is as bad as 30% defective. Draw the decision lines and plot the OC and ASN curves for the above sequential sampling plan. How many units would you require to arrive at a decision for the following sequence of inspected items :*

*"N N D N N N N N N N N N N N N N D*
*N N N N N N N N N D NNN, NNN, NN".*

*where D stands for defective item and N for non-defective item.*

***Solution.*** For the above sequential sampling plan, we have in the usual notations :
$p_0$ (AQL) = 0·15; $p_1$ (LTFD) = 0·30 ; α ((Producer's risk) = 0·01 ; β(Consumer's risk) = 0·02

$$a = \log A = \log\left(\frac{1-\beta}{\alpha}\right) = \log\left(\frac{0.98}{0.01}\right) = 1.99123$$

$$b = -\log B = \log\left(\frac{1}{B}\right) = \log\left(\frac{1-\alpha}{\beta}\right) = \log\left(\frac{.99}{.02}\right) = 1.69461$$

$$\log\left(\frac{p_1}{p_0}\right) = \log\left(\frac{0.30}{0.15}\right) = 0.30103$$

$$\log\left(\frac{1-p_1}{1-p_0}\right) = \log\left(\frac{0.70}{0.85}\right) = \log 0.8235 = \bar{1}.91567$$

$$g_1 = \log\frac{p_1}{p_0} = 0.30103$$

$$g_2 = \log\left(\frac{1-p_0}{1-p_1}\right) = -\log\left(\frac{1-p_1}{1-p_0}\right)$$

$$= -(\bar{1}.91567) = 0.08433$$

$$g_1 + g_2 = 0.30103 + 0.08433 = 0.38536$$

$$s = \frac{g_2}{g_1+g_2} = \frac{0.08433}{0.38536} = 0.2188$$

$$h_1 = \frac{b}{g_1+g_2} = \frac{1.69461}{0.38536} = 4.3975$$

$$h_2 = \frac{a}{g_1+g_2} = \frac{1.99123}{0.38536} = 5.1672$$

Hence, the acceptance and rejection lines are given by : [*c.f.* (1·24*e*) and *c.f.* (1·24*g*)]

*Acceptance Line* ($L_1$)

$$d_m = -h_1 + sm$$

$$\Rightarrow \quad d_m = -4.3975 + 0.2188m \qquad ...(*)$$

*Rejection Line* ($L_2$).

$$d_m = h_2 + sm$$

$$\Rightarrow \quad d_m = 5.1672 + 0.2188m \qquad ...(**)$$

For plotting the lines in (*) and (**), we need two points for each line which are obtained in the following table :

THE SEQUENTIAL SAMPLING PLAN

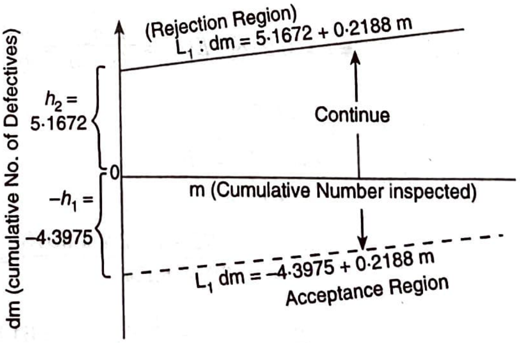


***Fig. 1·25***

| | $L_1$ | | $L_2$ | |
|---|---|---|---|---|
| $m$ | 0 | 10 | 0 | 10 |
| $d_m$ | –4·3975 | –2·2095 | 5·1672 | 7·3552 |

***OC Curve.*** When $p = s = 0{\cdot}2188$, $P(p) = \dfrac{h_1}{h_1 + h_2} = \dfrac{4{\cdot}3975}{9{\cdot}5647} = 0{\cdot}46$

5 POINTS OF THE OC CURVE

| $p$ (*Submitted lot quality*) | $L(p)$ (*Probability of Acceptance* ) |
|---|---|
| 0 | 1·00 |
| $p_0 = 0{\cdot}15$ | $1 - \alpha = 0{\cdot}99$ |
| $s = 0{\cdot}22$ | 0·46 |
| $p_2 = 0{\cdot}30$ | $\beta = 0{\cdot}02$ |
| 1·00 | 0 |

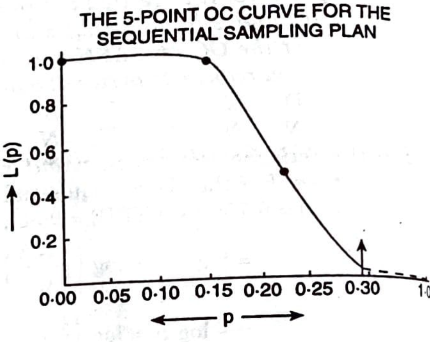


*Fig. 1·26*

The OC curve is drawn in adjoining *Fig. 1·26*.

***ASN Curve.*** The general five points on the ASN curve are obtained as follows :

When $\quad p = 0,\ E(n) = \dfrac{b}{g_2} = \dfrac{1{\cdot}69461}{0{\cdot}08433} = 20{\cdot}095$

$p = p_0 = 0{\cdot}15,$

When $E(n) = \dfrac{(1-\alpha)\,h_1 - \alpha h_2}{s - p_0} = \dfrac{0{\cdot}99 \times 4{\cdot}3975 - 0{\cdot}01 \times 5{\cdot}1672}{0{\cdot}2188 - 0{\cdot}15} = 62{\cdot}5262$

When $\quad p = s = 0{\cdot}2188,\ E(n) = \dfrac{h_1 h_2}{s(1-s)} = \dfrac{4{\cdot}3975 \times 5{\cdot}1672}{0{\cdot}2188 \times 0{\cdot}7812} = 132{\cdot}96$

For $\quad p = p_1 = 0{\cdot}30,$

$$E(n) = \frac{(1-\beta)\,h_2 - \beta h_1}{p_1 - s}$$

$$= \frac{0{\cdot}98 \times 5{\cdot}1672 - 0{\cdot}02 \times 4{\cdot}3975}{0{\cdot}30 - 0{\cdot}2188} = 61{\cdot}28$$

For $\quad p = 1,\ E(n) = \dfrac{a}{g_1} = \dfrac{1{\cdot}99123}{0{\cdot}30103} = 6{\cdot}61472$

Thus, the points of the ASN curve can be tabulated as follows:

| $p$ (Submitted lot quality) | ASN (Average Sample Number) |
|---|---|
| 0·00 | 20·0954 ≃ 20·10 |
| 0·15 | 62·5262 ≃ 62·53 |
| 0·2188 | 132·96 |
| 0·30 | 61·28 |
| 1·00 | 6·6172 ≃ 6·62 |

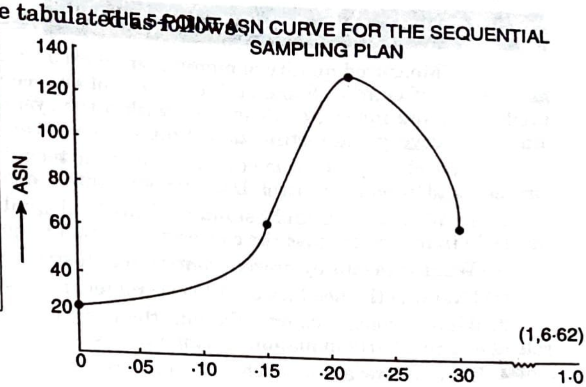


*Fig. 1·27*

## Sample Size for arriving at a Decision

SEQUENTIAL SAMPLING PLAN

$a_m = -4{\cdot}3975 + 0{\cdot}2188m$ ; $r_m = 5{\cdot}1672 + 0{\cdot}2188m$

| $m$ | $d_m$ | $a_m$ | $r_m$ | $m$ | $d_m$ | $a_m$ | $r_m$ |
|---|---|---|---|---|---|---|---|
| 1 | 0 | –4·1787 | 5·386 | 18 | 2 | –0·4591 | 9·1056 |
| 2 | 0 | –3·9599 | 5·6048 | 19 | 2 | –0·2403 | 9·3244 |
| 3 | 1 | –3·7411 | 5·8236 | 20 | 2 | –0·0215 | 9·5432 |
| 4 | 1 | –3·5223 | 6·0424 | 21 | 2 | 0·1973 | 9·7620 |
| 5 | 1 | –3·3035 | 6·2612 | 22 | 2 | 0·4161 | 9·9808 |
| 6 | 1 | –3·0847 | 6·4800 | 23 | 2 | 0·6349 | 10·1996 |
| 7 | 1 | –2·8659 | 6·6988 | 24 | 2 | 0·8537 | 10·4184 |
| 8 | 1 | –2·6471 | 6·9176 | 25 | 2 | 1·0725 | 10·6372 |
| 9 | 1 | –2·4283 | 7·1364 | 26 | 2 | 1·2913 | 10·856 |
| 10 | 1 | –2·2095 | 7·3552 | 27 | 3 | 1·5110 | 11·0748 |
| 11 | 1 | –1·9907 | 7·5740 | 28 | 3 | 1·7289 | 11·2936 |
| 12 | 1 | –1·7719 | 7·7928 | 29 | 3 | 1·9477 | 11·5124 |
| 13 | 1 | –1·5531 | 8·0116 | 30 | 3 | 2·1665 | 11·7312 |
| 14 | 1 | –1·3343 | 8·2304 | 31 | 3 | 2·3853 | 11·9500 |
| 15 | 1 | –1·1155 | 8·4492 | 32 | 3 | 2·6041 | 12·1688 |
| 16 | 1 | –0·8967 | 8·6680 | 33 | 3 | 2·8229 | 12·3876 |
| 17 | 2 | –0·6779 | 8·8868 | 34 | 3 | 3·0417 | 12·6064 |

**For $m = 34$, $d_m$ lies outside $a_m$ and $r_m$. In fact $d_m < a_m$ at $m = 34$. Hence, sequential sampling plan is terminated with the acceptance of the lot after inspecting the 34th item.**